Homework 5

Questions marked with an * are optional, i.e. not for credit.

1) What is $|\mathbf{C}_p^\times|_p$?

2) Show that Krasner's lemma can fail for polynomials with repeated roots, i.e. exhibit a reducible polynomial which is a limit of irreducible polynomials.

3) If p is a prime let

$$\Phi_p(X) = X^{p-1} + X^{p-2} + ... + X + 1$$

and

$$\Phi_{p^m}(X) = \Phi_p(X^{p^{m-1}})$$

for all $m \in \mathbf{Z}_{>0}$. Show that the roots of $\Phi_{p^m}(X)$ over a field of characteristic 0 are the primitive $(p^m)^{th}$ roots of 1.

(a) What is the constant term of $\Phi_{p^m}(1+Y)$? Show that

$$Y^{p^{m-1}}\Phi_{p^m}(1+Y) \equiv ((1+Y)^{p^{m-1}} - 1)\Phi_{p^m}(1+Y) \equiv (1+Y)^{p^m} - 1 \equiv Y^{p^m} \bmod p.$$

[Hint: Recall that over $\mathbf{F}_p$ we have $(X+Y)^p = X^p+Y^p$.] Use Eisenstein's criterion to deduce that $\Phi_{p^m}(1+Y)$ is irreducible over $\mathbf{Q}_p$. Conclude that $\Phi_{p^m}(X)$ is irreducible over $\mathbf{Q}_p$.

(b) If ζ is a root of $\Phi_{p^m}(X)$ show that $\Phi_{p^m}(X)$ splits over $\mathbf{Q}_p(\zeta)$. Deduce that $\mathbf{Q}_p(\zeta)/\mathbf{Q}_p$ is a Galois extension of degree $p^m - p^{m-1}$. Define a map

$$\chi : \mathrm{Gal}(\mathbf{Q}_p(\zeta)/\mathbf{Q}_p) \longrightarrow (\mathbf{Z}/p^m\mathbf{Z})^\times$$

by

$$\sigma(\zeta) = \zeta^{\chi(\sigma)}.$$

Show that χ is an isomorphism of groups.

(c) If $K/\mathbf{Q}_p$ is a finite extension, show that K contains only finitely many roots of unity. [Hint: consider the case of roots of unity of p-power order and of order prime to p separately.]

(d) What is the norm from $\mathbf{Q}_p(\zeta)$ to $\mathbf{Q}_p$ of $1-\zeta$? What is $|1-\zeta|_p$?

(e) Now suppose that $m = 1$ and that $\alpha \in \mathbf{Q}_p(\zeta)$ satisfies

$$|\alpha - 1|_p \leq p^{-(p+1)/(p-1)}.$$

Show that α has a p^{th} root in $\mathbf{Q}_p(\zeta_p)$. [Hint: consider the polynomial

$$(1 + (\zeta - 1)Y)^p - \alpha.]$$

4) Let

$$L(T) = T - T^2/2 + T^3/3 + ... \in \mathbf{Q}[[T]]$$

and

$$E[T] = 1 + T + T^2/2! + T^3/3! + ... \in \mathbf{Q}[[T]].$$

Also suppose that $K/\mathbf{Q}_p$ is a finite extension and let $\mu(K)$ denote the group of roots of unity in K.

(a) Show that

$$L(S + T + ST) - L(S) - L(T) = 0$$

in $\mathbf{Q}[[S,T]]$. [Hint: differentiate with respect to S and with respect to T.]

(b) Show that

$$L(E(T) - 1) = T$$

in $\mathbf{Q}[[T]]$. [Hint: differentiate with respect to T.]

(c) Show that $L(t)$ converges for all $t \in \mathbf{C}_p$ with $|t|_p < 1$. Show that if $|t|_p < 1$ and $|s|_p < 1$ then

$$L((1+t)(1+s) - 1) = L(s) + L(t).$$

(d) Show that $E(t)$ converges for $t \in \mathbf{C}_p$ with $|t|_p < p^{-1/(p-1)}$ and that for t in this region

$$L(E(t) - 1) = t.$$

[Hint: use question 8)(a) of homework 4.]

(e) Show that the zeros of $L(T)$ in $\{t \in \mathbf{C}_p : \ |t|_p < 1\}$ are exactly the $\zeta - 1$ as ζ runs over p-power roots of unity. [Hint: use question 6) of homework 4.]

(f) Show that

$$K^\times \cong \mathbf{Z} \times \mu(K) \times \mathbf{Z}_p^{[K:\mathbf{Q}_p]}$$

as a topological abelian group.

(g*) If $K/\mathbf{Q}_p$ is Galois, show moreover that

$$(\mathcal{O}_K^\times/\mu(K)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \cong \mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q}_p)]$$

as $\mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q}_p)]$-modules. [Hint: use the normal basis theorem.]