Homework 4: FACTORING POLYNOMIALS AND POWER SERIES

1) Let K be complete with respect to a non-trivial non-archimedean absolute value $|\ |$. Is every absolute value on $K(T)$ which extends $|\ |$ equivalent to $|\ |_c$ for some $c \in \mathbf{R}_{>0}$?

2) Let K be complete with respect to a non-trivial non-archimedean absolute value $|\ |$. Let $c \in \mathbf{R}_{>0}$ and let $K[T]_c \subset K[[T]]$ denote the subset of formal sums

$$f = \sum_{n=0}^{\infty} a_n T^n$$

with $|a_n|c^n \to 0$ as $n \to \infty$. [Thus $K[T]_c$ is the set of power series over K which "converge in a closed ball radius c".] If $f \in K[T]_c$ define

$$|f|_c = sup_n |a_n| c^n.$$

(a) Show that $|f|_c = 0$ if and only if $f = 0$.
(b) Show that $K[T]_c$ is closed under addition and that $|f+g|_c \le \max\{|f|_c, |g|_c\}$.
(c) Show that $K[T]_c$ is closed under multiplication and that $|fg|_c = |f|_c|g|_c$.
(d) Show that $K[T]_c$ is complete with respect to $|\ |_c$.
(e) Show that $K[T]$ is dense in $K[T]_c$.
(f) If $h \in K[T]_c$ and $f(T) = f_0 + f_1T + ... + f_nT^n \in K[T]$ satisfies $|f|_c = |f_nT^n|_c$ show that we can uniquely write

$$h = qf + r$$

where $q \in K[T]_c$, where $r \in K[T]$ has degree less than n, and where

$$|r|_c \le |h|_c \qquad \text{and} \qquad |q|_c \le |h|_c/|f|_c.$$

[Hint: use part (e).]

3) Which of the following polynomials are irreducible over $\mathbf{Q}_5$?

$$X^3 + 5X + 25, X^3 + 5X^2 + 25, X^4 + 5X^2 + 25, X^4 + 15X^2 + 25.$$

4) If p is an odd prime determine modulo p^3 the monic irreducible factors of $X^3 + 2pX^2 + pX + p^2$.

5) How many roots does $X^3 + 25X^2 + X - 9$ have in $\mathbf{Q}_p$ for $p = 2, 3, 5, 7$?

6) Keep the notation and assumptions of question 2). If

$$f = \sum_{n=0}^{\infty} a_n T^n \in K[T]_c,$$

let $\mathrm{NP}(f)$ denote the boundary of the smallest convex set containing the points $(n, v(a_n))$ for all $n \in \mathbf{Z}_{\geq 0}$ and $(0, y)$ for y any sufficiently large real number. Suppose $h \in K[T]_c$. Let $\mathcal{L}$ denote the lowest line of slope $\log_q c$ which meets $\mathrm{NP}(h)$. Let m denote the largest x-coordinate of a point of intersection of $\mathcal{L}$ with $\mathrm{NP}(h)$. Why is m an integer? Show that we can write

$$h = fg$$

where $f \in K[T]$ is a polynomial of degree m, where

$$\mathrm{NP}(f) = \mathrm{NP}(h)|_{[0,m]}$$

and where $g = \sum_{n=0}^{\infty} b_n T^n \in K[T]_c$ satisfies

$$|b_0| > |b_i| c^i$$

for all $i > 0$.

Deduce that there are only finitely many $\alpha \in K$ with $|\alpha| \leq c$ and $h(\alpha) = 0$. Also deduce that the number of zeros $\alpha \in K$ with $v(\alpha) = d \geq -\log_q c$ is zero unless $\mathrm{NP}(h)$ has a side of slope $-d$, in which case the number of such zeros is positive, but less than or equal the length of the x-axis below the side with slope $-d$.

7) Suppose that K is algebraically closed and complete with respect to a non-trivial non-archimedean absolute value $|\ |$. Also suppose that

$$f(T) = 1 + \sum_{n=1}^{\infty} a_n T^n \in K[[T]]$$

converges at all elements of K. Show that f has only finitely many zeros in any closed ball $\{t \in K : |t| \leq c\}$. Let $\alpha_1, \alpha_2, \ldots.$ be the zeros of f in K in order of increasing absolute value. Show that

$$f(T) = \lim_{N \to \infty} \prod_{i=1}^{N} (1 - T/\alpha_i)$$

where the limit is taken with respect to any one of the absolute values $|\ |_c$ with $c \in \mathbf{R}_{>0}$. Deduce that for any $t \in K$

$$f(t) = \lim_{N \to \infty} \prod_{i=1}^{N} (1 - t/\alpha_i).$$

8) (a) Show that

$$v_p(n!) = [n/p] + [n/p^2] + [n/p^3] + ...$$

where $[t]$ denotes the greatest integer less than or equal to a real number t. Deduce that

$$|n!|_p > p^{-n/(p-1)}.$$

(b) If $a \in \mathbf{Q}_p$ with $|a|_p < p^{-1/(p-1)}$ and if m is a positive integer recall that we have

$$(1+a)^m = 1 + \sum_{i=1}^{\infty} m(m-1)...(m+1-i)a^i/i!.$$

Show that this can be rewritten as

$$(1+a)^m = 1 + \sum_{j=1}^{\infty} \gamma_j(a)m^j,$$

where the $\gamma_j(a) \in \mathbf{Q}_p$ do not depend on m and tend to zero as $j \to \infty$.

(c) Show that $X^2 + X + 3$ splits as $(X-\alpha)(X-\beta)$ over $\mathbf{Q}_3$ with $|\alpha|_3 = 1$ and $|\beta|_3 = 1/3$. Calculate

$$\gamma_j(3\alpha) \bmod 27$$

and

$$\gamma_j(3\beta) \bmod 81$$

for all j.

(d) Consider the recurrence relation $u(n+2) = 3u(n+1) - 5u(n)$ with $u(0) = 1$ and $u(1) = 2$. Show that $u(2m+1) \equiv -1 \bmod 3$ for all non-negative integers m. Also show that

$$u(2m) = (\alpha(1+3\beta)^m - \beta(1+3\alpha)^m)/(\alpha - \beta).$$

Use this to write

$$u(2m) - 1 = \sum_{j=1}^{\infty} \delta_j m^j$$

where $\delta_j \in \mathbf{Q}_3$ and the sum converges 3-adically for all $m \in \mathbf{Z}_3$. Calculate each δ_j modulo 81. Show that $u(2m) = 1$ for at most three non-negative integers m. Find all non-negative integers m with $u(2m) = 1$.