Homework 1: VALUATIONS AND ABSOLUTE VALUES I

1) If k is a field and $f \neq 0$ and $g \neq 0$ are polynomials over k in a single variable T, define

$$\mathrm{ord}_\infty(f/g) = \deg\, g - \deg\, f.$$

Show that $\mathrm{ord}_\infty : k(T) \to \mathbf{Z}$ is a well defined valuation.

2) If $|\ |$ is an absolute value on a field K and $\zeta \in K$ is a root of unity, show that $|\zeta| = 1$.

3) Let k be a field. Let $k((T))$ denote the set of expressions

$$\sum_{n\in\mathbf{Z}} \alpha_n T^n$$

with $\alpha_n \in k$ and $\alpha_n = 0$ for $n << 0$ (i.e. for all n less than some integer N). Show that $k((T))$ is a field with respect to the operations:

$$\sum_n \alpha_n T^n + \sum_n \beta_n T^n = \sum_n (\alpha_n + \beta_n) T^n$$

and

$$\left(\sum_n \alpha_n T^n\right)\left(\sum_n \beta_n T^n\right) = \sum_n \left(\sum_i \alpha_i \beta_{n-i}\right) T^n.$$

[In particular you need to explain why these operations are well defined. The field $k((T))$ is called the field of Laurent series over k.] Also show that the subset $k[[T]] \subset k((T))$ consisting of sums $\sum_n \alpha_n T^n$ with $\alpha_n = 0$ for $n < 0$ is a sub-integral domain with field of fractions $k((T))$. [The ring $k[[T]]$ is called the ring of formal power series over k.]

Show that the function

$$\mathrm{ord} : k((T)) \longrightarrow \mathbf{Z}$$

defined by

$$\mathrm{ord} \sum_n \alpha_n T^n = \min\{n :\ \alpha_n \neq 0\}$$

is a valuation on $k((T))$. What are $\mathcal{O}_{k((T)),\mathrm{ord}}$, $\wp_{k((T)),\mathrm{ord}}$ and $k(k((T)), \mathrm{ord})$?

Explain why the natural inclusion $k[T] \hookrightarrow k[[T]]$ extends to an inclusion $k(T) \hookrightarrow k((T))$. What is the restriction of ord to $k(T)$?

Show that $k[T]$ is dense in $k[[T]]$, i.e. if $f \in k[[T]]$ then there exist $f_i \in k[T]$ with $\mathrm{ord}(f - f_i) \to \infty$ as $i \to \infty$. Deduce that $k(T)$ is dense in $k((T))$.

4) Show that any absolute value on a field of characteristic $p > 0$ is non-archimedean.

5) Show that an extension of an absolute value satisfying the triangle inequality also satisfies the triangle inequality. [Hint: you can use basically the same argument

that we used to show that if one can take $C = 2$ in the definition of absolute value then that absolute value satisfies the triangle inequality. However you will need to find a different method to bound the absolute value of binomial coefficients.]

6) Find a series of positive integers with 5-adic limit $-1/11$. [Hint: what is the decimal expansion of $1/11$?]

7) Suppose that k is a field of characteristic $p > 0$. Let

$$f(X) = 1 + \sum_{n=1}^{\infty} a_n T^n$$

be an element of $k[[T]]$. Show that

$$f(X)^p = 1 + \sum_{n=1}^{\infty} a_n^p T^{np} \in k[[T^p]].$$

[Hint: recall that if A is a ring in which $p = 0$ and if $a, b \in A$ then $(a+b)^p = a^p + b^p$. This follows from the observation that for $i = 1, ..., p-1$ we have

$$p \left| \binom{p}{i} \right. .]$$

Then show that the product

$$\prod_{n=0}^{\infty} f(T)^{p^n}$$

converges in $k((T))$ to an element $g(T) \in k[[T]]$ and that

$$g(T)^{1-p} = f(T).$$

8) Suppose that $K \supset \mathbf{C}$ is a field and that $|\ |$ is an absolute value on K extending $|\ |_{\mathbf{C}}^{1/2}$. If $\alpha \in K - \mathbf{C}$ show that there is $a \in \mathbf{C}$ such that $|\alpha - a| \leq |\alpha - z|$ for all $z \in \mathbf{C}$. If $|z|_{\mathbf{C}}^{1/2} < |\alpha - a|$ show that for all $n \in \mathbf{Z}_{>0}$ we have

$$|\alpha - (a+z)| \leq |(\alpha - a)^n - z^n|/|\alpha - a|^{n-1} \leq |\alpha - a|(1 + (|z|/|\alpha - a|)^n),$$

and deduce that $|\alpha - a| = |\alpha - (a+z)|$. [Hint: $(\alpha - a)^n - z^n = \prod_\zeta (\alpha - a - \zeta z)$, where ζ runs over all n^{th} roots of unity.] Conclude that $|\alpha - z|$ is constant for $z \in \mathbf{C}$. Finally deduce that in fact $K = \mathbf{C}$.