

# Math 128 Lecture 18

The spin representations.

# The Clifford algebras.

Let  $\mathfrak{p}$  be a vector space with a symmetric bilinear form  $(\ , \ )$ . The **Clifford algebra** associated to this data is the algebra

$$C(\mathfrak{p}) := T(\mathfrak{p})/I$$

where  $T(\mathfrak{p})$  denotes the tensor algebra

$$T(\mathfrak{p}) = k \oplus \mathfrak{p} \oplus (\mathfrak{p} \otimes \mathfrak{p}) \oplus \cdots$$

and where  $I$  denotes the ideal in  $T(\mathfrak{p})$  generated by all elements of the form

$$y_1 y_2 + y_2 y_1 - 2(y_1, y_2)\mathbf{1}, \quad y_1, y_2 \in \mathfrak{p}$$

and  $\mathbf{1}$  is the unit element of the tensor algebra. The space  $\mathfrak{p}$  injects as a subspace of  $C(\mathfrak{p})$  and generates  $C(\mathfrak{p})$  as an algebra.

# Clifford maps.

A linear map  $f$  of  $\mathfrak{p}$  to an associative algebra  $A$  with unit  $1_A$  is called a **Clifford map** if

$$f(y_1)f(y_2) + f(y_2)f(y_1) = 2(y_1, y_2)1_A, \quad \forall y_1, y_2 \in \mathfrak{p}$$

or what amounts to the same thing (by polarization since we are not over a field of characteristic 2) if

$$f(y)^2 = (y, y)1_A \quad \forall y \in \mathfrak{p}.$$

Any Clifford map gives rise to a unique algebra homomorphism of  $C(\mathfrak{p})$  to  $A$  whose restriction to  $\mathfrak{p}$  is  $f$ . The Clifford algebra is “universal” with respect to this property.

# The exterior algebra.

If the bilinear form is identically zero, then  $C(\mathbf{p}) = \wedge \mathbf{p}$ , the exterior algebra. But we will be interested in the opposite extreme, the case where the bilinear form is non-degenerate.

# The $\mathbf{Z}/2\mathbf{Z}$ gradation on the Clifford algebra.

The ideal  $I$  defining the Clifford algebra is not  $\mathbf{Z}$  homogeneous (unless the bilinear form is identically zero) since its generators  $y_1y_2 + y_2y_1 - 2(y_1, y_2)\mathbf{1}$  are “mixed”, being a sum of terms of degree two and degree zero in  $T(\mathbf{p})$ . But these terms are both even. So the  $\mathbf{Z}/2\mathbf{Z}$  gradation *is* preserved upon passing to the quotient. In other words,  $C(\mathbf{p})$  is a  $\mathbf{Z}/2\mathbf{Z}$  graded algebra:

$$C(\mathbf{p}) = C_0(\mathbf{p}) \oplus C_1(\mathbf{p})$$

where the elements of  $C_0(\mathbf{p})$  consist of sums of products of elements of  $\mathbf{p}$  with an even number of factors and  $C_1(\mathbf{p})$  consist of sums of terms each a product of elements of  $\mathbf{p}$  with an odd number of factors. The usual rules for multiplication of a graded algebra obtain:

$$C_0(\mathbf{p}) \cdot C_0(\mathbf{p}) \subset C_0(\mathbf{p}), \quad C_0(\mathbf{p}) \cdot C_1(\mathbf{p}) \subset C_1(\mathbf{p}), \quad C_1(\mathbf{p}) \cdot C_1(\mathbf{p}) \subset C_0(\mathbf{p}).$$

# The induced bilinear form on the exterior algebra.

Let  $\mathbf{p}$  be a vector space with a non-degenerate symmetric bilinear form. The exterior algebra,  $\wedge \mathbf{p}$  inherits a bilinear form which we continue to denote by  $(\ , \ )$ . Here the spaces  $\wedge^k(\mathbf{p})$  and  $\wedge^\ell(\mathbf{p})$  are orthogonal if  $k \neq \ell$  while

$$(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det((x_i, y_j)).$$

For  $v \in \mathbf{p}$  let  $\epsilon(v) \in \text{End}(\wedge \mathbf{p})$  denote exterior multiplication by  $v$  and  $\iota(v)$  be the transpose of  $\epsilon(v)$  relative to this bilinear form on  $\wedge \mathbf{p}$ .

So  $\iota(v)$  is interior multiplication by the element of  $\mathbf{p}^*$  corresponding to  $v$  under the map  $\mathbf{p} \rightarrow \mathbf{p}^*$  induced by  $(\ , \ )_{\mathbf{p}}$ .

# $\wedge \mathfrak{p}$ as a $C(\mathfrak{p})$ module.

So  $\iota(v)$  is interior multiplication by the element of  $\mathfrak{p}^*$  corresponding to  $v$  under the map  $\mathfrak{p} \rightarrow \mathfrak{p}^*$  induced by  $(\ , \ )_{\mathfrak{p}}$ . The map

$$\mathfrak{p} \rightarrow \text{End}(\wedge \mathfrak{p}), \quad v \mapsto \epsilon(v) + \iota(v)$$

is a Clifford map, i.e. satisfies

$$(\epsilon(v) + \iota(v))^2 = (v, v)_{\mathfrak{p}} \text{id}$$

and so extends to a homomorphism of

$$C(\mathfrak{p}) \rightarrow \text{End } \wedge \mathfrak{p}$$

making  $\wedge \mathfrak{p}$  into a  $C(\mathfrak{p})$  module. We let  $xy$  denote the product of  $x$  and  $y$  in  $C(\mathfrak{p})$ .

# Chevalley's linear identification of $C(\mathfrak{p})$ with $\wedge \mathfrak{p}$ .

Consider the linear map

$$C(\mathfrak{p}) \rightarrow \wedge \mathfrak{p}, \quad x \mapsto x1$$

where  $1 \in \wedge^0 \mathfrak{p}$  under the identification of  $\wedge^0 \mathfrak{p}$  with the ground field. The element  $x1$  on the extreme right means the image of 1 under the action of  $x \in C(\mathfrak{p})$ .

For elements  $v_1, \dots, v_k \in \mathfrak{p}$  this map sends

$$v_1 \mapsto v_1$$

$$v_1 v_2 \mapsto v_1 \wedge v_2 + (v_1, v_2)1$$

$$v_1 v_2 v_3 \mapsto v_1 \wedge v_2 \wedge v_3 + (v_1, v_2)v_3 - (v_1, v_3)v_2 + (v_2, v_3)v_1$$

$$v_1 v_2 v_3 v_4 \mapsto v_1 \wedge v_2 \wedge v_3 \wedge v_4 + (v_2, v_3)v_1 \wedge v_4 - (v_2, v_4)v_1 \wedge v_3$$

$$+ (v_3, v_4)v_1 \wedge v_2 + (v_1, v_2)v_3 \wedge v_4 - (v_1, v_3)v_1 \wedge v_4$$

$$+ (v_1, v_4)v_2 \wedge v_3 + (v_1, v_4)(v_2, v_3) - (v_1, v_3)(v_2, v_4) + (v_1, v_2)(v_3, v_4)$$

If the  $v$ 's form an “orthonormal” basis of  $\mathbf{p}$  then the products

$$v_{i_1} \cdots v_{i_k}, \quad i_1 < i_2 < \cdots < i_k, \quad k = 0, 1, \dots, n$$

form a basis of  $C(\mathbf{p})$  while the

$$v_{i_1} \wedge \cdots \wedge v_{i_k}, \quad i_1 < i_2 < \cdots < i_k, \quad k = 0, 1, \dots, n$$

form a basis of  $\wedge \mathbf{p}$ , and in fact

$$v_1 \cdots v_k \mapsto v_1 \wedge \cdots \wedge v_k \quad \text{if } (v_i, v_j) = 0 \quad \forall i \neq j. \quad (1)$$

In particular, the map  $C(\mathbf{p}) \rightarrow \wedge \mathbf{p}$  given above is an isomorphism of vector spaces, so we may identify  $C(\mathbf{p})$  with  $\wedge \mathbf{p}$  as a vector space if we choose, and then consider that  $\wedge \mathbf{p}$  has two products: the Clifford product which we denote by juxtaposition and the exterior product which we denote with a  $\wedge$ .

Notice that this identification preserves the  $\mathbf{Z}/2\mathbf{Z}$  gradation, an even element of the Clifford algebra is identified with an even element of the exterior algebra and an odd element is identified with an odd element.

# The canonical anti-automorphism.

The Clifford algebra has a canonical anti-automorphism  $a$  which is the identity map on  $\mathfrak{p}$ . In particular, for  $v_i \in \mathfrak{p}$  we have  $a(v_1v_2) = v_2v_1$ ,  $a(v_1v_2v_3) = v_3v_2v_1$ , etc. By abuse of language, we use the same letter  $a$  to denote the similar anti-automorphism on  $\wedge\mathfrak{p}$  and observe from the above computations (in particular from the corresponding choice of bases) that  $a$  commutes with our identifying map  $C(\mathfrak{p}) \rightarrow \wedge\mathfrak{p}$  so the notation is consistent. We have

$$a = (-1)^{\frac{1}{2}k(k-1)}\text{id} \quad \text{on } \wedge^k(\mathfrak{p}).$$

For small values of  $k$  we have

$k$	$(-1)^{\frac{1}{2}k(k-1)}$
0	1
1	1
2	-1
3	-1
4	1
5	1
6	-1.

We will use subscripts to denote the homogeneous components of elements of  $\wedge \mathbf{p}$ . Notice that if  $u \in \wedge^2 \mathbf{p}$  then  $au = -u$  by the above table, while  $a(u^2) = (au)^2 = u^2$ . Since  $u^2$  is even (and hence has only even homogeneous components) and since the maximum degree of the homogeneous component of  $u^2$  is 4, we conclude that

$$u^2 = (u^2)_0 + (u^2)_4 \quad \forall u \in \wedge^2 \mathbf{p}. \quad (2)$$

$k$	$(-1)^{\frac{1}{2}k(k-1)}$
0	1
1	1
2	-1
3	-1
4	1
5	1
6	-1.

$$u^2 = (u^2)_0 + (u^2)_4 \quad \forall u \in \wedge^2 \mathbf{p}. \quad (2)$$

For the same reason

$$v^2 = (v^2)_0 + (v^2)_4 \quad \forall v \in \wedge^3 \mathbf{p}. \quad (3)$$

We also claim the following:

$$(ww')_0 = (aw, w') = (-1)^{\frac{1}{2}k(k-1)}(w, w') \quad \forall w, w' \in \wedge^k(\mathbf{p}). \quad (4)$$

Indeed, it is sufficient to verify this for  $w, w'$  belonging to a basis of  $\wedge^k \mathbf{p}$ , say the basis given by all elements of the form (1), in which case both sides of (4) vanish unless  $w = w'$ . If  $w = w' = v_1 \wedge \cdots \wedge v_k$  (say) then

$$\begin{aligned} (ww)_0 &= \iota(v_1) \cdots \iota(v_k) v_1 \wedge \cdots \wedge v_k = \\ &(-1)^{\frac{1}{2}k(k-1)}(v_1, v_1) \cdots (v_k, v_k) = (-1)^{\frac{1}{2}k(k-1)}(w, w) \end{aligned}$$

proving (4).

As special cases that we will use later on, observe that

$$(uu')_0 = -(u, u') \quad \forall u, u' \in \wedge^2 \mathbf{p} \quad (5)$$

and

$$(vv')_0 = -(v, v') \quad \forall v, v' \in \wedge^3 \mathbf{p}. \quad (6)$$

# Commutator by an element of $\mathfrak{p}$ .

For any  $y \in \mathfrak{p}$  consider the linear map

$$w \mapsto [y, w] = yw - (-1)^k wy \quad \text{for } w \in \wedge^k \mathfrak{p}$$

which is (anti)commutator in the Clifford multiplication by  $y$ . We claim that

$$[y, w] = 2\iota(y)w. \quad (7)$$

In particular,  $[y, \cdot]$ , which is automatically a derivation for the Clifford multiplication, is also a derivation for the exterior multiplication. Alternatively, this equation says that  $\iota(y)$ , which is a derivation for the exterior algebra multiplication, is also a derivation for the Clifford multiplication.

# Proof of $[y, w] = 2\iota(y)w.$ (7)

To prove (7) write

$$wy = a(ya(w)).$$

Then

$$yw = y \wedge w + \iota(y)w, \quad wy = a(y \wedge a(w)) + a(\iota(y)aw) = w \wedge y + (a\iota(y)a)w.$$

We may assume that  $w \in \wedge^k \mathbf{p}$ . Then

$$y \wedge w - (-1)^k w \wedge y = 0,$$

so we must show that

$$a\iota(y)aw = (-1)^{k-1} \iota(y)w.$$

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$$a\iota(y)aw = (-1)^{k-1}\iota(y)w.$$

For this we may assume that  $y \neq 0$  and we may write

$$w = u \wedge z + z',$$

where  $\iota(y)u = 1$  and  $\iota(y)z = \iota(y)z' = 0$ . In fact, we may assume that  $z$  and  $z'$  are sums of products of linear elements all of which are orthogonal to  $y$ . Then  $\iota(y)az = \iota(y)az' = 0$  so

$$\iota(y)aw = (-1)^{k-1}az$$

since  $z$  has degree one less than  $w$  and hence

$$a\iota(y)aw = (-1)^{k-1}z = (-1)^{k-1}\iota(y)w. \quad QED$$

# Commutator by an element of $\wedge^2 \mathfrak{p}$ .

Suppose that

$$u \in \wedge^2 \mathfrak{p}.$$

Then for  $y \in \mathfrak{p}$  we have

$$[u, y] = -[y, u] = -2\iota(y)u. \quad (8)$$

In particular, if  $u = y_i \wedge y_j$  where  $y_i, y_j \in \mathfrak{p}$  we have

$$[u, y] = 2(y_j, y)y_i - 2(y_i, y)y_j \quad \forall y \in \mathfrak{p}. \quad (9)$$

If  $(y_i, y_j) = 0$  this is an “infinitesimal rotation” in the plane spanned by  $y_i$  and  $y_j$ . Since  $y_i \wedge y_j$ ,  $i < j$  form a basis of  $\wedge^2 \mathfrak{p}$  if  $y_1, \dots, y_n$  form an “orthonormal” basis of  $\mathfrak{p}$ , we see that the map

$$u \mapsto [u, \cdot]$$

gives an isomorphism of  $\wedge^2 \mathfrak{p}$  with the orthogonal algebra  $\mathfrak{o}(\mathfrak{p})$ . This identification differs by a factor of two from the identification that we had been using earlier.

# $\mathfrak{o}(\mathfrak{p})$ acting as derivations.

Now each element of  $\mathfrak{o}(\mathfrak{p})$  (in fact any linear transformation on  $\mathfrak{p}$ ) induces a derivation of  $\wedge\mathfrak{p}$ . We claim that under the above identification of  $\wedge^2\mathfrak{p}$  with  $\mathfrak{o}(\mathfrak{p})$ , the derivation corresponding to  $u \in \wedge^2\mathfrak{p}$  is Clifford commutation by  $u$ . In symbols, if  $\theta_u$  denotes this induced derivation, we claim that

$$\theta_u(w) = [u, w] = uw - wu \quad \forall w \in \wedge\mathfrak{p}. \quad (10)$$

To verify this, it is enough to check it on basis elements of the form (1), and hence by the derivation property for each  $v_j$ , where this reduces to (8).

$$[u, y] = -[y, u] = -2\iota(y)u. \quad (8)$$

# Orthogonal action of a Lie algebra.

Let  $\mathfrak{r}$  be a Lie algebra. Suppose that we have a representation of  $\mathfrak{r}$  acting as infinitesimal orthogonal transformations of  $\mathfrak{p}$  which means, in view of the identification of  $\wedge^2 \mathfrak{p}$  with  $\mathfrak{o}(\mathfrak{p})$  that we have a map

$$\nu : \mathfrak{r} \rightarrow \wedge^2 \mathfrak{p}$$

such that

$$x \cdot y = -2\iota(y)\nu(x) \tag{13}$$

where  $x \cdot y$  denotes the action of  $x \in \mathfrak{r}$  on  $y \in \mathfrak{p}$ .

It will be useful for us to write equation (13) in terms of a basis. So let  $y_1, \dots, y_n$  be a basis of  $\mathfrak{p}$  and let  $z_1, \dots, z_n$  be the dual basis relative to  $(\ , \ ) = (\ , \ )_{\mathfrak{p}}$ . We claim that

$$\nu(x) = -\frac{1}{4} \sum_j y_j \wedge (x \cdot z_j). \tag{14}$$

$$\nu : \mathbf{r} \rightarrow \wedge^2 \mathbf{p} \quad x \cdot y = -2\iota(y)\nu(x) \quad (13)$$

To prove:

$$\nu(x) = -\frac{1}{4} \sum y_j \wedge (x \cdot z_j). \quad (14)$$

Indeed, it suffices to verify (13) for each of the elements  $z_i$ .

$$\iota(z_i) \left( -\frac{1}{4} \sum_j y_j \wedge x \cdot z_j \right) = -\frac{1}{4} x \cdot z_i + \frac{1}{4} \sum_j (z_i, x \cdot z_j) y_j.$$

$$(z_i, x \cdot z_j) = -(x \cdot z_i, z_j)$$

So we can write the sum as

$$\frac{1}{4} \sum_j (z_i, x \cdot z_j) y_j = -\frac{1}{4} \sum_j (x \cdot z_i, z_j) y_j = -\frac{1}{4} x \cdot z_i \quad \text{yielding}$$

$$\iota(z_i) \left( -\frac{1}{4} \sum_j y_j \wedge x \cdot z_j \right) = -\frac{1}{2} x \cdot z_i$$

which is (13).

## The adjoint action of a reductive Lie algebra.

$$\nu(x) = -\frac{1}{4} \sum_j y_j \wedge (x \cdot z_j). \quad (14)$$

For future use we record here a special case of (14): Suppose that  $\mathfrak{p} = \mathfrak{r} = \mathfrak{g}$  is a reductive Lie algebra with an invariant symmetric bilinear form, and the action is the adjoint action, i.e.  $x \cdot y = [x, y]$ . Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and let  $\Phi$  denote the set of roots and suppose that we have chosen root vectors  $e_\phi, e_{-\phi}$ ,  $\phi \in \Phi$  so that

$$(e_\phi, e_{-\phi}) = 1.$$

Let  $h_1, \dots, h_s$  be a basis of  $\mathfrak{h}$  and  $k_1, \dots, k_s$  the dual basis. Let

$$\psi : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$$

be the map  $\nu$  when applied to this adjoint action. Then (14) becomes

$$\psi(x) = \frac{1}{4} \left( \sum_{i=1}^s h_i \wedge [k_i, x] + \sum_{\phi \in \Phi} e_{-\phi} \wedge [e_\phi, x] \right). \quad (15)$$

$$\psi(x) = \frac{1}{4} \left( \sum_{i=1}^s h_i \wedge [k_i, x] + \sum_{\phi \in \Phi} e_{-\phi} \wedge [e_{\phi}, x] \right). \quad (15)$$

In case  $x = h \in \mathfrak{h}$  this formula simplifies. The  $[k_i, h] = 0$ , and in the second sum we have

$$e_{-\phi} \wedge [e_{\phi}, h] = -\phi(h)e_{-\phi} \wedge e_{\phi}$$

which is invariant under the interchange of  $\phi$  and  $-\phi$ . So let us make a choice  $\Phi^+$  of positive roots. Then we can write (15) as

$$\psi(h) = -\frac{1}{2} \sum_{\phi \in \Phi^+} \phi(h)e_{-\phi} \wedge e_{\phi}, \quad h \in \mathfrak{h}. \quad (16)$$

$$\psi(h) = -\frac{1}{2} \sum_{\phi \in \Phi^+} \phi(h) e_{-\phi} \wedge e_{\phi}, \quad h \in \mathfrak{h}. \quad (16)$$

Now

$$e_{-\phi} \wedge e_{\phi} = -1 + e_{-\phi} e_{\phi}.$$

So if

$$\rho := \frac{1}{2} \sum_{\phi \in \Phi^+} \phi \quad (17)$$

is one half the sum of the positive roots we have

$$\psi(h) = \rho(h) - \frac{1}{2} \sum_{\phi \in \Phi^+} \phi(h) e_{-\phi} e_{\phi}, \quad h \in \mathfrak{h}. \quad (18)$$

In this equation, the multiplication on the right is in the Clifford algebra.

# The Clifford algebra of a direct sum.

If

$$\mathbf{p} = \mathbf{p}_1 \oplus \mathbf{p}_2$$

is a direct sum decomposition of a vector space  $\mathbf{p}$  with a symmetric bilinear form into two orthogonal subspaces then it follows from the definition of the Clifford algebra that

$$C(\mathbf{p}) = C(\mathbf{p}_1) \otimes C(\mathbf{p}_2)$$

where the multiplication on the tensor product is taken in the sense of superalgebras, that is

$$(a_1 \otimes a_2)(b_1 \otimes b_2) := a_1 b_1 \otimes a_2 b_2$$

if either  $a_2$  or  $b_1$  are even, but

$$(a_1 \otimes a_2)(b_1 \otimes b_2) := -a_1 b_1 \otimes a_2 b_2$$

if both  $a_2$  and  $b_1$  are odd. It costs a sign to move one odd symbol past another.

# The two dimensional split Clifford algebra.

Suppose that  $\mathbf{p}$  is even dimensional. If the metric is split (which is always the case if the metric is non-degenerate and we are over the complex numbers) then  $\mathbf{p}$  is a direct sum of two dimensional mutually orthogonal split spaces,  $\mathbf{W}_i$ , so let us examine first the case of a two dimensional split space  $\mathbf{p}$ , spanned by  $\iota, \epsilon$  with  $(\iota, \iota) = (\epsilon, \epsilon) = 0$ ,  $(\iota, \epsilon) = \frac{1}{2}$ . Let  $T$  be a one dimensional space with basis  $t$  and consider the linear map of  $\mathbf{p} \rightarrow \text{End}(\wedge T)$  determined by

$$\epsilon \mapsto \epsilon(t), \quad \iota \mapsto \iota(t^*)$$

where  $\epsilon(t)$  denotes exterior multiplication by  $t$  and  $\iota(t^*)$  denotes interior multiplication by  $t^*$ , the dual element to  $t$  in  $T^*$ . This is a Clifford map since

$$\epsilon(t)^2 = 0 = \iota(t^*)^2, \quad \epsilon(t)\iota(t^*) + \iota(t^*)\epsilon(t) = \text{id}.$$

# Explicit realization as matrices.

$$\epsilon \mapsto \epsilon(t), \quad \iota \mapsto \iota(t^*)$$

$$\epsilon(t)^2 = 0 = \iota(t^*)^2, \quad \epsilon(t)\iota(t^*) + \iota(t^*)\epsilon(t) = \text{id}.$$

This therefore extends to a map of  $C(\mathbf{p}) \rightarrow \text{End}(\wedge T)$ . Explicitly, if we use  $1 \in \wedge^0 T$ ,  $t \in \wedge^1 T$  as a basis of  $\wedge T$  this map is given by

$$\begin{aligned} 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \iota &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \epsilon &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \iota\epsilon &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This shows that the map is an isomorphism.

# Clifford algebras of split vector spaces.

If

$$\mathbf{p} = \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_m$$

is a direct sum of two dimensional split spaces, and we write

$$T = T_1 \oplus \cdots \oplus T_m$$

where the  $C(\mathbf{W}_i) \cong \text{End}(\wedge T_i)$  as above, then since

$$\wedge T = \wedge T_1 \otimes \cdots \otimes \wedge T_m$$

we see that

$$C(\mathbf{p}) \cong \text{End} (\wedge T).$$

# The spin module.

$$C(\mathfrak{p}) \cong \text{End} (\wedge T).$$

In particular,  $C(\mathfrak{p})$  is isomorphic to the full  $2^m \times 2^m$  matrix algebra and hence has a unique (up to isomorphism) irreducible module. One model of this is

$$S = \wedge T.$$

We can write

$$S = S_+ \oplus S_-$$

as a supervector space, where we choose the standard  $\mathbf{Z}_2$  grading on  $\wedge T$  to determine the grading on  $S$  if  $m$  is even, but use the opposite grading (for reasons which will become apparent in a moment) if  $m$  is odd.



# The half spin representations.

$$S = S_+ \oplus S_-$$

as a supervector space, where we choose the standard  $\mathbf{Z}_2$  grading on  $\wedge T$  to determine the grading on  $S$  if  $m$  is even, but use the opposite grading (for reasons which will become apparent in a moment) if  $m$  is odd.

The even part,  $C_0(\mathbf{p})$  of  $C(\mathbf{p})$  acts irreducibly on each of  $S_\pm$ . Since  $\wedge^2 \mathbf{p}$  together with the constants generates  $C_0(\mathbf{p})$  we see that the action of  $\wedge^2 \mathbf{p}$  on each of  $S_\pm$  is irreducible. Since  $\wedge^2 \mathbf{p}$  under Clifford commutation is isomorphic to  $o(\mathbf{p})$  the two modules  $S_\pm$  give irreducible modules for the even orthogonal algebra  $o(\mathbf{p})$ . These are the half spin representations of the even orthogonal algebras.

# The spin module as a left ideal.

We can identify  $S = S_+ \oplus S_-$  as a left ideal in  $C(\mathfrak{p})$  as follows: Suppose that we write

$$\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

where  $\mathfrak{p}_\pm$  are complementary isotropic subspaces. Choose a basis  $e_1^+, \dots, e_m^+$  of  $\mathfrak{p}_+$  and let

$$e_+ := e_1^+ \cdots e_m^+ = e_1^+ \wedge \cdots \wedge e_m^+ \in \wedge^m \mathfrak{p}_+.$$

We have

$$y_+ e_+ = 0, \quad \forall y_+ \in \mathfrak{p}_+$$

and hence

$$(\wedge \mathfrak{p}_+)_+ e_+ = 0.$$

In other words

$$\wedge \mathfrak{p}_+ e_+$$

consists of all scalar multiples of  $e_+$ .

$\wedge^{\mathbf{p}_+} e_+$  consists of all scalar multiples of  $e_+$ .

Since

$$\wedge^{\mathbf{p}_-} \otimes \wedge^{\mathbf{p}_+} \rightarrow C(\mathbf{p}), \quad w_- \otimes w_+ \mapsto w_- w_+$$

is a linear bijection, we see that

$$C(\mathbf{p})e_+ = \wedge^{\mathbf{p}_-} e_+.$$

This means that the left ideal generated by  $e_+$  in  $C(\mathbf{p})$  has dimension  $2^m$ , and hence must be isomorphic as a left  $C(\mathbf{p})$  module to  $S$ . In particular it is a minimal left ideal.

Let  $e_1^-, \dots, e_m^-$  be a basis of  $\mathbf{p}_-$  and for any subset  $J = \{i_1, \dots, i_j\}$ ,  $i_1 < i_2 < \dots < i_j$  of  $\{1, \dots, m\}$  let

$$e_-^J := e_{i_1}^- \wedge \dots \wedge e_{i_j}^- = e_{i_1}^- \cdots e_{i_j}^-.$$

Then the elements

$$e_-^J e_+$$

form a basis of this model of  $S$  as  $J$  ranges over all subsets of  $\{1, \dots, m\}$ .

# Weights of the spin module.

$$\nu(x) = -\frac{1}{4} \sum_j y_j \wedge (x \cdot z_j). \quad (14)$$

For example, suppose that we have a commutative Lie algebra  $\mathfrak{h}$  acting on  $\mathfrak{p}$  as infinitesimal isometries, so as to preserve each  $\mathfrak{p}_\pm$ , that the  $e_i^+$  are weight vectors corresponding to weights  $\beta_i$  and that the  $e_i^-$  form the dual basis, corresponding to the negative of these weights  $-\beta_i$ . Then it follows from (14) that the image,  $\nu(h) \in \wedge^2(\mathfrak{p}) \subset C(\mathfrak{p})$  of an element  $h \in \mathfrak{h}$  is given by

$$\nu(h) = \frac{1}{2} \sum \beta_i(h) e_i^+ \wedge e_i^- = \frac{1}{2} \sum \beta_i(h) (1 - e_i^- e_i^+).$$

Thus

$$\nu(h)e_+ = \rho_{\mathfrak{p}}(h)e_+ \quad (19)$$

where

$$\rho_{\mathfrak{p}} := \frac{1}{2}(\beta_1 + \cdots + \beta_m). \quad (20)$$

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For a subset  $J$  of  $\{1, \dots, m\}$  let us set  $\beta_J := \sum_{j \in J} \beta_j$ .

Then we have

$$[\nu(h), e_-^J] = -\beta_J(h)e_-^J$$

and so

$$\nu(h)(e_-^J e_+) = [\nu(h), e_-^J]e_+ + e_-^J \nu(h)e_+ = (\rho_{\mathbf{p}}(h) - \beta_J(h))e_-^J e_+.$$

So if we denote the action of  $\nu(h)$  on  $S_{\pm}$  by  $\text{Spin}_{\pm} \nu(h)$  and the action of  $\nu(h)$  on  $S = S_+ \oplus S_-$  by  $\text{Spin} \nu(h)$  we have proved that

$$\text{The } e_-^J e_+ \text{ are weight vectors of } \text{Spin } \nu \text{ with weights } \rho_{\mathbf{p}} - \beta_J. \quad (21)$$

The  $e_-^J e_+$  are weight vectors of Spin  $\nu$  with weights  $\rho_{\mathbf{p}} - \beta_J$ . (21)

It follows from (21) that the difference of the characters of Spin $_+\nu$  and Spin $_-\nu$  is given by

$$\text{ch}_{\text{Spin}_+\nu} - \text{ch}_{\text{Spin}_-\nu} = \prod_j \left( e(\frac{1}{2}\beta_j) - e(-\frac{1}{2}\beta_j) \right) = e(\rho_{\mathbf{p}}) \prod_j (1 - e(-\beta_j)). \quad (22)$$

There are two special cases which are of particular importance: First, this applies to the case where we take  $\mathfrak{h}$  to be a Cartan subalgebra of  $\mathfrak{o}(\mathbf{p}) = \mathfrak{o}(\mathbf{C}^{2k})$  itself, say the diagonal matrices in the block decomposition of  $\mathfrak{o}(\mathbf{p})$  given by the decomposition

$$\mathbf{C}^{2k} = \mathbf{C}^k \oplus \mathbf{C}^k$$

into two isotropic subspaces. In this case the  $\beta_i$  is just the  $i$ -th diagonal entry and (22) yields the standard formula for the difference of the characters of the spin representations of the even orthogonal algebras.

# The Weyl denominator.

$$\mathrm{ch}_{Spin_+ \nu} - \mathrm{ch}_{Spin_- \nu} = \prod_j \left( e\left(\frac{1}{2}\beta_j\right) - e\left(-\frac{1}{2}\beta_j\right) \right) = e(\rho_{\mathfrak{p}}) \prod_j (1 - e(-\beta_j)). \quad (22)$$

A second very important case is where we take  $\mathfrak{h}$  to be the Cartan subalgebra of a semi-simple Lie algebra  $\mathfrak{g}$ , and take

$$\mathfrak{p} := \mathfrak{n}_+ \oplus \mathfrak{n}_-$$

relative to a choice of positive roots. Then the  $\beta_j$  are just the positive roots, and we see that the right hand side of (22) is just the Weyl denominator, the denominator occurring in the Weyl character formula. This means that we can write the Weyl character formula as

$$\mathrm{ch}(\mathrm{Irr}(\lambda) \otimes S_+) - \mathrm{ch}(\mathrm{Irr}(\lambda) \otimes S_-) = \sum_{w \in W} (-1)^w e(w \bullet \lambda)$$

where

$$w \bullet \lambda := w(\lambda + \rho).$$

# The Weyl character formula.

If we let  $U_\mu$  denote the one dimensional module for  $\mathfrak{h}$  given by the weight  $\mu$  we can drop the characters from the preceding equation and simply write the Weyl character formula as an equation in virtual representations of  $\mathfrak{h}$ :

$$\text{Irr}(\lambda) \otimes S_+ - \text{Irr}(\lambda) \otimes S_- = \sum_{w \in W} (-1)^w U_{w \bullet \lambda}. \quad (23)$$