

Math 128 Lecture 17

The Kostant multiplicity formula.

Review: The Weyl character formula.

For any weight, μ we define

$$A_\mu := \sum_{w \in W} (-1)^w e(w\mu).$$

Then we can write the Weyl character formula as

$$\text{ch}_{\text{Irr}(\lambda)} = \frac{A_{\lambda+\rho}}{A_\rho}.$$

or

$$q \text{ch}_{\text{Irr}(\lambda)} = \sum_{w \in W} (-1)^w e(w(\lambda + \rho)). \quad (17)$$

$$q = \sum (-1)^w e(w\rho)$$

Rewriting the **WCF**.

$$q\text{ch}_{\text{Irr}(\lambda)} = \sum_{w \in W} (-1)^w e(w(\lambda + \rho)). \quad (17)$$

$$qp = e(\rho). \quad (16)$$

Let us multiply the fundamental equation (17) by $pe(-\rho)$ and use the fact (16) that $qpe(-\rho) = 1$ to obtain

$$\text{ch}_{\text{Irr}(\lambda)} = \sum_{w \in W} (-1)^w pe(-\rho)e(w(\lambda + \rho)).$$

The Kostant formula.

$$\text{ch}_{\text{Irr}(\lambda)} = \sum_{w \in W} (-1)^w p e(-\rho) e(w(\lambda + \rho)).$$

But

$$p e(-\rho) e(w(\lambda + \rho)) = p(\cdot - w(\lambda + \rho) + \rho)$$

or, in more pedestrian terms, the left hand side of this equation has, as its coefficient of $e(\mu)$ the value

$$p(\mu + \rho - w(\lambda + \rho)).$$

On the other hand, by definition,

$$\text{ch}_{\text{Irr}(\lambda)} = \sum \dim(\text{Irr}(\lambda))_{\mu} e(\mu).$$

We thus obtain Kostant's formula for the multiplicity of a weight μ in the irreducible module with highest weight λ :

$$\mathbf{KMF} \quad \dim(\text{Irr}(\lambda))_{\mu} = \sum_{w \in W} (-1)^w p(\mu + \rho - w(\lambda + \rho)). \quad (20)$$

Rewriting the **KMF**.

It will be convenient to introduce some notation which simplifies the appearance of the Kostant multiplicity formula: For $w \in W$ and $\mu \in \mathbf{L}$ (or in E for that matter) define

$$w \odot \mu := w(\mu + \rho) - \rho. \quad (21)$$

This defines another action of W on E where the “origin of the orthogonal transformations w has been shifted from 0 to $-\rho$ ”. Then we can rewrite the Kostant multiplicity formula as

$$\dim(\text{Irr}(\lambda))_{\mu} = \sum_{w \in W} (-1)^w P_K(w \odot \lambda - \mu) \quad (22)$$

or as

$$\text{ch}(\text{Irr}(\lambda)) = \sum_{w \in W} \sum_{\mu} (-1)^w P_K(w \odot \lambda - \mu) e(\mu), \quad (23)$$

where P_K is the original Kostant partition function.

A useful fact.

For the purposes of the next section it will be useful to record the following lemma:

Lemma 1 *If ν is a dominant weight and $e \neq w \in W$ then $w \odot \nu$ is not dominant.*

Proof. If ν is dominant, so lies in the closure of the positive Weyl chamber, then $\nu + \rho$ lies in the interior of the positive Weyl chamber. Hence if $w \neq e$, then $w(\nu + \rho)(h_i) < 0$ for some i , and so $w \odot \nu = w(\nu + \rho) - \rho$ is not dominant. QED

Tensoring two irreducibles.

Suppose that λ' and λ'' are dominant integral weights. Decompose $\text{Irr}(\lambda') \otimes \text{Irr}(\lambda'')$ into irreducibles, and let $n(\lambda) = n(\lambda, \lambda' \otimes \lambda'')$ denote the multiplicity of $\text{Irr}(\lambda)$ in this decomposition into irreducibles (with $n(\lambda) = 0$ if $\text{Irr}(\lambda)$ does not appear as a summand in the decomposition). In particular, $n(\nu) = 0$ if ν is not a dominant weight since $\text{Irr}(\nu)$ is infinite dimensional in this case, so can not appear as a summand in the decomposition. In terms of characters, we have

$$\text{ch}(\text{Irr}(\lambda')) \text{ch}(\text{Irr}(\lambda'')) = \sum_{\lambda} n(\lambda) \text{ch}(\text{Irr}(\lambda)).$$

$$\text{ch}(\text{Irr}(\lambda')) \text{ch}(\text{Irr}(\lambda'')) = \sum_{\lambda} n(\lambda) \text{ch}(\text{Irr}(\lambda)).$$

Steinberg's formula is a formula for $n(\lambda)$. To derive it, use the Weyl character formula

$$\text{ch}(\text{Irr}(\lambda'')) = \frac{A_{\lambda''+\rho}}{A_{\rho}}, \quad \text{ch}(\text{Irr}(\lambda)) = \frac{A_{\lambda+\rho}}{A_{\rho}}$$

in the above formula to obtain

$$\text{ch}(\text{Irr}(\lambda')) A_{\lambda''+\rho} = \sum_{\lambda} n(\lambda) A_{\lambda+\rho}.$$

$$\text{ch}(\text{Irr}(\lambda'))A_{\lambda''+\rho} = \sum_{\lambda} n(\lambda)A_{\lambda+\rho}.$$

Use the Kostant multiplicity formula (23) for λ' :

$$\text{ch}(\text{Irr}(\lambda')) = \sum_{w \in W} \sum_{\mu} (-1)^w P_K(w \odot \lambda' - \mu)e(\mu)$$

and the definition

$$A_{\lambda''+\rho} = \sum_{u \in W} (-1)^u e(u(\lambda'' + \rho))$$

and the similar expression for $A_{\lambda+\rho}$ to get

$$\sum_{\mu} \sum_{u, w \in W} (-1)^{uw} P_K(w \odot \lambda' - \mu)e(u(\lambda'' + \rho) + \mu) =$$

$$\sum_{\lambda} \sum_w n(\lambda)(-1)^w e(w(\lambda + \rho)).$$

$$\sum_{\mu} \sum_{u, w \in W} (-1)^{uw} P_K(w \odot \lambda' - \mu) e(u(\lambda'' + \rho) + \mu) =$$

$$\sum_{\lambda} \sum_w n(\lambda) (-1)^w e(w(\lambda + \rho)).$$

Let us make a change of variables on the right hand side, writing

$$\nu = w \odot \lambda$$

so the right hand side becomes

$$\sum_{\nu} \sum_w (-1)^w n(w^{-1} \odot \nu) e(\nu + \rho).$$

If ν is a dominant weight, then by Lemma 1 $w^{-1} \odot \nu$ is not dominant if $w^{-1} \neq e$. So $n(w^{-1} \odot \nu) = 0$ if $w \neq 1$ and so the coefficient of $e(\nu + \rho)$ is precisely $n(\nu)$ when ν is dominant.

$$\sum_{\mu} \sum_{u, w \in W} (-1)^{uw} P_K(w \odot \lambda' - \mu) e(u(\lambda'' + \rho) + \mu) = \sum_{\nu} \sum_w (-1)^w n(w^{-1} \odot \nu) e(\nu + \rho).$$

$n(w^{-1} \odot \nu) = 0$ if $w \neq 1$ and so the coefficient of $e(\nu + \rho)$ is precisely $n(\nu)$ when ν is dominant.

On the left hand side let

$$\mu = \nu - u \odot \lambda''$$

to obtain

$$\sum_{\nu, u, w} (-1)^{uw} P_K(w \odot \lambda' + u \odot \lambda'' - \nu) e(\nu + \rho).$$

Comparing coefficients for ν dominant gives

$$n(\nu) = \sum_{u, w} (-1)^{uw} P_K(w \odot \lambda' + u \odot \lambda'' - \nu). \quad (24)$$

Steinberg's formula.

$$\text{ch}(\text{Irr}(\lambda')) \text{ch}(\text{Irr}(\lambda'')) = \sum_{\lambda} n(\lambda) \text{ch}(\text{Irr}(\lambda)).$$

where

$$n(\nu) = \sum_{u,w} (-1)^{uw} P_K(w \odot \lambda' + u \odot \lambda'' - \nu).$$

Back to the proof of the **WDF**.

We return to the study of a semi-simple Lie algebra \mathfrak{g} and get a refinement of the Weyl dimension formula by looking at the next order term in the expansion we used to derive the Weyl dimension formula from the Weyl character formula.

By definition, the Killing form restricted to the Cartan subalgebra \mathfrak{h} is given by

$$\kappa(h, h') = \sum_{\alpha} \alpha(h)\alpha(h')$$

where the sum is over all roots. If $\mu, \lambda \in \mathfrak{h}^*$ with t_{μ}, t_{λ} the elements of H corresponding to them under the Killing form, we have

$$(\lambda, \mu)_{\kappa} = \kappa(t_{\lambda}, t_{\mu}) = \sum_{\alpha} \alpha(t_{\lambda})\alpha(t_{\mu})$$

so

$$(\lambda, \mu)_{\kappa} = \sum_{\alpha} (\lambda, \alpha)_{\kappa} (\mu, \alpha)_{\kappa}. \quad (25)$$

For each λ in the weight lattice L we have let $e(\lambda)$ denote the “formal exponential” so $\mathbf{Z}_{fin}(L)$ is the space spanned by the $e(\lambda)$ and we have defined the homomorphism

$$\Psi_\rho : \mathbf{Z}_{fin}(\Lambda) \rightarrow \mathbf{C}[[t]], \quad e(\lambda) \mapsto e^{(\lambda \cdot \rho)_\kappa t}.$$

Let N and D be the images under Ψ_ρ of the Weyl numerator and denominator. So

$$\begin{aligned} N(t) &= \prod_{\alpha > 0} \left(e^{(\lambda + \rho, \alpha)_\kappa t/2} - e^{-(\lambda + \rho, \alpha)_\kappa t/2} \right) \\ &= \prod \left((\lambda + \rho, \alpha)_\kappa t \left[1 + \frac{1}{24} (\lambda + \rho, \alpha)_\kappa^2 t^2 + \dots \right] \right) \end{aligned}$$

with a similar formula for D . Then $N/D \rightarrow d(\lambda) =$ the dimension of the representation as $t \rightarrow 0$ is the usual proof (that we reproduced above) of the Weyl dimension formula. Sticking this in to N/D gives

$$\frac{N}{D} = d(\lambda) \left(1 + \frac{1}{24} \sum_{\alpha > 0} [(\lambda + \rho, \alpha)_\kappa^2 - (\rho, \alpha)_\kappa^2] t^2 + \dots \right).$$

$$(\lambda, \mu)_\kappa = \kappa(t_\lambda, t_\mu) = \sum_\alpha \alpha(t_\lambda) \alpha(t_\mu)$$

so

$$(\lambda, \mu)_\kappa = \sum_\alpha (\lambda, \alpha)_\kappa (\mu, \alpha)_\kappa. \quad (25)$$

where the sum is over all roots.

$$\frac{N}{D} = d(\lambda) \left(1 + \frac{1}{24} \sum_{\alpha > 0} [(\lambda + \rho, \alpha)_\kappa^2 - (\rho, \alpha)_\kappa^2] t^2 + \dots \right).$$

For any weight μ we have $(\mu, \mu)_\kappa = \sum (\mu, \alpha)_\kappa^2$ by (25), where the sum is over *all* roots so

$$\frac{N}{D} = d \left(1 + \frac{1}{48} [(\lambda + \rho, \lambda + \rho)_\kappa - (\rho, \rho)_\kappa] t^2 + \dots \right),$$

and we recognize the coefficient of $\frac{1}{48} t^2$ in the above expression as $\chi_\lambda(\text{Cas}^\kappa)$, the scalar giving the value of the Casimir associated to the Killing form in the representation with highest weight λ .

$$\frac{N}{D} = d \left(1 + \frac{1}{48} [(\lambda + \rho, \lambda + \rho)_\kappa - (\rho, \rho)_\kappa] t^2 + \dots \right),$$

and we recognize the coefficient of $\frac{1}{48}t^2$ in the above expression as $\chi_\lambda(\text{Cas}^\kappa)$, the scalar giving the value of the Casimir associated to the Killing form in the representation with highest weight λ .

On the other hand, the image under Ψ_ρ of the character of the irreducible representation with highest weight λ is

$$\sum_{\mu} e^{(\mu, \rho)_\kappa t} = \sum_{\mu} \left(1 + (\mu, \rho)_\kappa t + \frac{1}{2} (\mu, \rho)_\kappa^2 t^2 + \dots \right)$$

where the sum is over all weights in the irreducible representation counted with multiplicity. Comparing coefficients gives

$$\sum_{\mu} (\mu, \rho)_\kappa^2 = \frac{1}{24} d(\lambda) \chi_\lambda(\text{Cas}^\kappa).$$

The Freudenthal - de Vries formula.

$$\sum_{\mu} (\mu, \rho)_{\kappa}^2 = \frac{1}{24} d(\lambda) \chi_{\lambda}(\text{Cas}^{\kappa}).$$

Applied to the adjoint representation the left hand side becomes $(\rho, \rho)_{\kappa}$ by (25), while $d(\lambda)$ is the dimension of the Lie algebra. On the other hand, $\chi_{\lambda}(\text{Cas}^{\kappa}) = 1$ since $\text{tr ad}(\text{Cas}^{\kappa}) = \dim(\mathfrak{g})$ by the definition of Cas^{κ} . So we get

$$(\rho, \rho)_{\kappa} = \frac{1}{24} \dim \mathfrak{g} \tag{26}$$

for any semisimple Lie algebra \mathfrak{g} .

Reductive Lie algebras.

An algebra which is the direct sum a commutative Lie and a semi-simple Lie algebra is called reductive. The previous result of Freudenthal and deVries has been generalized by Kostant from a semi-simple Lie algebra to all reductive Lie algebras: Suppose that \mathfrak{g} is merely reductive, and that we have chosen a symmetric bilinear form on \mathfrak{g} which is invariant under the adjoint representation, and denote the associated Casimir element by $\text{Cas}_{\mathfrak{g}}$. We claim that (26) generalizes to

$$\frac{1}{24} \text{tr ad}(\text{Cas}_{\mathfrak{g}}) = (\rho, \rho). \quad (27)$$

(Notice that if \mathfrak{g} is semisimple and we take our symmetric bilinear form to be the Killing form $(\ , \)_{\kappa}$ (27) becomes (26).) To prove (27) observe that both sides decompose into sums as we decompose \mathfrak{g} into a sum of its center and its simple ideals, since this must be an orthogonal decomposition for our invariant scalar product. The contribution of the center is zero on both sides, so we are reduced to proving (27) for a simple algebra.

Then our symmetric biinear form

(\cdot, \cdot) must be a scalar multiple of the Killing form:

$$(\cdot, \cdot) = c^2(\cdot, \cdot)_\kappa$$

for some non-zero scalar c . If z_1, \dots, z_N is an orthonormal basis of \mathfrak{g} for $(\cdot, \cdot)_\kappa$ then $z_1/c, \dots, z_N/c$ is an orthonormal basis for (\cdot, \cdot) . Thus

$$\text{Cas}_{\mathfrak{g}} = \frac{1}{c^2} \text{Cas}^\kappa.$$

So

$$\text{tr ad}(\text{Cas}_{\mathfrak{g}}) = \frac{1}{c^2} \text{tr ad}(\text{Cas}^\kappa) = \frac{1}{c^2} \frac{1}{24} \dim \mathfrak{g}.$$

But on \mathfrak{h}^* we have the dual relation

$$(\rho, \rho) = \frac{1}{c^2} (\rho, \rho)_\kappa.$$

Combining the last two equations shows that (27) becomes (26).

Notice that the same proof shows that we can generalize (8) as

$$\chi_\lambda(\text{Cas}) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho) \quad (28)$$

valid for any reductive Lie algebra equipped with a symmetric bilinear form invariant under the adjoint representation.

Fundamental representations.

We let ω_i denote the weight which satisfies

$$\omega_i(h_j) = \delta_{ij}$$

so that the ω_i form an integral basis of \mathbf{L} and are dominant. We call these the **basic** weights. If (V, ρ) and (W, σ) are two finite dimensional irreducible representations with highest weights λ and σ , then $V \otimes W, \rho \otimes \sigma$ contains the irreducible representation with highest weight $\lambda + \mu$, and highest weight vector $v_\lambda \otimes w_\mu$, the tensor product of the highest weight vectors in V and W . Taking this “highest” component in the tensor product is known as the **Cartan product** of the two irreducible representations.

Let (V_i, ρ_i) be the irreducible representations corresponding to the basic weight ω_i . Then every finite dimensional irreducible representation of \mathfrak{g} can be obtained by Cartan products from these, and for that reason they are called the **fundamental representations**.

For the case of $A_n = sl(n + 1)$ we have already verified that the fundamental representations are $\wedge^k(V)$ where $V = \mathbf{C}^{n+1}$ and where the basic weights are

$$\omega_i = L_1 + \cdots + L_i$$

We now sketch the results for the other classical simple algebras, leaving the details as an exercise in the use of the Weyl character formula.

$$\omega_i = L_1 + \cdots + L_i$$

For $C_n = sp(2n)$ it is immediate to check that these same expressions give the basic weights. However while $V = \mathbf{C}^{2n} = \wedge^1(V)$ is irreducible, the higher order exterior powers are not: Indeed, the symplectic form $\Omega \in \wedge^2(V^*)$ is preserved, and hence so is the the map

$$\wedge^j(V) \rightarrow \wedge^{j-2}(V)$$

given by contraction by Ω . It is easy to check that the image of this map is surjective (for $j = 2, \dots, n$). the kernel is thus an invariant subspace of dimension

$$\binom{2n}{j} - \binom{2n}{2j-2}$$

and a (not completely trivial) application of the Weyl dimension formula will show that these are indeed the dimensions of the irreducible representations with highest weight ω_j . Thus these kernels are the fundamental representations of C_n . Here are some of the details:

$$\rho = \omega_1 + \cdots + \omega_n = \sum (n - i + 1)L_i.$$

The most general dominant weight is of the form

$$\sum k_i \omega_i = a_1 L_1 + \cdots + a_n L_n$$

where

$$a_1 = k_1 + \cdots + k_n, \quad a_2 = k_2 + \cdots + k_n, \quad \cdots \quad a_n = k_n$$

where the k_i are non-negative integers. So we can equally well use any decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ of integers to parameterize the irreducible representations. We have

$$(\rho, L_i - L_j) = j - i, \quad (\rho, L_i + L_j) = 2n + 2 - i - j.$$

Multiplying these all together gives the denominator in the Weyl dimension formula.

Similarly the numerator becomes

$$\prod_{i < j} (l_i - l_j) \prod_{i \leq j} (l_i + l_j)$$

where

$$l_i := a_i + n - i + 1.$$

If we set $m_i := n - i + 1$ then we can write the Weyl dimension formula as

$$\dim V(a_1, \dots, a_n) = \prod_{i < j} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \prod_i \frac{l_i}{m_i},$$

where for the case $i = j$ we have taken out a common factor of 2^n from the numerator and the denominator.

$$\dim V(a_1, \dots, a_n) = \prod_{i < j} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \prod_i \frac{l_i}{m_i},$$

An easy induction shows that

$$\prod_{i < j} (m_i^2 - m_j^2) \prod_i m_i = (2n - 1)!(2n - 3)! \cdots 1!.$$

so if we set

$$r_i = l_i - 1 = a + n - i$$

then

$$\dim V(a_1, \dots, a_n) = \frac{\prod_{i < j} (r_i - r_j)(r_i + r_j + 2) \prod_i (r_i + 1)}{(2n - 1)!(2n - 3)! \cdots 1!}.$$

$$\dim V(a_1, \dots, a_n) = \frac{\prod_{i < j} (r_i - r_j)(r_i + r_j + 2) \prod_i (r_i + 1)}{(2n - 1)!(2n - 3)! \cdots 1!}.$$

For example, suppose we want to compute the dimension of the fundamental representation corresponding to $\lambda_2 = L_1 + L_2$ so $a_1 = a_2 = 1, a_i = 0, i > 2$. In applying the preceding formula, all of the terms with $2 < i$ are the same as for the trivial representation, as is $r_1 - r_2$. The ratios of the remaining factors to those of the trivial representation are

$$\prod_{j=3}^n \frac{j}{j-1} \cdot \prod_{j=3}^n \frac{j}{j-2} = \prod_{j=3}^n \frac{j}{j-2}$$

coming from the $r_i - r_j$ terms, $i = 1, 2$. Similarly the $r_i + r_j$ terms give a factor

$$\frac{2n+1}{2n-1} \prod_{j=3}^n \frac{2n+2-j}{2n-j}$$

and the terms $r_1 + 1, r_2 + 1$ contribute a factor

$$\frac{n+1}{n-1}.$$

In multiplying all of these terms together there is a huge cancellation and what is left for the dimension of this fundamental representation is

$$\frac{(2n+1)(2n-2)}{2}.$$

Notice that this equals

$$\binom{2n}{2} - 1 = \dim \wedge^2 V - 1.$$

For B_n it is easy to check that $\omega_i := L_1 + \cdots + L_i$ ($i \leq n - 1$), and $\omega_n = \frac{1}{2}(L_1 + \cdots + L_n)$ are the basic weights and the Weyl dimension formula gives the value $\binom{2n+1}{j}$ for $j \leq n - 1$ as the dimensions of the irreducibles with these weight, so that they are $\wedge^j(V)$, $j = 1, \dots, n - 1$ while the dimension of the irreducible corresponding to ω_n is 2^n . This is the spin representation which we will study later.

Finally, for $D_n = o(2n)$ the basic weights are

$$\omega_j = L_1 + \cdots + L_j, \quad j \leq n - 2,$$

and

$$\omega_{n-1} := \frac{1}{2}(L_1 + \cdots + L_{n-1} + L_n) \text{ and } \omega_n := \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n).$$

The Weyl dimension formula shows that the the first $n - 2$ fundamental representations are in fact the representation on $\wedge^j(V)$, $j = 1, \dots, n - 2$ while the last two have dimension 2^{n-1} . These are the half spin representations which we will also study later.