Math 128 Lecture 15.

Cyclic highest weight modules

Weights and weight vectors.

In this chapter, $\mathbf{g}$ will denote a semi-simple Lie algebra for which we have chosen a Cartan subalgebra, $\mathbf{h}$ and a base Δ for the roots $\Phi = \Phi^+ \cup \Phi^-$ of $\mathbf{g}$.

We will be interested in describing its finite dimensional irreducible representations. If W is a finite dimensional module for $\mathbf{g}$, then $\mathbf{h}$ has at least one simultaneous eigenvector; that is there is a $\mu \in \mathbf{h}^*$ and a $w \neq 0 \in W$ such that

$$hw = \mu(h)w \quad \forall\ h \in \mathbf{h}. \tag{1}$$

The linear function μ is called a **weight** and the vector v is called a **weight vector**. If $x \in \mathbf{g}_\alpha$,

$$hxw = [h,x]w + xhw = (\mu + \alpha)(h)xw.$$

Highest weight vectors.

$$hxw = [h, x]w + xhw = (\mu + \alpha)(h)xw.$$

This shows that the space of all vectors w satisfying an equation of the type (1) (for varying μ) spans an invariant subspace. If W is irreducible, then the weight vectors (those satisfying an equation of the type (1)) must span all of W. Furthermore, since W is finite dimensional, there must be a vector v and a linear function λ such that

$$hv = \lambda(h)v \ \ \forall\ h \in \mathbf{h}, \quad e_\alpha v = 0,\ \forall \alpha \in \Phi^+. \tag{2}$$

Using irreducibility again, we conclude that

$$W = U(\mathbf{g})v.$$

The module is **cyclic** generated by v. In fact we can be more precise:

Using PBW.

Let $h_1, \ldots, h_\ell$ be the basis of $\mathbf{h}$ corresponding to the choice of simple roots, let $e_i \in \mathbf{g}_{\alpha_i}$, $f_i \in \mathbf{g}_{-\alpha_i}$ where $\alpha_1, \ldots, \alpha_m$ are all the positive roots. (We can choose them so that each e and f generate a little $sl(2)$.) Then

$$\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+,$$

where $e_1, \ldots, e_m$ is a basis of $\mathbf{n}_+$, where $h_1, \ldots, h_\ell$ is a basis of $\mathbf{h}$, and $f_1, \ldots, f_m$ is a basis of $\mathbf{n}_-$. The Poincaré-Birkhoff-Witt theorem says that monomials of the form

$$f_1^{i_1} \cdots f_m^{i_m} h_1^{j_1} \cdots h_\ell^{j_\ell} e_1^{k_1} \cdots e_m^{k_m}$$

form a basis of $U(\mathbf{g})$. Here we have chosen to place all the e's to the extreme right, with the h's in the middle and the f's to the left. It now follows that the elements

$$f_1^{i_1} \cdots f_m^{i_m} v \qquad \text{span } W.$$

Possible weights.

the elements

$$f_1^{i_1} \cdots f_m^{i_m} v$$

span W. Every such element, if non-zero, is a weight vector with weight

$$\lambda - (i_1\alpha_1 + \cdots + i_m\alpha_m).$$

Recall that

$$\mu \prec \lambda \quad \text{means that } \lambda - \mu = \sum k_i \alpha_i, \quad \alpha_i > 0,$$

where the k_i are non-negative integers. We have shown that every weight μ of W satisfies

$$\mu \prec \lambda.$$

Cyclic highest weight modules.

So we make the definition: A cyclic highest weight module for $\mathbf{g}$ is a module (not necessarily finite dimensional) which has a vector v_+ such that

$$x_+v_+ = 0, \ \ \forall\, x_+ \in \mathbf{n}_+, \quad hv_+ = \lambda(h)v_+ \ \forall h \in \mathbf{h}$$

and

$$V = U(\mathbf{g})v_+.$$

In any such cyclic highest weight module every submodule is a direct sum of its weight spaces (by van der Monde). The weight spaces V_μ all satisfy

$$\mu \prec \lambda$$

and we have

$$V = \bigoplus V_\mu.$$

Proper submodules of a cyclic highest weight module.

$$V = \bigoplus V_\mu .$$

Any proper submodule can not contain the highest weight vector, and so the sum of two proper submodules is again a proper submodule. Hence any such V has a unique maximal submodule and hence a unique irreducible quotient. The quotient of any highest weight module by an invariant submodule, if not zero, is again a cyclic highest weight module with the same highest weight.

Verma modules.

There is a "biggest" cyclic highest weight module, associated with any $\lambda \in \mathbf{h}^*$ called the **Verma module**. It is defined as follows: Let us set

$$\mathbf{b} := \mathbf{h} \oplus \mathbf{n}_+.$$

Given any $\lambda \in \mathbf{h}^*$ let $\mathbf{C}_\lambda$ denote the one dimensional vector space $\mathbf{C}$ with basis z_+ and with the action of $\mathbf{b}$ given by

$$(h + \sum_{\beta \succ 0} x_\beta) z_+ := \lambda(h) z_+.$$

So it is a left $U(\mathbf{b})$ module. By the Poincaré Birkhoff Witt theorem, $U(\mathbf{g})$ is a free right $U(\mathbf{b})$ module with basis $\{f_1^{i_1} \cdots f_\ell^{i_m}\}$, and so we can form the **Verma module**

$$\mathrm{Verm}(\lambda) := U(\mathbf{g}) \otimes_{U(\mathbf{b})} \mathbf{C}_\lambda$$

which is a cyclic module with highest weight vector $v_+ := 1 \otimes z_+$.

Irreducible highest weight modules.

Furthermore, any two *irreducible* cyclic highest weight modules with the same highest weight are isomorphic. Indeed, if V and W are two such with highest weight vector v_+, u_+, consider $V \oplus W$ which has (v_+, u_+) as a maximal weight vector with weight λ, and hence $Z := U(\mathbf{g})(v_+, u_+)$ is cyclic and of highest weight λ. Projections onto the first and second factors give non-zero homomorphisms which must be surjective. But Z has a unique irreducible quotient. Hence these must induce isomorphisms on this quotient, V and W are isomorphic.

Hence, up to isomorphism, there is a unique irreducible cyclic highest weight module with highest weight λ. We call it

$$\mathrm{Irr}(\lambda).$$

In short, we have constructed a "largest" highest weight module $\mathrm{Verm}(\lambda)$ and a "smallest" highest weight module $\mathrm{Irr}(\lambda)$.

When is $\dim \mathrm{Irr}(\lambda) < \infty$?

If $\mathrm{Irr}(\lambda)$ is finite dimensional, then it is finite dimensional as a module over any subalgebra, in particular over any subalgebra isomorphic to $sl(2)$. Applied to the subalgebra $sl(2)_i$ generated by e_i, h_i, f_i we conclude that

$$\lambda(h_i) \in \mathbf{Z}.$$

Such a weight is is called **integral**. Furthermore the representation theory of $sl(2)$ says that the maximal weight for any finite dimensional representation must satisfy

$$\lambda(h_i) = \langle \lambda, \alpha_i \rangle \geq 0$$

so that λ lies in the closure of the fundamental Weyl chamber. Such a weight is called **dominant**. So a necessary condition for $\mathrm{Irr}(\lambda)$ to be finite dimensional is that λ be dominant integral. We now show that conversely, $\mathrm{Irr}(\lambda)$ is finite dimensional whenever λ is dominant integral.

For this we recall that in the universal enveloping algebra $U(\mathbf{g})$ we have

1. $[e_j, f_i^{k+1}] = 0$, if $i \neq j$
2. $[h_j, f_i^{k+1}] = -(k+1)\alpha_i(h_j) f_i^{k+1}$
3. $[e_i, f_i^{k+1}] = -(k+1) f_i^k (k \cdot 1 - h_i)$

where the first two equations are consequences of the fact that ad is a derivation and

$$[e_i, f_j] = 0 \text{ if } i \neq j \quad \text{since } \alpha_i - \alpha_j \text{ is not a root}$$

and

$$[h_j, f_j] = -\alpha_j(h_i) f_j.$$

The last is a the fact about $sl(2)$ which we have proved in Chapter II. Notice that it follows from 1.) that $e_j(f_i^k)v_+ = 0$ for all k and all $i \neq j$ and from 3.) that

$$e_i f_i^{\lambda(h_i)+1} v_+ = 0$$

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so that $f_i^{\lambda(h_i)+1} v_+$ is a maximal weight vector. If it were non-zero, the cyclic module it generates would be a proper submodule of $\mathrm{Irr}(\lambda)$ contradicting the irreducibility. Hence

$$f_i^{\lambda(h_i)+1} v_+ = 0.$$

So for each i the subspace spanned by $v_+, f_i v_+, \cdots, f_i^{\lambda(h_i)} v_+$ is a finite dimensional $sl(2)_i$ module. In particular $\mathrm{Irr}(\lambda)$ contains some finite dimensional $sl(2)_i$ modules. Let V' denote the sum of all such. If W is a finite dimensional $sl(2)_i$ module, then $e_\alpha W$ is again finite dimensional, thus so their sum, which is a finite dimensional $sl(2)_i$ module. Hence V' is $\mathbf{g}$- stable, hence all of $\mathrm{Irr}(\lambda)$.

Enter the Weyl group.

In particular, the e_i and the f_i act as locally nilpotent operators on $\mathrm{Irr}(\lambda)$. So the operators $\tau_i := (\exp e_i)(\exp -f_i)(\exp e_i)$ are well defined and

$$\tau_i(\mathrm{Irr}(\lambda))_\mu = \mathrm{Irr}(\lambda)_{s_i\mu}$$

so

$$\dim \mathrm{Irr}(\lambda)_{w\mu} = \dim \mathrm{Irr}(\lambda)_\mu \quad \forall w \in \mathcal{W} \tag{3}$$

where $\mathcal{W}$ denotes the Weyl group. These are all finite dimensional subspaces: Indeed their dimension is at most the corresponding dimension in the Verma module $\mathrm{Verm}(\lambda)$, since $\mathrm{Irr}(\lambda)_\mu$ is a quotient space of $\mathrm{Verm}(\lambda)_\mu$. But $\mathrm{Verm}(\lambda)_\mu$ has a basis consisting of those $f_1^{k_1}\cdots f_m^{k_m}v_+$. The number of such elements is the number of ways of writing

$$\lambda - \mu = k_1\alpha_1 + \cdots k_m\alpha_m.$$

The Kostant partition function.

$\text{Verm}(\lambda)_\mu$ has a basis consisting of those $f_1^{k_1} \cdots f_m^{k_m} v_+$. The number of such elements is the number of ways of writing

$$\lambda - \mu = k_1\alpha_1 + \cdots k_m\alpha_m.$$

So dim $\text{Verm}(\lambda)_\mu$ is the number of m-tuplets of non-negative integers $(k_1, \ldots, k_m)$ such that the above equation holds. This number is clearly finite, and is known as $P_K(\lambda - \mu)$, the Kostant partition function of $\lambda - \mu$, which will play a central role in what follows.

Conjugating to the fundamental Weyl chamber.

Now every element of E is conjugate under W to an element of the closure of the fundamental Weyl chamber, i.e. to a μ satisfying

$$(\mu, \alpha_i) \geq 0$$

i.e. to a μ that is dominant. We claim that there are only finitely many dominant weights μ which are $\prec \lambda$, which will complete the proof of finite dimensionality. Indeed, the sum of two dominant weights is dominant, so $\lambda + \mu$ is dominant. On the other hand, $\lambda - \mu = \sum k_i \alpha_i$ with the $k_i \geq 0$. So

$$(\lambda, \lambda) - (\mu, u) = (\lambda + \mu, \lambda - \mu) = \sum k_i (\lambda + \mu, \alpha_i) \geq 0.$$

So μ lies in the intersection of the ball of radius $\sqrt{(\lambda, \lambda)}$ with the discrete set of weights $\prec \lambda$ which is finite.

A useful fact.

$$\dim \mathrm{Irr}(\lambda)_{w\mu} = \dim \mathrm{Irr}(\lambda)_{\mu} \quad \forall w \in \mathcal{W} \tag{3}$$

We record a consequence of (3) which is useful under very special circumstances. Suppose we are given a finite dimensional representation of **g** with the property that each weight space is one dimensional and all weights are conjugate under $\mathcal{W}$. Then this representation must be irreducible.

Example.

For example, take $\mathbf{g} = sl(n+1)$ and consider

the representation of $\mathbf{g}$ on $\wedge^k(\mathbf{C}^{n+1}), 1 \leq k \leq n$. In terms of the standard basis $e_1, \ldots, e_{n+1}$ of $\mathbf{C}^{n+1}$ the elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$ are weight vectors with weights $L_{i_1} + \cdots + L_{i_k}$, Where $\mathbf{h}$ consists of all diagonal traceless matrices and L_i is the linear function which assigns to each diagonal matrix its i-th entry.

These weight spaces are all one dimensional and conjugate under the Weyl group. Hence these representations are irreducible with highest weight

$$\omega_i := L_1 + \cdots + L_k$$

in terms of the usual choice of base, $h_1, \ldots, h_n$ where h_j is the diagonal matrix with 1 in the j-th position, -1 in the $j+1$-st position and zeros elsewhere.

Example, continued.

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in terms of the usual choice of base, $h_1, \ldots, h_n$ where h_j is the diagonal matrix with 1 in the j-th position, -1 in the $j+1$-st position and zeros elsewhere. Notice that

$$\omega_i(h_j) = \delta_{ij}$$

so that the ω_i form a basis of the "weight lattice" consisting of those $\lambda \in \mathbf{h}^*$ which take integral values on $h_1, \ldots, h_n$.

The center of U(**g**).

Recall that our basis of $U(\mathbf{g})$ consists of the elements

$$f_1^{i_1} \cdots f_m^{i_m} h_1^{j_1} \cdots h_\ell^{j_\ell} e_1^{k_1} \cdots e_m^{k_m}.$$

The elements of $U(\mathbf{h})$ are then the ones with no e or f component in their expression. So we have a vector space direct sum decomposition

$$U(\mathbf{g}) = U(\mathbf{h}) \oplus (U(\mathbf{g})\mathbf{n}_+ + \mathbf{n}_- U(\mathbf{g})),$$

where $\mathbf{n}_+$ and $\mathbf{n}_-$ are the corresponding nilpotent subalgebras. Let γ denote projection onto the first factor in this decomposition. Now suppose $z \in Z(\mathbf{g})$, the center of the universal enveloping algebra. In particular, $z \in U(\mathbf{g})^{\mathbf{h}}$.

The center of U(**g**), continued.

$$U(\mathbf{g}) = U(\mathbf{h}) \oplus \left(U(\mathbf{g})\mathbf{n}_+ + \mathbf{n}_- U(\mathbf{g})\right),$$

where $\mathbf{n}_+$ and $\mathbf{n}_-$ are the corresponding nilpotent subalgebras. Let γ denote projection onto the first factor in this decomposition. Now suppose $z \in Z(\mathbf{g})$, the center of the universal enveloping algebra. In particular, $z \in U(\mathbf{g})^{\mathbf{h}}$. The eigenvalues of the monomial above under the action of $h \in \mathbf{h}$ are

$$\sum_{s=1}^{m} (k_s - i_s)\alpha_s(h).$$

So any monomial in the expression for z can not have f factors alone. We have proved that

$$z - \gamma(z) \in U(\mathbf{g})\mathbf{n}_+, \quad \forall\, z \in Z(\mathbf{g}). \tag{4}$$

For any $\lambda \in \mathbf{h}^*$, the element $z \in Z(\mathbf{g})$ acts as a scalar, call it $\chi_\lambda(z)$ on the Verma module associated to λ.

The center of U($\mathbf{g}$), continued.

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For any $\lambda \in \mathbf{h}^*$, the element $z \in Z(\mathbf{g})$ acts as a scalar, call it $\chi_\lambda(z)$ on the Verma module associated to λ.

In particular, if λ is a dominant integral weight, it acts by this same scalar on the irreducible finite dimensional module associated to λ.

On the other hand, the linear map $\lambda : \mathbf{h} \to \mathbf{C}$ extends to a homomorphism, which we will also denote by λ of $U(\mathbf{h}) = S(\mathbf{h}) \to \mathbf{C}$. Explicitly, if we think of elements of $U(\mathbf{h}) = S(\mathbf{h})$ as polynomials on $\mathbf{h}^*$, then $\lambda(P) = P(\lambda)$ for $P \in S(\mathbf{h})$. Since $\mathbf{n}_+ v = 0$ if v is the maximal weight vector, we conclude from (4) that

$$\chi_\lambda(z) = \lambda(\gamma(z)) \quad \forall\, z \in Z(\mathbf{g}). \tag{5}$$

The value of the Casimir.

The Casimir element (associated to the Killing form) belongs to Z(**g**). The rest of this lecture will be devoted to proving the proving the following key formula: Let

$$\rho := \frac{1}{2}\sum_{\alpha>0}\alpha.$$

Then

$$\chi_\lambda(\mathrm{Cas}^\kappa) = (\lambda+\rho,\lambda+\rho)_\kappa - (\rho,\rho)_\kappa. \tag{8}$$

The value of the Casimir, I.

$$\chi_\lambda(z) = \lambda(\gamma(z)) \quad \forall\, z \in Z(\mathbf{g}). \tag{5}$$

We want to apply this formula to the second order Casimir element associated to the Killing form κ. So let $k_1, \ldots, k_\ell \in \mathbf{h}$ be the dual basis to $h_1, \ldots, h_\ell$ relative to κ, i.e.

$$\kappa(h_i, k_j) = \delta_{ij}.$$

Let $x_\alpha \in \mathbf{g}_\alpha$ be a basis (i.e. non-zero) element and $z_\alpha \in \mathbf{g}_{-\alpha}$ be the dual basis element to x_α under the Killing form, so the second order Casimir element is

$$\mathrm{Cas}^\kappa = \sum h_i k_i + \sum_\alpha x_\alpha z_\alpha.$$

where the second sum on the right is over *all* roots.

The value of the Casimir, 2.

$$\chi_\lambda(z) = \lambda(\gamma(z)) \quad \forall\, z \in Z(\mathbf{g}). \tag{5}$$

$$\mathrm{Cas}^\kappa = \sum h_i k_i + \sum_\alpha x_\alpha z_\alpha.$$

where the second sum on the right is over *all* roots. We might choose the $x_\alpha = e_\alpha$ for positive roots, and then the corresponding z_α is some multiple of the f_α. (And, for present purposes we might even choose $f_\alpha = z_\alpha$ for positive α.) The problem is that the z_α for positive α in the above expression for Cas^κ are written to the right, and we must move them to the left. So we write

$$\mathrm{Cas}^\kappa = \sum_i h_i k_i + \sum_{\alpha>0} [x_\alpha, z_\alpha] + \sum_{\alpha>0} z_\alpha x_\alpha + \sum_{\alpha<0} x_\alpha z_\alpha.$$

This expression for Cas^κ has all the $\mathbf{n}^+$ elements moved to the right; in particular, all of the summands in the last two sums annihilate v_λ. Hence

$$\gamma(\mathrm{Cas}^\kappa) = \sum_i h_i k_i + \sum_{\alpha>0} [x_\alpha, z_\alpha]$$

The value of the Casimir, 3.

$$\chi_\lambda(z) = \lambda(\gamma(z)) \quad \forall\, z \in Z(\mathbf{g}). \tag{5}$$

$$\mathrm{Cas}^\kappa = \sum h_i k_i + \sum_\alpha x_\alpha z_\alpha.$$

$$\mathrm{Cas}^\kappa = \sum_i h_i k_i + \sum_{\alpha>0} [x_\alpha, z_\alpha] + \sum_{\alpha>0} z_\alpha x_\alpha + \sum_{\alpha<0} x_\alpha z_\alpha.$$

$$\gamma(\mathrm{Cas}^\kappa) = \sum_i h_i k_i + \sum_{\alpha>0} [x_\alpha, z_\alpha]$$

and

$$\chi_\lambda(\mathrm{Cas}^\kappa) = \sum_i \lambda(h_i)\lambda(k_i) + \sum_{\alpha>0} \lambda([x_\alpha, z_\alpha]).$$

The value of the Casimir, 4.

$$\chi_\lambda(\mathrm{Cas}^\kappa) = \sum \lambda(h_i)\lambda(k_i) + \sum \lambda([x_\alpha, z_\alpha]).$$

For any $h \in \mathbf{h}$ we have

$$\kappa(h, [x_\alpha, z_\alpha]) = \kappa([h, x_\alpha], z_\alpha) = \alpha(h)\kappa(x_\alpha, z_\alpha) = \alpha(h) \qquad \textbf{so}$$

$$[x_\alpha, z_\alpha] = t_\alpha$$

where $t_\alpha \in \mathbf{h}$ is uniquely determined by

$$\kappa(t_\alpha, h) = \alpha(h) \quad \forall\, h \in \mathbf{h}.$$

Let $(\ ,\ )_\kappa$ denote the bilinear form on $\mathbf{h}^*$ obtained from the identification of $\mathbf{h}$ with $\mathbf{h}^*$ given by κ. Then

$$\sum_{\alpha>0} \lambda([x_\alpha, z_\alpha]) = \sum_{\alpha>0} \lambda(t_\alpha) = \sum_{\alpha>0} (\lambda, \alpha)_\kappa = 2(\lambda, \rho)_\kappa \tag{6}$$

where

$$\rho := \frac{1}{2} \sum_{\alpha>0} \alpha.$$

The value of the Casimir, 5.

On the other hand, let the constants a_i be defined by

$$\lambda(h) = \sum_i a_i \kappa(h_i, h) \quad \forall\ h \in \mathbf{h}.$$

In other words λ corresponds to $\sum a_i h_i$ under the isomorphism of $\mathbf{h}$ with $\mathbf{h}^*$ so

$$(\lambda, \lambda)_\kappa = \sum_{i,j} a_i a_j \kappa(h_i, h_j).$$

Since $\kappa(h_i, k_j) = \delta_{ij}$ we have

$$\lambda(k_i) = a_i.$$

Combined with $\lambda(h_i) = \sum_j a_j \kappa(h_j, h_i)$ this gives

$$(\lambda, \lambda)_\kappa = \sum_i \lambda(h_i)\lambda(k_i). \tag{7}$$

The value of the Casimir - the key formula.

$$\chi_\lambda(\mathrm{Cas}^\kappa) = \sum \lambda(h_i)\lambda(k_i) + \sum \lambda([x_\alpha, z_\alpha]).$$

$$\sum_{\alpha>0} \lambda([x_\alpha, z_\alpha]) = \sum_{\alpha>0} \lambda(t_\alpha) = \sum_{\alpha>0} (\lambda, \alpha)_\kappa = 2(\lambda, \rho)_\kappa \tag{6}$$

where

$$\rho := \frac{1}{2} \sum_{\alpha>0} \alpha.$$

$$(\lambda, \lambda)_\kappa = \sum_i \lambda(h_i)\lambda(k_i). \tag{7}$$

Combined with (6) this yields

$$\chi_\lambda(\mathrm{Cas}^\kappa) = (\lambda + \rho, \lambda + \rho)_\kappa - (\rho, \rho)_\kappa. \tag{8}$$