Math 128 Lecture 14

Serre's theorem.

We have classified all the possibilities for an irreducible Cartan matrix via the classification of the possible Dynkin diagrams. The four major series in our classification correspond to the classical simple algebras we introduced in Chapter III. The remaining five cases also correspond to simple algebras - the "exceptional algebras". Each deserves a discussion on its own. However a theorem of Serre guarantees that starting with any Cartan matrix, there is a corresponding semi-simple Lie algebra. So even before studying each of the simple algebras in detail, we know in advance that they exist. We present Serre's theorem in this chapter.

Unbroken strings of roots.

Recall that if α and β are roots,

$$\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$

and the string of roots of the form $\beta + j\alpha$ is unbroken and extends from

$$\beta - r\alpha \quad \text{to} \quad \beta + q\alpha \quad \text{where} \quad r - q = \langle \beta, \alpha \rangle.$$

In particular, if $\alpha, \beta \in \Delta$ so that $\beta - \alpha$ is not a root, the string is

$$\beta, \beta+\alpha, \ldots, \beta + q\alpha$$

where

$$q = -\langle \beta, \alpha \rangle.$$

Thus

$$(\mathrm{ad} e_\alpha)^{-\langle \beta,\alpha \rangle + 1} e_\beta = 0,$$

for $e_\alpha \in \mathbf{g}_\alpha,\ e_\beta \in \mathbf{g}_\beta$ but

$$(\mathrm{ad}\ e_\alpha)^k e_\beta \neq 0 \quad \text{for } 0 \leq k \leq -\langle \beta, \alpha \rangle,$$

if $e_\alpha \neq 0,\ e_\beta \neq 0$.

The Serre relations.

So if $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ we may choose

$$e_i \in \mathbf{g}_{\alpha_i},\ f_i \in \mathbf{g}_{-\alpha_i}$$

so that

$$e_1, \ldots, , e_\ell,\ f_1, \ldots, f_\ell$$

generate the algebra and

$$\begin{aligned}
[h_i, h_j] &= 0, \quad 1 \leq i, j, \leq \ell && (1)\\
[e_i, f_i] &= h_i && (2)\\
[e_i, f_j] &= 0 \quad i \neq j && (3)\\
[h_i, e_j] &= \langle \alpha_j, \alpha_i \rangle e_j && (4)\\
[h_i, f_j] &= -\langle \alpha_j, \alpha_i \rangle f_i && (5)\\
(\mathrm{ad}\ e_i)^{-\langle \alpha_j, \alpha_i \rangle + 1} e_j &= 0 \qquad i \neq j && (6)\\
(\mathrm{ad}\ f_i)^{-\langle \alpha_j, \alpha_i \rangle + 1} f_j &= 0 \quad i \neq j. && (7)
\end{aligned}$$

Serre's theorem.

$$[h_i, h_j] = 0, \quad 1 \leq i, j, \leq \ell \qquad (1)$$

$$[e_i, f_i] = h_i \qquad (2)$$

$$[e_i, f_j] = 0 \quad i \neq j \qquad (3)$$

$$[h_i, e_j] = \langle \alpha_j, \alpha_i \rangle e_j \qquad (4)$$

$$[h_i, f_j] = -\langle \alpha_j, \alpha_i \rangle f_i \qquad (5)$$

$$(\text{ad } e_i)^{-\langle \alpha_j, \alpha_i \rangle + 1} e_j = 0 \quad i \neq j \qquad (6)$$

$$(\text{ad } f_i)^{-\langle \alpha_j, \alpha_i \rangle + 1} f_j = 0 \quad i \neq j. \qquad (7)$$

Serre's theorem says that this is a presentation of a (semi-)simple Lie algebra. In particular, the Cartan matrix gives a presentation of a simple Lie algebra, showing that for every Dynkin diagram there exists a unique simple Lie algebra.

The first five relations.

Let $\mathbf{f}$ be the free Lie algebra on 3ℓ generators,

$$X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell.$$

If $\mathbf{g}$ is a semi-simple Lie algebra with generators and relations (1)–(7), we have a unique homomorphism $\mathbf{f} \to \mathbf{g}$ where $X_i \to e_i$, $Y_i \to f_i, Z_i \to h_i$. We want to consider an intermediate algebra, $\mathbf{m}$, where we make use of all but the last two sets of relations. So let $\mathbf{I}$ be the ideal in $\mathbf{f}$ generated by the elements

$$[Z_i, Z_j],\ [X_i, Y_j] - \delta_{ij} Z_i,\ [Z_i, X_j] - \langle \alpha_j, \alpha_i \rangle X_j,\ [Z_i, Y_j] + \langle \alpha_j, \alpha_i \rangle Y_j.$$

We let $\mathbf{m} := \mathbf{f}/\mathbf{I}$ and denote the image of X_i in $\mathbf{m}$ by x_i etc.

We will first exhibit $\mathbf{m}$ as Lie subalgebra of the algebra of endomorphisms of a vector space. We will then find that there is a homomorphism of $\mathbf{m}$ onto our desired Lie algebra sending $x \mapsto e$, $y \mapsto f$, $z \mapsto h$.

The action of the Z.

So consider a vector space with basis $v_1, \ldots, v_\ell$ and let A be the tensor algebra over this vector space. We drop the tensor product signs in the algebra, so write

$$v_{i_1} v_{i_2} \cdots v_{i_t} := v_{i_1} \otimes \cdots v_{i_t}$$

for any finite sequence of integers with values from 1 to ℓ. We make A into an $\mathbf{f}$ module as follows: We let the Z_i act as derivations of A, determined by its actions on generators by

$$Z_i 1 = 0, \quad Z_j v_i = -\langle \alpha_i, \alpha_j \rangle v_j.$$

So if we define

$$c_{ij} := \langle \alpha_i, \alpha_j \rangle$$

we have

$$Z_j(v_{i_1} \cdots v_{i_t}) = -(c_{i_1 j} + \cdots + c_{i_t j})(v_{i_1} \cdots v_{i_t}).$$

The action of Y.

$$Z_j(v_{i_1} \cdots v_{i_t}) = -(c_{i_1 j} + \cdots + c_{i_t j})(v_{i_1} \cdots v_{i_t}).$$

The action of the Z_i is diagonal in this basis, so their actions commute. We let the Y_i act by left multiplication by v_i. So

$$Y_j v_{i_1} \cdots v_{i_t} := v_j v_{i_1} \cdots v_{i_t}$$

and hence

$$[Z_i, Y_j] = -c_{ji} Y_j = -\langle \alpha_j, \alpha_i \rangle Y_j$$

as desired. We now want to define the action of the X_i so that the relations analogous to (2) and (3) hold.

The action of X.

We now want to define the action of the X_i so that the relations analogous to (2) and (3) hold. Since $Z_i 1 = 0$ these relations will hold when applied to the element 1 if we set

$$X_j 1 = 0 \quad \forall j$$

and

$$X_j v_i = 0 \quad \forall i, j.$$

Suppose we define

$$X_j(v_p v_q) = -\delta_{jp} c_{qj} v_q.$$

Then

$$Z_i X_j(v_p v_q) = \delta_{jp} c_{qj} c_{qi} v_q = -c_{qi} X_j(v_p v_q)$$

while

$$X_j Z_i(v_p v_q) = \delta_{jp} c_{qj}(c_{pi} + c_{qi}) v_q = -(c_{pi} + c_{qi}) X_j(v_j v_q).$$

Thus

$$[Z_i, X_j](v_p v_q) = c_{ji} X_j(v_p v_q)$$

In general, define

$$X_j(v_{p_1}\cdots v_{p_t}) := v_{p_1}\left(X_j(v_{p_2}\cdots v_{p_t})\right) - \delta_{p_1 j}(c_{p_2 j}+\cdots+c_{p_t j})(v_{p_2}\cdots v_{p_t}) \tag{8}$$

for $t \geq 2$. We claim that

$$Z_i X_j\left(v_{p_1}\cdots v_{p_t}\right) = -(c_{p_1 i}+\cdots+c_{p_t i}-c_{ji})X_j(v_{p_1}\cdots v_{p_t}).$$

Indeed, we have verified this for the case $t=2$. By induction, we may assume that $X_j(v_{p_2}\cdots v_{p_t})$ is an eigenvector of Z_i with eigenvalue $c_{p_2 i}+\cdots+c_{p_t i}-c_{ji}$. Multiplying this on the left by v_{p_1} produces the first term on the right of (8). On the other hand, this multiplication produces an eigenvector of Z_i with eigenvalue $c_{p_1 i}+\cdots+c_{p_t i}-c_{ji}$. As for the second term on the right of (8), if $j \neq p_1$ it does not appear. If $j = p_1$ then $c_{p_1 i}+\cdots+c_{p_t i}-c_{ji} = c_{p_2 i}+\cdots+c_{p_t i}$. So in either case, the right hand side of (8) is an eigenvector of Z_i with eigenvalue $c_{p_1 i}+\cdots+c_{p_t i}-c_{ji}$. But then

$$[Z_i, X_j] = \langle \alpha_j, \alpha_i \rangle X_j$$

The action of **m** on A.

We have defined an action of $\mathbf{f}$ on A whose kernel contains $\mathbf{I}$, hence descends to an action of $\mathbf{m}$ on A.

Let $\phi : \mathbf{m} \to \mathrm{End}\, A$ denote this action. Suppose that $z := a_1 z_1 + \cdots + a_\ell z_\ell$ for some complex numbers $a_1, \ldots, a_\ell$ and that $\phi(z) = 0$. The operator $\phi(z)$ has eigenvalues

$$-\sum a_j c_{ij}$$

when acting on the subspace V of A. All of these must be zero. But the Cartan matrix is non-singular. Hence all the $a_i = 0$. This shows that the space spanned by the z_i is in fact ℓ-dimensional and spans an ℓ-dimensional abelian subalgebra of $\mathbf{m}$. Call this subalgebra $\mathbf{z}$.

The image of the X, Y, and Z.

Now consider the 3ℓ-dimensional subspace of $\mathbf{f}$ spanned by the X_i, Y_i and Z_i, $i = 1, \ldots, \ell$. We wish to show that it projects onto a 3ℓ dimensional subspace of $\mathbf{m}$ under the natural passage to the quotient $\mathbf{f} \to \mathbf{m} = \mathbf{f}/\mathbf{i}$. The image of this subspace is spanned by x_i, y_i and z_i. Since $\phi(x_i) \neq 0$ and $\phi(y_i) \neq 0$ we know that $x_i \neq 0$ and $y_i \neq 0$. Suppose we had a linear relation of the form

$$\sum a_i x_i + \sum b_i y_i + z = 0.$$

Choose some $z' \in \mathbf{z}$ such that $\alpha_i(z') \neq 0$ and $\alpha_i(z') \neq \alpha_j(z')$ for any $i \neq j$. This is possible since the α_i are all linearly independent. Bracketing the above equation by z' gives

$$\sum \alpha_(z') a_i x_i - \sum \alpha_i(z') b_i y_i = 0$$

by the relations (4) and (5). Repeated bracketing by z' and using the van der Monde (or induction) argument shows that $a_i = 0, b_i = 0$ and hence that $z = 0$.

We have proved that the elements x_i, y_j, z_k in $\mathbf{m}$ are linearly independent.

The element

$$[x_{i_1}, [x_{i_2}, [\cdots [x_{i_{t-1}}, x_{i_t}] \cdots]]]$$

is an eigenvector of z_i with eigenvalue

$$c_{i_1 i} + \cdots + c_{i_t i}.$$

For any pair of elements μ and λ of $\mathbf{z}^*$ (or of $\mathbf{h}^*$)

$$\mu \prec \lambda$$

denotes the fact that $\lambda - \mu = \sum k_i \alpha_i$ where the k_i are all non-negative integers.

For any $\lambda \in \mathbf{z}^*$ let $\mathbf{m}_\lambda$ denote the set of all $m \in \mathbf{m}$ satisfying

$$[z, m] = \lambda(z)m \ \ \forall z \in \mathbf{z}.$$

Then we have shown that the subalgebra $\mathbf{x}$ of $\mathbf{m}$ generated by $x_1, \ldots, x_\ell$ is contained in

$$\mathbf{m}_+ := \bigoplus_{0 \prec \lambda} \mathbf{m}_\lambda.$$

A decomposition of **m**.

$$\mathbf{m}_+ := \bigoplus_{0 \prec \lambda} \mathbf{m}_\lambda.$$

Similarly, the subalgebra $\mathbf{y}$ of $\mathbf{m}$ generated by the y_i lies in

$$\mathbf{m}_- := \bigoplus_{\lambda \prec 0} \mathbf{m}_\lambda.$$

In particular, the vector space sum

$$\mathbf{y} + \mathbf{z} + \mathbf{x}$$

is direct since $\mathbf{z} \subset \mathbf{m}_0$. We claim that this is in fact all of $\mathbf{m}$. First of all, observe that it is a subalgebra. Indeed, $[y_i, x_j] = -\delta_{ij} z_i$ lies in this subspace, and hence

$$[y_i, [x_{j_1}, [\cdots [x_{j_{t-1}}, x_{j_t}] \cdots] \in \mathbf{x} \quad \text{for } t \geq 2.$$

The decomposition of $\mathbf{m}$.

Thus the subspace $\mathbf{y}+\mathbf{z}+\mathbf{x}$ is closed under ad y_i and hence under any product of these operators. Similarly for ad x_i. Since these generate the algebra $\mathbf{m}$ we see that $\mathbf{y}+\mathbf{z}+\mathbf{x}=\mathbf{m}$ and hence

$$\mathbf{x}=\mathbf{m}_+ \quad \text{and} \quad \mathbf{y}=\mathbf{m}_-.$$

We have shown that

$$\mathbf{m}=\mathbf{m}_- \oplus \mathbf{z} \oplus \mathbf{m}_+$$

where $\mathbf{z}$ is an abelian subalgebra of dimension ℓ, where the subalgebra $\mathbf{m}_+$ is generated by $x_1,\dots,x_\ell$, where the subalgebra $\mathbf{m}_-$ is generated by $y_1,\dots,y_\ell$, and where the 3ℓ elements $x_1,\dots,x_\ell,y_1\dots,y_\ell,z_1,\dots,z_\ell$ are linearly independent.

An identity in $\mathbf{m}$.

There is a further property of $\mathbf{m}$ which we want to use in the next section in the proof of Serre's theorem. For all $i \neq j$ between 1 and ℓ define the elements x_{ij} and y_{ij} by

$$x_{ij} := (\mathrm{ad}\ x_i)^{-c_{ji}+1}(x_j), \quad y_{ij} := (\mathrm{ad}\ y_i)^{-c_{ji}+1}(y_j).$$

Conditions (6) and (7) amount to setting these elements, and hence the ideal that they generate equal to zero. We claim that for all k and all $i \neq j$ between 1 and ℓ we have

$$\mathrm{ad}\ x_k(y_{ij}) = 0 \tag{9}$$

and

$$\mathrm{ad}\ y_k(x_{ij}) = 0. \tag{10}$$

Proof of

$$\text{ad } x_k(y_{ij}) = 0 \qquad (9)$$

where

$$y_{ij} := (\text{ad } y_i)^{-c_{ji}+1}(y_j).$$

By symmetry, it is enough to prove the first of these equations. If $k \neq i$ then $[x_k, y_i] = 0$ by (3) and hence

$$\text{ad } x_k(y_{ij}) = (\text{ad } y_i)^{-c_{ji}+1}[x_k, y_j] = (\text{ad } y_i)^{-c_{ji}+1}\delta_{kj}h_j$$

by (2) and (3). If $k \neq j$ this is zero. If $k = j$ we can write this as

$$(\text{ad } y_i)^{-c_{ji}}(\text{ad } y_i)h_j = (\text{ad } y_i)^{-c_{ji}}c_{ij}y_i.$$

If $c_{ij} = 0$ there is nothing to prove. If $c_{ij} \neq 0$ then $c_{ji} \neq 0$ and in fact is strictly negative since the angles between all elements of a base are obtuse. But then

$$(\text{ad } y_i)^{-c_{ji}}y_i = 0.$$

Completion of the proof.

It remains to consider the case where $k = i$. The algebra generated by $x_i, y_, z_i$ is isomorphic to $sl(2)$ with $[x_i, y_i] = z_i, [z_i, x_i] = 2x_i, [z_i, y_i] = -2y_i$. We have a decomposition of $\mathbf{m}$ into weight spaces for all of $\mathbf{z}$, in particular into weight spaces for this little $sl(2)$. Now $[x_i, y_j] = 0$ (from (3)) so y_j is a maximal weight vector for this $sl(2)$ with weight $-c_{ji}$ and (9) is just a standard property of a maximal weight module for $sl(2)$ with non-negative integer maximal weight.

The ideals **i**, **j**, and **k**.

Let $\mathbf{k}$ be the ideal of $\mathbf{m}$ generated by the x_{ij} and y_{ij} as defined above. We wish to show that

$$\mathbf{g} := \mathbf{m}/\mathbf{k}$$

is a semi-simple Lie algebra with Cartan subalgebra $\mathbf{h} = \mathbf{z}/\mathbf{k}$ and root system Φ. For this purpose, let $\mathbf{i}$ now denote the ideal in $\mathbf{m}_+$ generated by the x_{ij} and $\mathbf{j}$ be the ideal in $\mathbf{m}_-$ generated by the y_{ij} so that

$$\mathbf{i} + \mathbf{j} \subset \mathbf{k}.$$

$\mathbf{j}$ is an ideal of $\mathbf{m}$.

We claim that $\mathbf{j}$ is an ideal of $\mathbf{m}$. Indeed, each y_{ij} is a weight vector for $\mathbf{z}$, and $[\mathbf{z}, \mathbf{m}_-] \subset \mathbf{m}_-$, hence $[\mathbf{z}, \mathbf{j}] \subset \mathbf{j}$. On the other hand, we know that $[x_k, \mathbf{m}_-] \subset \mathbf{m}_- + \mathbf{z}$ and $[x_k, y_{ij}] = 0$ by (9). So $(\mathrm{ad}\ x_k)\mathbf{j} \subset \mathbf{j}$ by Jacobi. Since the x_k generate $\mathbf{m}_+$ Jacobi then implies that $[\mathbf{m}_+, \mathbf{j}] \subset \mathbf{j}$ as well, hence $\mathbf{j}$ is an ideal of $\mathbf{m}$. Similarly, $\mathbf{i}$ is an ideal of $\mathbf{m}$. Hence $\mathbf{i} + \mathbf{j}$ is an ideal of $\mathbf{m}$, and since it contains the generators of $\mathbf{k}$, it must coincide with $\mathbf{k}$, i.e.

$$\mathbf{k} = \mathbf{i} + \mathbf{j}.$$

In particular, $\mathbf{z} \cap \mathbf{k} = \{0\}$ and so $\mathbf{z}$ projects isomorphically onto an ℓ-dimensional abelian subalgebra of $\mathbf{g} = \mathbf{m}/\mathbf{k}$. Furthermore, since $\mathbf{j} \cap \mathbf{m}_+ = \{0\}$ and $\mathbf{i} \cap \mathbf{m}_- = \{0\}$ we have

A decomposition of $\mathbf{g}$.

$$\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+ \tag{11}$$

as a vector space where

$$\mathbf{n}_- = \mathbf{m}_-/\mathbf{j}, \quad \text{and} \quad \mathbf{n}_+ = \mathbf{m}_+/\mathbf{i},$$

and $\mathbf{n}_+$ is a a sum of weight spaces of $\mathbf{h}$, summed over $\lambda \succ 0$ while $\mathbf{n}_-$ is a sum of weight spaces of $\mathbf{h}$ with $\lambda \prec 0$. We have to see which weight spaces survive the passage to the quotient. The $sl(2)$ generated by x_i, y_i, z_i is not sent into zero by the projection of $\mathbf{m}$ onto $\mathbf{g}$ since z_i is not sent into zero. Since $sl(2)$ is simple, this means that the projection map is an isomorphism when restricted to this $sl(2)$. Let us denote the images of x_i, y_i, z_i by e_i, f_i, h_i. Thus $\mathbf{g}$ is generated by the 3ℓ elements

$$e_1, \ldots, e_\ell, f_1, \ldots, f_\ell, h_1, \ldots, h_\ell$$

and all the axioms (1)-(7) are satisfied.

Enter the Weyl group.

We must show that $\mathbf{g}$ is finite dimensional, semi-simple, and has Φ as its root system.

First observe that ad e_i acts nilpotently on each of the generators of the algebra $\mathbf{g}$, and hence acts locally nilpotently on all of $\mathbf{g}$. Similarly for ad f_i. Hence the automorphism

$$\tau_i := (\exp \operatorname{ad}\, e_i)(\operatorname{ad}\, -f_i)(\exp \operatorname{ad}\, e_i)$$

is well defined on all of $\mathbf{g}$. So if s_i denotes the reflection in the Weyl group W corresponding to i, we have

$$\tau_i(\mathbf{g}_\lambda) = \mathbf{g}_{s_i\lambda}.$$

$\mathbf{g}_\lambda$ is finite dimensional

Notice that each of the $\mathbf{m}_\lambda$ is finite dimensional, since the dimension of $\mathbf{m}_\lambda$ for $\lambda \succ 0$ is at most the number of ways to write λ as a sum of successive α_i, each such sum corresponding to the element $[x_{i_1}, [x_{i_2}, [\cdots, x_{i_t}]\cdots]$. (In particular $\mathbf{m}_{k\alpha} = \{0\}$ for $k > 1$.) Similarly for $\lambda \prec 0$. So it follows that each of the $\mathbf{g}_\lambda$ is finite dimensional, that

$$\dim \mathbf{g}_{w\lambda} = \dim \mathbf{g}_\lambda \quad \forall w \in W$$

and that

$$\mathbf{g}_{k\lambda} = 0 \quad \text{for } k \neq -1, 0, 1.$$

Furthermore, $\mathbf{g}_{\alpha_i}$ is one dimensional, and since every root is conjugate to a simple root, we conclude that

$$\dim \mathbf{g}_\alpha = 1 \quad \forall \alpha \in \Phi.$$

$$\mathbf{g}_\lambda = \{0\} \text{ for } \lambda \neq 0, \quad \lambda \notin \Phi.$$

Indeed, suppose that $\mathbf{g}_\lambda \neq \{0\}$. We know that λ is not a multiple of α for any $\alpha \in \Phi$, since we know this to be true for simple roots, and the dimensions of the $\mathbf{g}_\lambda$ are invariant under the Weyl group, each root being conjugate to a simple root. So $\lambda^\perp$ does not coincide with any hyperplane orthogonal to any root. So we can find a $\mu \in \lambda^\perp$ such that $(\alpha, \mu) \neq 0$ for all roots. We may find a $w \in W$ which maps μ into the positive Weyl chamber for Δ so that $(\alpha_i, \mu) \geq 0$ and hence $(\alpha_i, w\mu) > 0$ for $i = 1, \ldots, \ell$. Now

$$\dim \mathbf{g}_{w\lambda} = \dim \mathbf{g}_\lambda$$

and for the latter to be non-zero, we must have

$$w\lambda = \sum k_i \alpha_i$$

with the coefficients all non-negative or non-positive integers.

$$w\lambda = \sum k_i \alpha_i$$

with the coefficients all non-negative or non-positive integers. But

$$0 = (\lambda, \mu) = (w\lambda, w\mu) = \sum k_i (\alpha_i, \mu)$$

with $(\alpha_i, \mu) > 0 \ \forall i$. Hence all the $k_i = 0$.

Completion of the proof of Serre's theorem.

So

$$\dim \mathbf{g} = \ell + \text{ Card } \Phi.$$

We conclude the proof if we show that $\mathbf{g}$ is semi-simple, i.e. contains no abelian ideals. So suppose that $\mathbf{a}$ is an abelian ideal. Since $\mathbf{a}$ is an ideal, it is stable under $\mathbf{h}$ and hence decomposes into weight spaces. If $\mathbf{g}_\alpha \cap \mathbf{a} \neq \{0\}$, then $\mathbf{g}_\alpha \subset \mathbf{a}$ and hence $[\mathbf{g}_{-\alpha}, \mathbf{g}_\alpha] \subset \mathbf{a}$ and hence the entire $sl(2)$ generated by $\mathbf{g}_\alpha$ and $\mathbf{g}_{-\alpha}$ is contained in $\mathbf{a}$ which is impossible since $\mathbf{a}$ is abelian and $sl(2)$ is simple. So $\mathbf{a} \subset \mathbf{h}$. But then $\mathbf{a}$ must be annihilated by all the roots, which implies that $\mathbf{a} = \{0\}$ since the roots span $\mathbf{h}^*$. QED