Math 128 Lecture 13

Classification of the simple Lie algebras.

Goal.

In this chapter we classify all possible root systems of simple Lie algebras. A consequence, as we shall see, is the classification of the simple Lie algebras themselves. The amazing result - due to Killing with some repair work by Élie Cartan - is that with only five exceptions, the root systems of the classical algebras that we studied in Chapter III exhaust all possibilities.

Throughout this chapter we will be dealing with semi-simple Lie algebras over the complex numbers.

Outline.

The logical structure of this chapter is as follows: We first show that the root system of a simple Lie algebra is irreducible (definition below). We then develop some properties of the of the root structure of an irreducible root system, in particular we will introduce its extended Cartan matrix. We then use the Perron-Frobenius theorem to classify all possible such matrices.

From the extended diagrams it is an easy matter to get all possible bases of irreducible root systems. We then develop a few more facts about root systems which allow us to conclude that an isomorphism of irreducible root systems implies an isomorphism of the corresponding Lie algebras. We postpone the the proof of the existence of the exceptional Lie algebras until Chapter VII, where we prove Serre's theorem which gives a unified presentation of all the simple Lie algebras in terms of generators and relations derived directly from the Cartan integers of the simple root system.

Simple Lie algebras and irreducible root systems.

We choose a Cartan subalgebra **h** of a semi-simple Lie algebra **g**, so we have the corresponding set Φ of roots and the real (Euclidean) space E that they span. We say that Φ is **irreducible** if Φ can *not* be partitioned into two disjoint subsets

$$\Phi = \Phi_1 \cup \Phi_2$$

such that every element of Φ_1 is orthogonal to every element of Φ_2.

Proposition 1 *If* $\mathbf{g}$ *is simple then* Φ *is irreducible.*

Proof. Suppose that Φ is not irreducible, so we have a decomposition as above. If $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ then

$$(\alpha + \beta, \alpha) = (\alpha, \alpha) > 0 \quad \text{and} \quad (\alpha + \beta, \beta) = (\beta, \beta) > 0$$

which means that $\alpha + \beta$ can not belong to either Φ_1 or Φ_2 and so is not a root. This means that

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = 0.$$

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In other words, the subalgebra $\mathbf{g}_1$ of $\mathbf{g}$ generated by all the $\mathbf{g}_\alpha,\ \alpha \in \Phi_1$ is centralized by all the $\mathbf{g}_\beta$, so $\mathbf{g}_1$ is a proper subalgebra of $\mathbf{g}$, since if $\mathbf{g}_1 = \mathbf{g}$ this would say that $\mathbf{g}$ has a non-zero center, which is not true for any semi-simple Lie algebra. The above equation also implies that the normalizer of $\mathbf{g}_1$ contains all the $\mathbf{g}_\gamma$ where γ ranges over all the roots. But these $\mathbf{g}_\gamma$ generate $\mathbf{g}$. So $\mathbf{g_1}$ is a proper ideal in $\mathbf{g}$, contradicting the assumption that $\mathbf{g}$ is simple. QED

Let us choose a base Δ for the root system Φ of a semi-simple Lie algebra. We say that Δ is irreducible if we can not partition Δ into two non-empty mutually orthogonal sets as in the definition of irreducibility of Φ as above.

Proposition 2 *Φ is irreducible if and only if Δ is irreducible.*

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Proof. Suppose that Φ is not irreducible, so has a decomposition as above. This induces a partition of Δ which is non-trivial unless Δ is wholly contained in Φ_1 or Φ_2. If $\Delta \subset \Phi_1$ say, then since E is spanned by Δ, this means that all the elements of Φ_2 are orthogonal to E which is impossible. So if Δ is irreducible so is Φ. Conversely, suppose that

$$\Delta = \Delta_1 \cup \Delta_2$$

is a partition of Δ into two non-empty mutually orthogonal subsets. We have proved that every root is conjugate to a simple root by an element of the Weyl group W which is generated by the simple reflections. Let Φ_1 consist of those roots which are conjugate to an element of Δ_1 and Φ_2 consist of those roots which are conjugate to an element of Δ_2. The reflections $s_\beta, \beta \in \Delta_2$ commute with the reflections $s_\alpha,\ \alpha \in \Delta_1$, and furthermore

$$s_\beta(\alpha) = \alpha$$

since $(\alpha, \beta) = 0$.

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$$s_\beta(\alpha) = \alpha$$

since $(\alpha, \beta) = 0$. So any element of Φ_1 is conjugate to an element of Δ_1 by an element of the subgroup W_1 generated by the $s_\alpha,\ \alpha \in \Delta_1$. But each such reflection adds or subtracts α. So Φ_1 is in the subspace E_1 of E spanned by Δ_1 and so is orthogonal to all the elements of Φ_2. So if Φ_1 is irreducible so is Δ. QED

We are now into the business of classifying irreducible bases.

The maximal root and the minimal root.

Suppose that Φ is an irreducible root system and Δ a base, so irreducible. Recall that once we have chosen Δ, every root β is an integer combination of the elements of Δ with all coefficients non-negative, or with all coefficients non-positive. We write $\beta \succ 0$ in the first case, and $\beta \prec 0$ in the second case. This defines a partial order on the elements of E by

$$\mu \prec \lambda \text{ if and only if } \lambda - \mu = \sum_{\alpha \in \Delta} k_\alpha \alpha, \tag{1}$$

where the k_α are non-negative integers. This partial order will prove very important to us in representation theory.

Also, for any $\beta = \sum k_\alpha \alpha \in \Phi^+$ we define its **height** by

$$\mathrm{ht}\ \beta = \sum_\alpha k_\alpha. \tag{2}$$

$$\mu \prec \lambda \text{ if and only if } \lambda - \mu = \sum_{\alpha \in \Delta} k_\alpha \alpha, \tag{1}$$

Proposition 3 *Suppose that* Φ *is an irreducible root system and* Δ *a base. Then*

- *There exists a unique* $\beta \in \Phi^+$ *which is maximal relative to the ordering* $\prec$.
- *This* $\beta = \sum k_\alpha \alpha$ *where all the* k_α *are positive.*
- $(\beta, \alpha) \geq 0$ *for all* $\alpha \in \Delta$ *and* $(\beta, \alpha) > 0$ *for at least one* $\alpha \in \Delta$.

Proof. Choose a $\beta = \sum k_\alpha \alpha$ which is maximal relative to the ordering. At least one of the $k_\alpha > 0$. We claim that *all* the $k_\alpha > 0$. Indeed, suppose not. This partitions Δ into Δ_1, the set of α for which $k_\alpha > 0$ and Δ_2, the set of α for which $k_\alpha = 0$. Now the scalar product of any two distinct simple roots is ≤ 0. (Recall that this followed from the fact that if $(\alpha_1, \alpha_2) > 0$, then $s_2(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_2 \rangle \alpha_2$ would be a root whose α_1 coefficient is positive and whose α_2 coefficient is negative which is impossible.) In particular, all the $(\alpha_1, \alpha_2) \leq 0$, $\alpha_1 \in \Delta_1$, $\alpha_2 \in \Delta_2$ and so

$$(\beta, \alpha_2) \leq 0, \quad \forall\, \alpha_2 \in \Delta_2.$$

The irreducibility of Δ implies that $(\alpha_1, \alpha_2) \neq 0$ for at least one pair $\alpha_1 \in \Delta_1$, $\alpha_2 \in \Delta_2$. But this scalar product must then be negative. So $(\beta, \alpha_2) < 0$ and hence $s_{\alpha_2}\beta = \beta - \langle \beta, \alpha_2 \rangle \alpha_2$

is a root with

$$s_{\alpha_2}\beta - \beta \succ 0$$

contradicting the maximality of β. So we have proved that all the k_α are positive.

Furthermore, this same argument shows that $(\beta, \alpha) \geq 0$

for all $\alpha \in \Delta$. Since the elements of Δ form a basis of E, at least one of the scalar products must not vanish, and so be positive. We have established the second and third items in the proposition for any maximal β. We will now show that this maximal weight is unique.

- *This* $\beta = \sum k_\alpha \alpha$ *where all the* k_α *are positive.*
- $(\beta, \alpha) \geq 0$ *for all* $\alpha \in \Delta$ *and* $(\beta, \alpha) > 0$ *for at least one* $\alpha \in \Delta$.

Uniqueness of the maximal root.

Suppose there were two, β and β'. Write $\beta' = \sum k'_\alpha \alpha$ where all the $k'_\alpha > 0$. Then $(\beta, \beta') > 0$ since $(\beta, \alpha) \geq 0$ for all α and > 0 for at least one. Since $s_\beta \beta'$ is a root, this would imply that $\beta - \beta'$ is a root, unless $\beta = \beta'$. But if $\beta - \beta'$ is a root, it is either positive or negative, contradicting the maximality of one or the other. QED

The minimal root.

Let us label the elements of Δ as $\alpha_1, \ldots, \alpha_\ell$, and let us set

$$\alpha_0 := -\beta$$

so that α_0 is the minimal root. From the second and third items in the proposition we know that

$$\alpha_0 + k_1\alpha_1 + \cdots + k_\ell\alpha_\ell = 0 \qquad (3)$$

and that

$$\langle \alpha_0, \alpha_i \rangle \leq 0$$

for all i and < 0 for some i.

The extended matrix.

$$\alpha_0 + k_1\alpha_1 + \cdots + k_\ell\alpha_\ell = 0 \qquad (3)$$

Let us take the left hand side (call it γ) of (3) and successively compute $\langle \gamma, \alpha_i \rangle$, $i = 0, 1, \ldots, \ell$. We obtain

$$\begin{pmatrix} 2 & \langle \alpha_1, \alpha_0 \rangle & \cdots & \langle \alpha_\ell, \alpha_0 \rangle \\ \langle \alpha_0, \alpha_1 \rangle & 2 & \cdots & \langle \alpha_\ell, \alpha_1 \rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle \alpha_0, \alpha_\ell \rangle & \cdots & \langle \alpha_{\ell-1}, \alpha_\ell \rangle & 2 \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \\ \vdots \\ k_\ell \end{pmatrix} = 0.$$

This means that if we write the matrix on the left of this equation as $2I - A$, then A is a matrix with 0 on the diagonal and whose i, j entry is $-\langle \alpha_j, \alpha_i \rangle$.

Properties of the extended matrix.

$$\begin{pmatrix} 2 & \langle\alpha_1,\alpha_0\rangle & \cdots & \langle\alpha_\ell,\alpha_0\rangle \\ \langle\alpha_0,\alpha_1\rangle & 2 & \cdots & \langle\alpha_\ell,\alpha_1\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle\alpha_0,\alpha_\ell\rangle & \cdots & \langle\alpha_{\ell-1},\alpha_\ell\rangle & 2 \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \\ \vdots \\ k_\ell \end{pmatrix} = 0.$$

This means that if we write the matrix on the left of this equation as $2I - A$, then A is a matrix with 0 on the diagonal and whose i, j entry is $-\langle\alpha_j, \alpha_i\rangle$.

So A a non-negative matrix with integer entries with the properties

- if $A_{ij} \neq 0$ then $A_{ji} \neq 0$,
- The diagonal entries of A are 0,
- A is irreducible in the sense that we can not partition the indices into two non-empty subsets I and J such that $A_{ij} = 0 \ \forall\ i \in I,\ j \in J$ and
- A has an eigenvector of eigenvalue 2 with all its entries positive.

Extended diagrams.

This means that A is a matrix corresponding to one of the diagrams in problem set # 5, and furthemore that the minimal root corresponds to a vertex on this diagram with the integer 1 attached. But if we remove such a vertex (leaving a connected graph) from one of the graphs in the second or third figure, we can get the same graph by removing a vertex from a graph in the first figure.

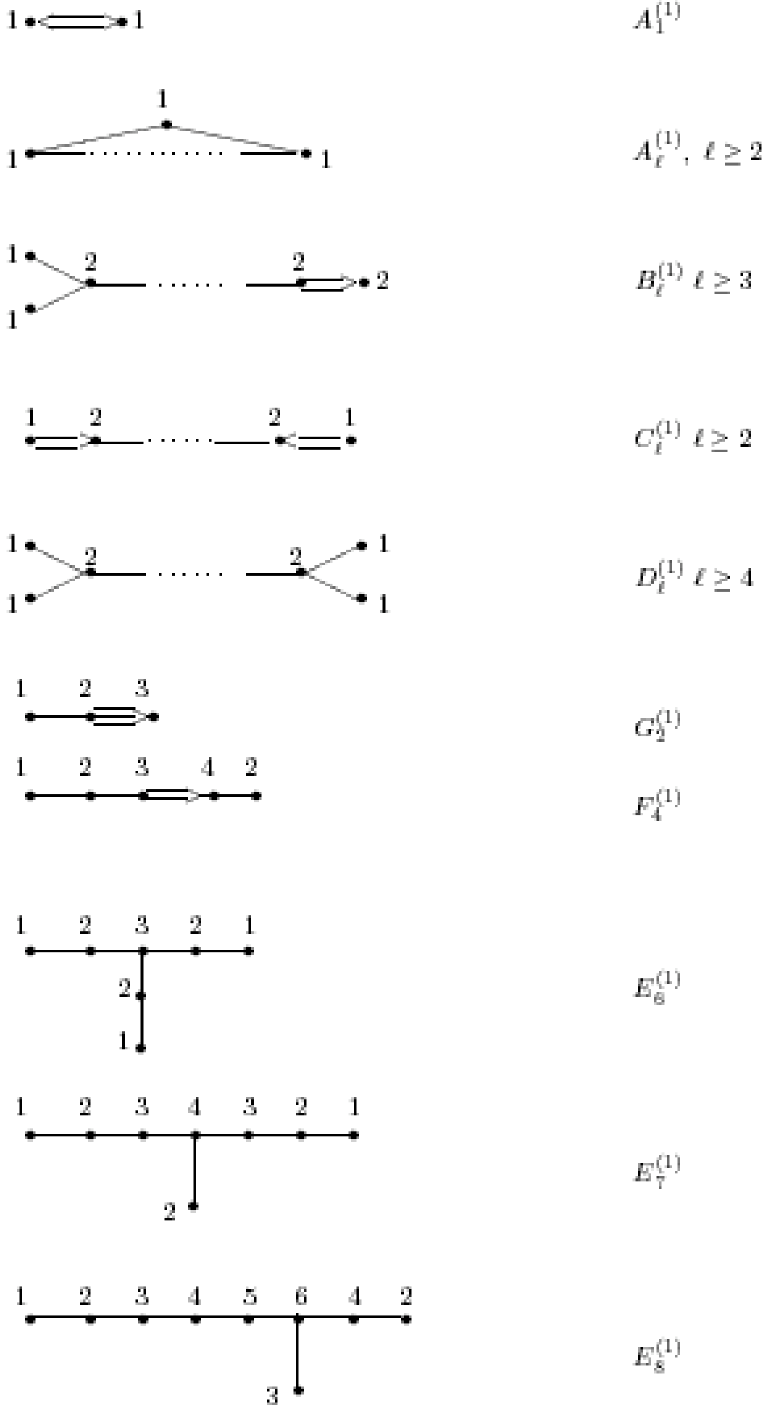


Figure 1: Aff 1.

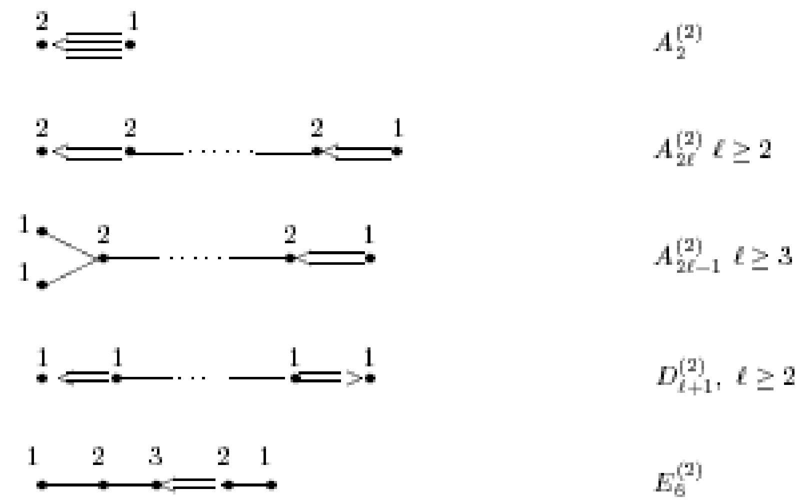


Figure 2: Aff 2

Figure 3: Aff 3

Notice that if we remove a vertex labeled 1 (and the bonds emanating from it) from any of the diagrams in **Aff 2** or **Aff 3** we obtain a diagram which can also be obtained by removing a vertex labeled 1 from one of the diagrams in**Aff 1**. (In the diagram so obtained we ignore the remaining labels.) Indeed, removing the right hand vertex labeled 1 from $D_4^{(3)}$ yields A_2 which is obtained from $A_2^{(1)}$ by removing a vertex. Removing the left vertex marked 1 gives G_2, the diagram obtained from $G_2^{(1)}$ by removing the vertex marked 1.

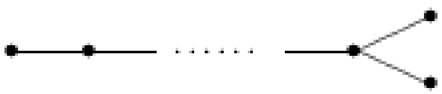
Removing a vertex from $A_2^{(2)}$ gives A_1. Removing the vertex labeled 1 from $A_{2\ell}^{(2)}$ yields $B_{2\ell}$, obtained by removing one of the vertices labeled 1 from $B_\ell^{(1)}$. Removing a vertex labeled 1 from $A_{2\ell-1}^{(2)}$ yields $D_{2\ell}$ or $C_{2\ell}$, removing a vertex labeled 1 from $D_{\ell+1}^{(2)}$ yields $B_{\ell+1}$ and removing a vertex labeled 1 from $E_6^{(2)}$ yields F_4 or C_4.

Thus all irreducible Δ correspond to graphs obtained by removing a vertex labeled 1 from the table**Aff 1**. So we have classified all possible Dynkin diagrams of all irreducible Δ . They are given in the table labeled Dynkin diagrams.

$A_\ell,\ \ell \geq 1$

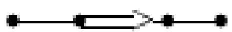
$B_\ell\ \ell \geq 2$

$C_\ell\ \ell \geq 2$

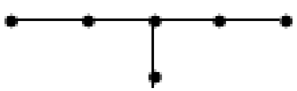


$D_\ell\ \ell \geq 4$

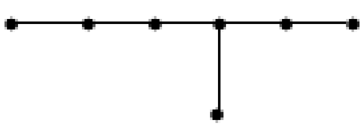
G_2



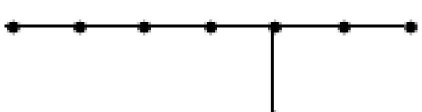
F_4



E_6



E_7



E_8

Dynkin diagrams.

Root systems.

It is useful to introduce here some notation due to Bourbaki: A subset Φ of a Euclidean space E is called a **root system** if the following axioms hold:

- Φ is finite, spans E and does not contain 0.
- If $\alpha \in \Phi$ then the only multiples of α which are in Φ are $\pm\alpha$.
- If $\alpha \in \Phi$ then the reflection s_α in the hyperplane orthogonal to α sends Φ into itself.
- If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbf{Z}$,

Recall that

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

so that the reflection s_α is given by

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

Unbroken chains of roots.

We have shown that each semi-simple Lie algebra gives rise to a root system, and derived properties of the root system. If we go back to the various arguments, we will find that most of them apply to a "general" root system according to the above definition. The one place where we used Lie algebra arguments directly, was in showing that if $\beta \neq \pm\alpha$ is a root then the collection of j such that $\beta + j\alpha$ is a root forms an unbroken chain going from $-r$ to q where $r - q = \langle \beta, \alpha \rangle$. For this we used the representation theory of $sl(2)$. So we now pause to give an alternative proof of this fact based solely on that axioms above, and in the process derive some additional useful information about roots.

For any two non-zero vectors α and β in E, the cosine of the angle between them is given by

$$\|\alpha\|\|\beta\| \cos\theta = (\alpha, \beta).$$

So

$$\langle \beta, \alpha \rangle = 2\frac{\|\beta\|}{\|\alpha\|} \cos\theta.$$

Interchanging the role of α and β and multiplying gives

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta.$$

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4 \cos^2 \theta.$$

The right hand side is a non-negative integer between 0 and 4. So assuming that $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$ The possibilities are listed in the following table:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$0 \leq \theta \leq \pi$	$\|\beta\|^2 / \|\alpha\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-1	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Proposition 4 *If $\alpha \neq \pm\beta$ and if $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.*

Proof. The second assertion follows from the first by replacing β by $-\beta$. So we need to prove the first assertion. From the table, one or the other of $\langle \beta, \alpha \rangle$ or $\langle \alpha, \beta \rangle$ equals one. So either $s_\alpha \beta = \beta - \alpha$ is a root or $s_\beta \alpha = \alpha - \beta$ is a root. But roots occur along with their negatives so in either event $\alpha - \beta$ is a root. QED

Unbroken chains of roots, 2.

Proposition 5 *Suppose that $\alpha \neq \pm\beta$ are roots. Let r be the largest integer such that $\beta - r\alpha$ is a root, and let q be the largest integer such that $\beta + q\alpha$ is a root. Then $\beta + i\alpha$ is a root for all $-r \leq i \leq q$. Furthermore $r - q = \langle \beta, \alpha \rangle$ so in particular $|q - r| \leq 3$.*

Proof. Suppose not. Then we can find a p and an s such that $-r \leq p < s \leq q$ such that $\beta + p\alpha$ is a root, but $\beta + (p+1)\alpha$ is not a root, and $\beta + s\alpha$ is a root

but $\beta + (s-1)\alpha$ is not. The preceding proposition then implies that

$$\langle \beta + p\alpha, \alpha \rangle \geq 0 \quad \text{while} \quad \langle \beta + s\alpha, \alpha \rangle \leq 0$$

which is impossible since $\langle \alpha, \alpha \rangle = 2 > 0$.

This proves the first assertion.

Now s_α adds a multiple of α to any root, and so preserves the string of roots $\beta - r\alpha, \beta - (r-1)\alpha, \ldots, \beta + q\alpha$. Furthermore

$$s_\alpha(\beta + i\alpha) = \beta - (\langle \beta, \alpha \rangle + i)\alpha$$

so s_α reverses the order of the string. In particular

$$s_\alpha(\beta + q\alpha) = b - r\alpha.$$

The left hand side is $\beta - (\langle \beta, \alpha \rangle + q)\alpha$ so $r - q = \langle \beta, \alpha \rangle$ as stated in the proposition.

Long and short roots.

Since every root is conjugate to a simple root, we can use the Dynkin diagrams to conclude that in an irreducible root system, either all roots have the same length (cases A, D, E) or there are two root lengths - the remaining cases. Furthermore, if β denotes a long root and α a short root, the ratios $\|\beta\|^2/\|\alpha\|^2$ are 2 in the cases B, C, and F_4, and 3 for the case G_2.

Irreducibility again.

Proposition 6 *In an irreducible root system, the Weyl group W acts irreducibly on E. In particular, the W-orbit of any root spans E.*

Proof. Let E' be a proper invariant subspace. Let E'' denote its orthogonal complement, so

$$E = E' \oplus E''.$$

For any root α, If $e \in E'$ then $s_\alpha e = e - \langle e, \alpha \rangle \alpha \in E'$. So either $(e, \alpha) = 0$ for all e, and so $\alpha \in E''$ or $\alpha \in E'$. Since the roots span, they can't all belong to the same subspace. This contradicts the irreducibility. QED

Long and short roots,2.

Proposition 7 *If there are two distinct root lengths in an irreducible root system, then all roots of the same length are conjugate under the Weyl group. Also, the maximal weight is long.*

Proof. Suppose that α and β have the same length. We can find a Weyl group element W such that $w\beta$ is not orthogonal to α by the preceding proposition. So we may assume that $\langle \beta, \alpha \rangle \neq 0$. Since α and β have the same length, by the table above we have $\langle \beta, \alpha \rangle = \pm 1$. Replacing β by $-\beta = s_\beta$ we may assume that $\langle \beta, \alpha \rangle = 1$. Then

$$\begin{aligned} (s_\beta s_\alpha s_\beta)(\alpha) &= (s_\beta s_\alpha)(\alpha - \beta) \\ &= s_\beta(-\alpha - \beta + \alpha) \\ &= s_\beta(-\beta) \\ &= \beta. \text{ QED} \end{aligned}$$

Isomorphism of root systems.

Let (E, Φ) and (E', Φ') be two root systems. We say that a linear map $f : E \to E'$ is an **isomorphism** from the root system (E, Φ) to the root system (E', Φ') if f is a linear isomorphism of E onto E' with $f(\Phi) = \Phi'$ and

$$\langle f(\beta), f(\alpha)\rangle = \langle \beta, \alpha \rangle$$

for all $\alpha, \beta \in \Phi$.

Theorem 2 *Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a base of Φ. Suppose that (E', Φ') is a second root system with base $\Delta' = \{\alpha'_1, \ldots, \alpha'_\ell\}$ and that*

$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle, \quad \forall\, 1 \leq i, j \leq \ell.$$

Then the bijection

$$\alpha_i \mapsto \alpha'_i$$

extends to a unique isomorphism $f : (E, \Phi) \to (E', \Phi')$. In other words, the Cartan matrix A of Δ determines Φ up to isomorphism. In particular, The Dynkin diagrams characterize all possible irreducible root systems.

Proof. Since Δ is a basis of E and Δ' is a basis of E', the map $\alpha_i \mapsto \alpha_i'$ extends to a unique linear isomorphism of E onto E'. The equality in the theorem implies that for $\alpha, \beta \in \Delta$ we have

$$s_{f(\alpha)} f(\beta) = f(\beta) - \langle f(\beta), f(\alpha)\rangle f(\alpha) = f(s_\alpha \beta).$$

Since the Weyl groups are generated by these simple reflections, this implies that the map

$$w \mapsto f \circ w \circ f^{-1}$$

is an isomorphism of W onto W'. Every $\beta \in \Phi$ is of the form $w(\alpha)$ where $w \in W$ and α is a simple root. Thus

$$f(\beta) = f \circ w \circ f^{-1} f(\alpha) \in \Phi'$$

so $f(\Phi) = \Phi'$. Since $s_\alpha(\beta) = \beta - \langle \beta, \alpha\rangle \alpha$, the number $\langle \beta, \alpha\rangle$ is determined by the reflection s_α acting on β. But then the corresponding formula for Φ' together with the fact that

$$s_{f(\alpha)} = f \circ s_\alpha \circ f^{-1}$$

implies that

$$\langle f(\beta), f(\alpha)\rangle = \langle \beta, \alpha\rangle.$$

QED

Classification of the simple Lie algebras.

Suppose that $\mathbf{g}, \mathbf{h}$, is a pair consisting of a semi-simple Lie algebra $\mathbf{g}$, and a Cartan subalgebra $\mathbf{h}$. This determines the corresponding Euclidean space E and root system Φ. Suppose we have a second such pair $(\mathbf{g}', \mathbf{h}')$. We would like to show that an isomorphism of (E, Φ) with (E', Φ') determines a Lie algebra isomorphism of $\mathbf{g}$ with $\mathbf{g}'$. This would then imply that the Dynkin diagrams classify all possible simple Lie algebras. We would still be left with the problem of showing that the exceptional Lie algebras exist. We will defer this until Chapter VIII where we prove Serre's theorem with gives a direct construction of all the simple Lie algebras in terms of generators and relations determined by the Cartan matrix.

We need a few preliminaries.

Proposition 8 *Every positive root can be written as a sum of simple roots*

$$\alpha_{i_1} + \cdots \alpha_{i_k}$$

in such a way that every partial sum is again a root.

Preliminaries.

Proposition 8 *Every positive root can be written as a sum of simple roots*

$$\alpha_{i_1} + \cdots \alpha_{i_k}$$

in such a way that every partial sum is again a root.

Proof. By induction (on say the height) it is enough to prove that for every positive root β which is not simple, there is a simple root α such that $\beta - \alpha$ is a root. We can not have $(\beta, \alpha) \leq 0$ for all $\alpha \in \Delta$ for this would imply that the set $\{\beta\} \cup \Delta$ is independent (by the same method that we used to prove that Δ was independent). So $(\beta, \alpha) > 0$ for some $\alpha \in \Delta$ and so $\beta - \alpha$ is a root. Since β is not simple, its height is at least two, and so subtracting α will not be zero or a negative root, hence positive. QED

More preliminaries.

Proposition 9 *Let* $\mathbf{g}, \mathbf{h}$ *be a semi-simple Lie algebra with a choice of Cartan subalgebra, Let* Φ *be the corresponding root system, and let* Δ *be a base. Then* $\mathbf{g}$ *is generated as a Lie algebra by the subspaces* $\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}, \alpha \in \Delta$.

From the representation theory of $sl(2)_\alpha$ we know that $[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{g}_{\alpha+\beta}$ if $\alpha + \beta$ is a root. Thus from the preceding proposition, we can successively obtain all the $\mathbf{g}_\beta$ for β positive by bracketing the $\mathbf{g}_\alpha$, $\alpha \in \Delta$. Similarly we can get all the $\mathbf{g}_\beta$ for β negative from the $\mathbf{g}_{-\alpha}$. So we can get all the root spaces. But $[\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}] = \mathbf{C}h_\alpha$ so we can get all of $\mathbf{h}$. The decomposition

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\gamma \in \Phi} \mathbf{g}_\gamma$$

then shows that we have generated all of $\mathbf{g}$.

Classification!

Theorem 3 *Let* $\mathbf{g}, \mathbf{h}$ *and* $\mathbf{g}', \mathbf{h}'$ *be simple Lie algebras with choices of Cartan subalgebras, and let* Φ, Φ' *be the corresponding root systems. Suppose there is an isomorphism*

$$f : (E, \Phi) \to (E', \Phi')$$

which is an isometry of Euclidean spaces. Extend f *to an isomorphism of*

$$\mathbf{h}^* \to \mathbf{h}'^*$$

via complexification. Let $f : \mathbf{h} \to \mathbf{h}'$ *denote the corresponding isomorphism on the Cartan subalgebras obtained by identifying* $\mathbf{h}$ *and* $\mathbf{h}'$ *with their duals using the Killing form.*

Fix a base Δ *of* Φ *and* Δ' *of* Φ'. *Choose* $0 \neq x_\alpha \in \mathbf{g}_\alpha, \alpha \in \Delta$ *and* $0 \neq x'_{\alpha'} \in \mathbf{g}'_{\alpha'}$. *Extend* f *to a linear map*

$$f : \mathbf{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbf{g}_\alpha \to \mathbf{h}' \oplus \bigoplus_{\alpha' \in \Delta'} \mathbf{g}_{\alpha'}$$

by

$$f(x_\alpha) = x'_{\alpha'}.$$

Then f *extends to a unique isomorphism of* $\mathbf{g} \to \mathbf{g}'$.

Proof. The uniqueness is easy. Given x_α there is a unique $y_\alpha \in \mathbf{g}_{-\alpha}$ for which $[x_\alpha, y_\alpha] = h_\alpha$ so f, if it exists, is determined on the y_α and hence on all of $\mathbf{g}$ since the x_α and y_α generate $\mathbf{g}$ be the preceding proposition.

To prove the existence, we will construct the graph of this isomorphism. That is, we will construct a subalgebra $\mathbf{k}$ of $\mathbf{g} \oplus \mathbf{g}'$ whose projections onto the first and onto the second factor are isomorphisms:

Use the x_α and y_α as above, with the corresponding elements $x'_{\alpha'}$ and $y'_{\alpha'}$ in $\mathbf{g}'$. Let

$$\overline{x}_\alpha := x_\alpha \oplus x'_{\alpha'} \in \mathbf{g} \oplus \mathbf{g}'$$

and similarly define

$$\overline{y}_\alpha := y_\alpha \oplus y'_{\alpha'},$$

and

$$\overline{h_\alpha} := h_\alpha \oplus h'_{\alpha'}.$$

Let β be the (unique) maximal root of $\mathbf{g}$, and choose $x \in \mathbf{g}_\beta$. Make a similar choice of $x' \in \mathbf{g}'_{\beta'}$ where β' is the maximal root of $\mathbf{g}'$. Set

$$\overline{x} := x \oplus x'.$$

$$\overline{x} := x \oplus x'.$$

Let $\mathbf{m} \subset \mathbf{g} \oplus \mathbf{g}'$ be the subspace spanned by all the

$$\mathrm{ad}\,\overline{y}_{\alpha_{i_1}} \cdots \mathrm{ad}\,\overline{y}_{\alpha_{i_m}} \overline{x}.$$

The element $\mathrm{ad}\,\overline{y}_{\alpha_{i_1}} \cdots \mathrm{ad}\,\overline{y}_{\alpha_{i_m}} \overline{x}$ belongs to $\mathbf{g}_{\beta - \sum \alpha_{i_j}} \oplus \mathbf{g}'_{\beta' - \sum \alpha'_{i_j}}$ so

$$\mathbf{m} \cap (\mathbf{g}_{\beta} \oplus \mathbf{g}'_{\beta'}) \ \text{ is one dimensional .}$$

In particular $\mathbf{m}$ is a proper subspace of $\mathbf{g} \oplus \mathbf{g}'$.

Let $\mathbf{k}$ denote the subalgebra of $\mathbf{g} \oplus \mathbf{g}'$ generated by the $\overline{x}_\alpha$ the $\overline{y}_\alpha$ and the $\overline{h}_\alpha$. We claim that

$$[\mathbf{k}, \mathbf{m}] \subset \mathbf{m}.$$

Indeed, it is enough to prove that $\mathbf{m}$ is invariant under the adjoint action of the generators of $\mathbf{k}$. For the $\mathrm{ad}\,\overline{y}_\alpha$ this follows from the definition. For the $\mathrm{ad}\,\overline{h}_\alpha$ we use the fact that

$$[h, y_\alpha] = -\alpha(h) y_\alpha$$

to move the $\mathrm{ad}\,\overline{h}_\alpha$ past all the $\mathrm{ad}\,\overline{y}_\gamma$ at the cost of introducing some scalar multiple, while

$$\mathrm{ad}\,\overline{h}_\alpha \overline{x} = \langle \beta, \alpha \rangle x_\beta + \langle \beta', \alpha' \rangle x'_{\beta'} = \langle \beta, \alpha \rangle \overline{x}$$

because f is an isomorphism of root systems.

Finally $[x_{\alpha_1}, y_{\alpha_2}] = 0$ if $\alpha_1 \neq \alpha_2 \in \Delta$ since $\alpha_1 - \alpha_2$ is not a root. On the other hand $[x_\alpha, y_\alpha] = h_\alpha$. So we can move the $\mathrm{ad}\,\overline{x}_\alpha$ past the $\mathrm{ad}\,\overline{y}_\gamma$ at the expense of introducing an $\mathrm{ad}\,\overline{h}_\alpha$ every time $\gamma = \alpha$. Now $\alpha + \beta$ is not a root, since β is the maximal root. So $[x_\alpha, x_\beta] = 0$. Thus $\mathrm{ad}\,\overline{x}_\alpha \overline{x} = 0$, and we have proved that $[\mathbf{k}, \mathbf{m}] \subset \mathbf{m}$. But since $\mathbf{m}$ is a proper subspace of $\mathbf{g} \oplus \mathbf{g}'$, this implies that $\mathbf{k}$ is a proper subalgebra, since otherwise $\mathbf{m}$ would be a proper ideal, and the only proper ideals in $\mathbf{g} \oplus \mathbf{g}'$ are $\mathbf{g}$ and $\mathbf{g}'$.

Now the subalgebra $\mathbf{k}$ can not contain any element of the form $z \oplus 0, z \neq 0$, for it if did, it would have to contain all of the elements of the form $u \oplus 0$ since we could repeatedly apply $\mathrm{ad}\,x_\alpha$'s until we reached the maximal root space and then get all of $\mathbf{g} \oplus 0$, which would mean that $\mathbf{k}$ would also contain all of $0 \oplus \mathbf{g}'$ and hence all of $\mathbf{g} \oplus \mathbf{g}'$ which we know not to be the case. Similarly $\mathbf{k}$ can not contain any element of the form $0 \oplus z'$. So the projections of $\mathbf{k}$ onto $\mathbf{g}$ and onto $\mathbf{g}'$ are linear isomorphisms. By construction they are Lie algebra homomorphisms. Hence the inverse of the projection of $\mathbf{k}$ onto $\mathbf{g}$ followed by the projection of $\mathbf{k}$ onto $\mathbf{g}'$ is a Lie algebra isomorphism of $\mathbf{g}$ onto $\mathbf{g}'$. By construction it sends x_α to $x'_{\alpha'}$ and h_α to $h_{\alpha'}$ and so is an extension of f. QED