Math 128 Lecture 12

Conjugacy of Borel subalgebras of a semi-simple Lie algebra, part 2.

Review: Roots

The set of $\alpha \in \mathbf{h}^*$, $\alpha \neq 0$ for which $\mathbf{g}_\alpha \neq 0$ is called the set of **roots** and is denoted by Φ. We have

- Φ spans $\mathbf{h}^*$ for otherwise $\exists h \neq 0 : \alpha(h) = 0 \ \forall \alpha \in \Phi$ implying that $[h, \mathbf{g}_\alpha] = 0 \ \forall \alpha$ so $[h, \mathbf{g}] = 0$.
- $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$ for otherwise $\mathbf{g}_\alpha \perp \mathbf{g}$.
- $x \in \mathbf{g}_\alpha, y \in \mathbf{g}_{-\alpha}, \alpha \in \Phi \Rightarrow [x, y] = \kappa(x, y) t_\alpha$.
- $[\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}]$ is one dimensional with basis t_α.
- $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.
- Choose $e_\alpha \in \mathbf{g}_\alpha, f_\alpha \in \mathbf{g}_{-\alpha}$ with $\kappa(e_\alpha, f_\alpha) = \dfrac{2}{\kappa(t_\alpha, t_\alpha)}$.

  Set

$$h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha.$$

  Then $e_\alpha, f_\alpha, h_\alpha$ span a subalgebra isomorphic to $sl(2)$.

- Consider the action of $sl(2)_\alpha$ on the subalgebra $\mathbf{m} := \mathbf{h} \oplus \bigoplus \mathbf{g}_{n\alpha}$ where $n \in \mathbf{Z}$. The zero eigenvectors of h_α consist of $\mathbf{h} \subset \mathbf{m}$. One of these corresponds to the adjoint representation of $sl(2)_\alpha \subset \mathbf{h}$. The orthocomplement of $h_\alpha \in \mathbf{h}$ gives dim $\mathbf{h} - 1$ trivial representations of $sl(2)_\alpha$. This must exhaust all the even maximal weight representations, as we have accounted for all the zero weights of $sl(2)_\alpha$ acting on $\mathbf{g}$. In particular, dim $\mathbf{g}_\alpha = 1$ and no integer multiple of α other than $-\alpha$ is a root. Now consider the subalgebra $\mathbf{p} := \mathbf{h} \oplus \bigoplus \mathbf{g}_{c\alpha}$, $c \in \mathbf{C}$. This is a module for $sl(2)_\alpha$. Hence all such c's must be multiples of $1/2$. But $1/2$ can not occur, since the double of a root is not a root. Hence the $\pm\alpha$ are the only multiples of α which are roots. consider $\beta \in \Phi$, $\beta \neq \pm\alpha$.

Let q be the maximal integer so that $\beta + q\alpha \in \Phi$, and r the maximal integer so that $\beta - r\alpha \in \Phi$. Then the entire string

$$\beta - r\alpha, \beta - (r-1)\alpha, \ldots \beta + q\alpha$$

are roots, and

$$\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q) \quad \text{so} \quad \beta(h_\alpha) = r - q \in \mathbf{Z}.$$

These integers are called the **Cartan integers**.

Review: the space E.

We can transfer the bilinear form κ from $\mathbf{h}$ to $\mathbf{h}^*$ by defining

$$(\gamma, \delta) = \kappa(t_\gamma, t_\delta).$$

So

$$\begin{aligned} \beta(h_\alpha) &= \kappa(t_\beta, h_\alpha) \\ &= \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} \\ &= \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \end{aligned} \quad \text{So} \quad \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = r - q \in \mathbf{Z}.$$

Let E be the *real* vector space spanned by the $\alpha \in \Phi$. Then (,) restricts to a real scalar product on E. Also, for any $\lambda \neq 0 \in E$,

$$(\lambda, \lambda) > 0.$$

So the scalar product (,) on E is positive definite. E is a Euclidean space.

Review: reflection in a hyperplane perpendicular to a root.

In the string of roots, β is q steps down from the top, so q steps up from the bottom is also a root, so

$$\beta - (r-q)\alpha$$

is a root, or

$$\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi.$$

But

$$\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = s_\alpha(\beta)$$

where s_α denotes Euclidean reflection in the hyperplane perpendicular to α. In other words, for every $\alpha \in \Phi$

$$s_\alpha : \Phi \to \Phi. \tag{6}$$

Review: the Weyl group.

The subgroup of the orthogonal group of E generated by these reflections is called the **Weyl group** and is denoted by W. We have thus associated to every semi-simple Lie algebra, and to every choice of Cartan subalgebra a finite subgroup of the orthogonal group generated by reflections. (This subgroup is finite, because all the generating reflections, s_α, and hence the group they generate, preserve the finite set of all roots, which span the space.) Once we will have completed the proof of the conjugacy theorem for Cartan subalgebras of a semi-simple algebra, then we will know that the Weyl group is determined, up to isomorphism, by the semi-simple algebra, and does not depend on the choice of Cartan subalgebra.

We define

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

So

$$\begin{aligned} \langle \beta, \alpha \rangle &= \beta(h_\alpha) \\ &= r - q \in \mathbf{Z} \end{aligned}$$

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle.$$

Review: Bases

$\Delta \subset \Phi$ is a called a **Base** if it is a basis of E (so $\#\Delta = \ell = \dim_{\mathbf{R}} E = \dim_{\mathbf{C}} \mathbf{h}$) and every $\beta \in \Phi$ can be written as $\sum_{\alpha \in \Delta} k_\alpha \alpha,\ k_\alpha \in \mathbf{Z}$ with either all the coefficients $k_\alpha \geq 0$ or all ≤ 0. Roots are accordingly called positive or negative and we define the height of a root by

$$\text{ht } \beta := \sum_{\alpha} k_\alpha.$$

Given a base, we get partial order on E by defining $\lambda \succ \mu$ iff $\lambda - \mu$ is a sum of positive roots or zero. We have

$$(\alpha, \beta) \leq 0, \quad \alpha, \beta \in \Delta \tag{12}$$

Review: construction of bases.

To construct a base, choose a $\gamma \in E, | (\gamma, \beta) \neq 0 \; \forall \; \beta \in \Phi$. Such an element is called **regular**. Then every root has positive or negative scalar product with γ, dividing the set of roots into two subsets:

$$\Phi = \Phi^+ \cup \Phi^-, \quad \Phi^- = -\Phi^+.$$

A root $\beta \in \Phi^+$ is called **decomposable** if $\beta = \beta_1 + \beta_2, \beta_1, \beta_2 \in \Phi^+$, indecomposable otherwise. Let $\Delta(\gamma)$ consist of the indecomposable elements of $\Phi^+(\gamma)$.

Theorem 3 $\Delta(\gamma)$ *is a base, and every base is of the form* $\Delta(\gamma)$ *for some* γ.

Review: Weyl chambers.

Define $P_\beta := \beta^\perp$. Then $E - \bigcup P_\beta$ is the union of **Weyl chambers** each consisting of regular γ's with the same Φ^+. So the Weyl chambers are in one to one correspondence with the bases, and the Weyl group permutes them.

Fix a base, Δ. Our goal in this section is to prove that the reflections s_α, $\alpha \in \Delta$ generate the Weyl group, W, and that W acts simply transitively on the Weyl chambers.

Each s_α, $\alpha \in \Delta$ sends $\alpha \mapsto -\alpha$. But acting on $\lambda = \sum c_\beta \beta$, the reflection s_α does not change the coefficient of any other element of the base. If $\lambda \in \Phi^+$ and $\lambda \neq \alpha$, we must have $c_\beta > 0$ for some $\beta \neq \alpha$ in the base Δ. Then the coefficient of β in the expansion of $s_\alpha(\lambda)$ is positive, and hence all its coefficients must be non-negative. So $s_\alpha(\lambda) \in \Phi^+$. In short, the only element of Φ^+ sent into Φ^- is α. So if

$$\delta := \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \text{ then } s_\alpha \delta = \delta - \alpha.$$

A fact about reflections.

Let γ be any vector in a Euclidean space, and let s_γ denote reflection in the hyperplane orthogonal to γ. Let R be any orthogonal transformation. Then

$$s_{R\gamma} = R s_\gamma R^{-1} \tag{13}$$

as follows immediately from the definition.

Deletion and exchange.

$$s_{R\gamma} = R s_\gamma R^{-1} \tag{13}$$

Let $\alpha_1, \ldots, \alpha_i \in \Delta$, and, for short, let us write $s_i := s_{\alpha_i}$.

Lemma 10 *If* $s_1 \cdots s_{i-1}\alpha_i < 0$ *then* $\exists j < i, j \geq 1$ *so that*

$$s_1 \cdots s_i = s_1 \cdots s_{j-1} s_{j+1} \cdots s_{i-1}.$$

Proof. Set $\beta_{i-1} := \alpha_i$, $\beta_j := s_{j+1} \cdots s_{i-1}\alpha_i$, $j < i-1$. Since $\beta_{i-1} \in \Phi^+$ and $\beta_0 \in \Phi^-$ there must be some j for which $\beta_j \in \Phi^+$ and $s_j\beta_j = \beta_{j-1} \in \Phi^-$ implying that that $\beta_j = \alpha_j$ so by (13) with $R = s_{j+1} \cdots s_{i-1}$ we conclude that

$$s_j = (s_{j+1} \cdots s_{i-1}) s_i (s_{j+1} \cdots s_{i-1})^{-1}$$

or

$$s_j s_{j+1} \cdots s_i = s_{j+1} \cdots s_{i-1}$$

implying the lemma. QED

Simple reflections.

Lemma 10 *If* $s_1 \cdots s_{i-1}\alpha_i < 0$ *then* $\exists j < i, j \geq 1$ *so that*

$$s_1 \cdots s_i = s_1 \cdots s_{j-1} s_{j+1} \cdots s_{i-1}.$$

As a consequence, if $s = s_1 \cdots s_t$ is a shortest expression for s, then, since $s_t\alpha_t \in \Phi^-$, we must have $s\alpha_t \in \Phi^-$.

Keeping Δ fixed in the ensuing discussion, we will call the elments of Δ **simple** roots, and the corresponding reflections **simple** reflections. Let W' denote the subgroup of W generated by the simple reflections, $s_\alpha, \alpha \in \Delta$. (Eventually we will prove that this is all of W.) It now follows that if $s \in W'$ and $s\Delta = \Delta$ then $s = id$. Indeed, if $s \neq id$, write s in a minimal fashion as a product of simple reflections. By what we have just proved, it must send some simple root into a negative root. So W' permutes the Weyl chambers without fixed points. We now show that W' acts transitively on the Weyl chambers:

W' acts transitively on the Weyl chambers.

Let $\gamma \in E$ be a regular element. We claim

$$\exists \;\; s \in W' \text{ with } (s(\gamma), \alpha) > 0 \; \forall \; \alpha \in \Delta.$$

Indeed, choose $s \in W'$ so that $(s(\gamma), \delta)$ is as large as possible. Then

$$\begin{aligned} (s(\gamma), \delta) &\geq (s_\alpha s(\gamma), \delta) \\ &= (s(\gamma), s_\alpha \delta) \\ &= (s(\gamma), \delta) - (s(\gamma), \alpha) \text{ so} \\ (s(\gamma), \alpha) &\geq 0 \;\; \forall \; \alpha \in \Delta. \end{aligned}$$

We can't have equality in this last inequality since $s(\gamma)$ is not orthogonal to any root. This proves that W' acts transitively on all Weyl chambers and hence on all bases.

W'=W.

We next claim that every root belongs to at least one base. Choose a (non-regular) $\gamma' \perp \alpha$, but $\gamma' \notin P_\beta$, $\beta \neq \alpha$. Then choose γ close enough to γ' so that $(\gamma, \alpha) > 0$ and $(\gamma, \alpha) < |(\gamma, \beta)| \ \forall \ \beta \neq \alpha$. Then in $\Phi^+(\gamma)$ the element α must be indecomposable. If β is any root, we have shown that there is an $s' \in W'$ with $s'\beta = \alpha_i \in \Delta$. By (13) this implies that every reflection $s_\beta$in W is conjugate by an element of W' to a simple reflection: $s_\beta = s' s_i {s'}^{-1} \in W'$. Since W is generated by the s_β, this shows that $W' = W$.

$$s_{R\gamma} = R s_\gamma R^{-1} \qquad (13)$$

Length.

Define the length of an element of W as the minimal word length in its expression as a product of simple roots. Define $n(s)$ to be the number of positive roots made negative by s. We know that $n(s) = \ell(s)$ if $\ell(s) = 0$ or 1. We claim that

$$\ell(s) = n(s)$$

in general.

Proof by induction on $\ell(s)$. Write $s = s_1 \cdots s_i$ in reduced form and let $\alpha = \alpha_i$. We have $s\alpha \in \Phi^-$. Then $n(ss_i) = n(s) - 1$ since s_i leaves all positive roots positive except α. Also $\ell(ss_i) = \ell(s) - 1$. So apply induction. QED

The closures of the Weyl chambers.

Let $C = C(\Delta)$ be the Weyl chamber associated to the base Δ. Let $\overline{C}$ denote its closure.

Lemma 11 *If $\lambda, \mu \in \overline{C}$ and $s \in W$ satisfies $s\lambda = \mu$ then s is a product of simple reflections which fix λ. In particular, $\lambda = \mu$. So $\overline{C}$ is a fundamental domain for the action of W on E.*

Proof. By induction on $\ell(s)$. If $\ell(s) = 0$ then $s = id$ and the assertion is clear with the empty product. So we may assume that $n(s) > 0$, so s sends some positive root to a negative root, and hence must send some simple root to a negative root. So let $\alpha \in \Delta$ be such that $s\alpha \in \Phi^-$. Since $\mu \in \overline{C}$, we have $(\mu, \beta) \geq 0$, $\forall \beta \in \Phi^+$ and hence $(\mu, s\alpha) \leq 0$.

So

$$\begin{aligned} 0 &\geq (\mu, s\alpha) \\ &= (s^{-1}\mu, \alpha) \\ &= (\lambda, \alpha) \\ &\geq 0. \end{aligned}$$

So $(\lambda, \alpha) = 0$ so $s_\alpha \lambda = \lambda$ and hence $ss_\alpha \lambda = \mu$. But $n(ss_\alpha) = n(s) - 1$ since $s_\alpha = -\alpha$ and s_α permutes all the other positive roots. So $\ell(ss_\alpha) = \ell(s) - 1$ and we can apply induction to conclude that $s = (ss_\alpha)s_\alpha$ is a product of simple reflections which fix λ.

Standard Borel subalgebras.

Define a **standard** Borel subalgebra (relative to a choice of **CSA** $\mathbf{h}$ and a system of simple roots, Δ) to be

$$\mathbf{b}(\Delta) := \mathbf{h} \oplus \bigoplus_{\beta \in \Phi^+(\Delta)} \mathbf{g}_\beta.$$

Define the corresponding nilpotent Lie algebra by

$$\mathbf{n}_+(\Delta) := \bigoplus_{\beta \in \Phi^+} \mathbf{g}_\beta.$$

Since each s_α can be realized as $(\exp e_\alpha)(\exp -f_\alpha)(\exp e_\alpha)$ every element of W can be realized as an element of $\mathcal{E}(\mathbf{g})$. Hence all standard Borel subalgebras relative to a given Cartan subalgebra are conjugate.

A **BSA** is its own normalizer.

Notice that if x normalizes a Borel subalgebra, $\mathbf{b}$, then

$$[\mathbf{b} + \mathbf{C}x, \mathbf{b} + \mathbf{C}x] \subset \mathbf{b}$$

and so $\mathbf{b} + \mathbf{C}x$ is a solvable subalgebra containing $\mathbf{b}$ and hence must coincide with $\mathbf{b}$:

$$N_{\mathbf{g}}(\mathbf{b}) = \mathbf{b}.$$

In particular, if $x \in \mathbf{b}$ then its semi-simple and nilpotent parts lie in $\mathbf{b}$.

Goal and strategy.

From now on, fix a standard **BSA**, $\mathbf{b}$. We want to prove that any other **BSA**, $\mathbf{b}'$ is conjugate to $\mathbf{b}$. We may assume that the theorem is known for Lie algebras of smaller dimension, or for $\mathbf{b}'$ with $\mathbf{b} \cap \mathbf{b}'$

of greater dimension, since if $\dim\, \mathbf{b} \cap \mathbf{b}' = \dim\, \mathbf{b}$, so that $\mathbf{b}' \supset \mathbf{b}$, we must have $\mathbf{b}' = \mathbf{b}$ by maximality. Therefore we can proceed by downward induction on the dimension of the intersection $\mathbf{b} \cap \mathbf{b}'$.

Suppose $\mathbf{b} \cap \mathbf{b}' \neq 0$. Let $\mathbf{n}'$ be the set of nilpotent elements in $\mathbf{b} \cap \mathbf{b}'$. So $\mathbf{n}' = \mathbf{n}^+ \cap \mathbf{b}'$.

Also $[\mathbf{b} \cap \mathbf{b}', \mathbf{b} \cap \mathbf{b}'] \subset \mathbf{n}^+ \cap \mathbf{b}' = \mathbf{n}'$ so $\mathbf{n}'$ is a nilpotent ideal in $\mathbf{b} \cap \mathbf{b}'$. Suppose that $\mathbf{n}' \neq 0$. Then since $\mathbf{g}$ contains no solvable ideals,

$$\mathbf{k} := N_{\mathbf{g}}(\mathbf{n}') \neq \mathbf{g}.$$

Consider the action of $\mathbf{n}'$ on $\mathbf{b}/(\mathbf{b} \cap \mathbf{b}')$. By Engel, there exists a $y \notin \mathbf{b} \cap \mathbf{b}'$ with $[x, y] \in \mathbf{b} \cap \mathbf{b}'$ $\forall x \in \mathbf{n}'$. But $[x, y] \in [\mathbf{b}, \mathbf{b}] \subset \mathbf{n}^+$ and so $[x, y] \in \mathbf{n}'$. So $y \in \mathbf{k}$. Thus $y \in \mathbf{k} \cap \mathbf{b}$, $y \notin \mathbf{b} \cap \mathbf{b}'$. Similarly, we can interchange the roles of $\mathbf{b}$ and $\mathbf{b}'$ in the above argument, replacing $\mathbf{n}^+$ by the nilpotent subalgebra $[\mathbf{b}', \mathbf{b}']$ of $\mathbf{b}'$, to conclude that there exists a $y' \in \mathbf{k} \cap \mathbf{b}'$, $y' \notin \mathbf{b} \cap \mathbf{b}'$. In other words, the inclusions

$$\mathbf{k} \cap \mathbf{b} \supset \mathbf{b} \cap \mathbf{b}', \quad \mathbf{k} \cap \mathbf{b}' \supset \mathbf{b} \cap \mathbf{b}'$$

are strict.

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Both $\mathbf{b} \cap \mathbf{k}$ and $\mathbf{b}' \cap \mathbf{k}$ are solvable subalgebras of $\mathbf{k}$. Let $\mathbf{c}, \mathbf{c}'$ be **BSA**'s containing them. By induction, there is a $\sigma \in \mathcal{E}(\mathbf{k}) \subset \mathcal{E}(\mathbf{g})$ with $\sigma(\mathbf{c}') = \mathbf{c}$. Now let $\mathbf{b}''$ be a **BSA** containing $\mathbf{c}$. We have

$$\mathbf{b}'' \cap \mathbf{b} \supset \mathbf{c} \cap \mathbf{b} \supset \mathbf{k} \cap \mathbf{b} \supset \mathbf{b}' \cap \mathbf{b}$$

with the last inclusion strict. So by induction there is a $\tau \in \mathcal{E}(\mathbf{g})$ with $\tau(\mathbf{b}'') = \mathbf{b}$. Hence

$$\tau\sigma(\mathbf{c}') \subset \mathbf{b}.$$

Then

$$\mathbf{b} \cap \tau\sigma(\mathbf{b}') \supset \tau\sigma(\mathbf{c}') \cap \tau\sigma(\mathbf{b}') \supset \tau\sigma(\mathbf{b}' \cap \mathbf{k}) \supset \tau\sigma(\mathbf{b} \cap \mathbf{b}')$$

with the last inclusion strict. So by induction we can further conjugate $\tau\sigma\mathbf{b}'$ into $\mathbf{b}$.

So we must now deal with the case that $\mathbf{n}' = 0$, but we will still assume that $\mathbf{b} \cap \mathbf{b}' \neq 0$. Since any Borel subalgebra contains both the semi-simple and nilpotent parts of any of its elements, we conclude that $\mathbf{b} \cap \mathbf{b}'$ consists entirely of semi-simple elements, and so is a toral subalgebra, call it $\mathbf{t}$. If $x \in \mathbf{b}, t \in \mathbf{t} = \mathbf{b} \cap \mathbf{b}'$ and $[x, t] \in \mathbf{t}$, then we must have $[x, t] = 0$, since all elements of $[\mathbf{b}, \mathbf{b}]$ are nilpotent. So

$$N_{\mathbf{b}}(\mathbf{t}) = C_{\mathbf{b}}(\mathbf{t}).$$

Let $\mathbf{c}$ be a **CSA** of $C_{\mathbf{b}}(\mathbf{t})$. Since a Cartan subalgebra is its own normalizer, we have $\mathbf{t} \subset \mathbf{c}$. So we have

$$\mathbf{t} \subset \mathbf{c} \subset C_{\mathbf{b}}(\mathbf{t}) = N_{\mathbf{b}}(\mathbf{t}) \subset N_{\mathbf{b}}(\mathbf{c}).$$

Let $t \in \mathbf{t}$, $n \in N_{\mathbf{b}}(\mathbf{c})$. Then $[t, n] \in \mathbf{c}$ and successive brackets by t will eventually yield 0, since $\mathbf{c}$ is nilpotent. Thus $(\operatorname{ad} t)^k n = 0$ for some k, and since t is semi-simple, $[t, n] = 0$. Thus $n \in C_{\mathbf{b}}(\mathbf{t})$ and hence $n \in \mathbf{c}$ since $\mathbf{c}$ is its own normalizer in $C_{\mathbf{b}}(\mathbf{t})$. Thus $\mathbf{c}$ is a **CSA** of $\mathbf{b}$. We can now apply the conjugacy theorem for **CSA**'s of solvable algebras to conjugate $\mathbf{c}$ into $\mathbf{h}$.

So we may assume from now on that $\mathbf{t} \subset \mathbf{h}$. If $\mathbf{t} = \mathbf{h}$, then decomposing $\mathbf{b}'$ into root spaces under $\mathbf{h}$, we find that the non-zero root spaces must consist entirely of negative roots, and there must be at least one such, since $\mathbf{b}' \neq \mathbf{h}$. But then we can find a τ_α which conjugates this into a positive root, preserving $\mathbf{h}$, and then $\tau_\alpha(\mathbf{b}') \cap \mathbf{b}$ has larger dimension and we can further conjugate into $\mathbf{b}$.

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If

$$\mathbf{b}' \subset C_{\mathbf{g}}(\mathbf{t})$$

then since we also have $\mathbf{h} \subset C_{\mathbf{g}}(\mathbf{t})$, we can find a **BSA**, $\mathbf{b}''$ of $C_{\mathbf{g}}(\mathbf{t})$ containing $\mathbf{h}$, and conjugate $\mathbf{b}'$ to $\mathbf{b}''$, since we are assuming that $\mathbf{t} \neq 0$ and hence $C_{\mathbf{g}}(\mathbf{t}) \neq \mathbf{g}$. Since $\mathbf{b}'' \cap \mathbf{b} \supset \mathbf{h}$ has bigger dimension than $\mathbf{b}' \cap \mathbf{b}$, we can further conjugate to $\mathbf{b}$ by the induction hypothesis.

So we may assume that

$$\mathbf{t} \subset \mathbf{h} \qquad\qquad \mathbf{b}' \not\subset C_{\mathbf{g}}(\mathbf{t})$$

is strict.

then there is a common non-zero eigenvector for ad t in $\mathbf{b}'$, call it x. So there is a $t' \in \mathbf{t}$ such that $[t', x] = c'x,\ c' \neq 0$. Setting

$$t := \frac{1}{c'}t'$$

we have $[t, x] = x$. Let $\Phi_t \subset \Phi$ consist of those roots for which $\beta(t)$ is a positive rational number. Then

$$\mathbf{s} := \mathbf{h} \oplus \bigoplus_{\beta \in \Phi_t} \mathbf{g}_\beta$$

is a solvable subalgebra and so lies in a **BSA**, call it $\mathbf{b}''$. Since $\mathbf{t} \subset \mathbf{b}'', x \in \mathbf{b}''$ we see that $\mathbf{b}'' \cap \mathbf{b}'$ has strictly larger dimension than $\mathbf{b} \cap \mathbf{b}'$. Also $\mathbf{b}'' \cap \mathbf{b}$ has strictly larger dimension than $\mathbf{b} \cap \mathbf{b}'$ since $\mathbf{h} \subset \mathbf{b} \cap \mathbf{b}''$. So we can conjugate $\mathbf{b}'$ to $\mathbf{b}''$ and then $\mathbf{b}''$ to $\mathbf{b}$.

Conclusion of proof.

This leaves only the case $\mathbf{b} \cap \mathbf{b}' = 0$ which we will show is impossible. Let $\mathbf{t}$ be a maximal toral subalgebra of $\mathbf{b}'$. We can not have $\mathbf{t} = 0$, for then $\mathbf{b}'$ would consist entirely of nilpotent elements, hence nilpotent by Engel, and also self-normalizing as is every **BSA**. Hence it would be a **CSA** which is impossible since every **CSA** in a semi-simple Lie algebra is toral. So choose a **CSA**, $\mathbf{h}''$ containing $\mathbf{t}$, and then a standard **BSA** containing $\mathbf{h}''$. By the preceding, we know that $\mathbf{b}'$ is conjugate to $\mathbf{b}''$ and, in particular has the same dimension as $\mathbf{b}''$. But the dimension of each standard **BSA** (relative to any Cartan subalgebra) is strictly greater than half the dimension of $\mathbf{g}$, contradicting the hypothesis $\mathbf{g} \supset \mathbf{b} \oplus \mathbf{b}'$. QED