Math 128 Lecture 11

Conjugacy of Borel subalgebras of a semi-simple Lie algebra, part 1.

Roots.

Until further notice in this lecture **g** is semi-simple and **h** is a Cartan subalgebra, so we have the decomposition

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha \neq 0} \mathbf{g}_\alpha$$

We have proved that the restriction of κ to $\mathbf{h}$ is non-degenerate. This allows us to associate to every linear function ϕ on $\mathbf{h}$ the unique element $t_\phi \in \mathbf{h}$ given by

$$\phi(h) = \kappa(t_\phi, h).$$

The set of $\alpha \in \mathbf{h}^*$, $\alpha \neq 0$ for which $\mathbf{g}_\alpha \neq 0$ is called the set of **roots** and is denoted by Φ. We have

- Φ spans $\mathbf{h}^*$ for otherwise $\exists h \neq 0 : \alpha(h) = 0\ \forall \alpha \in \Phi$ implying that $[h, \mathbf{g}_\alpha] = 0\ \forall \alpha$ so $[h, \mathbf{g}] = 0$.

More facts about roots.

- $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$ for otherwise $\mathbf{g}_\alpha \perp \mathbf{g}$.
- $x \in \mathbf{g}_\alpha, y \in \mathbf{g}_{-\alpha}, \alpha \in \Phi \Rightarrow [x,y] = \kappa(x,y)t_\alpha$. Indeed,

$$\begin{aligned} \kappa(h,[x,y]) &= \kappa([h,x],y) \\ &= \kappa(t_\alpha,h)\kappa(x,y) \\ &= \kappa(\kappa(x,y)t_\alpha,h). \end{aligned}$$

- $[\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}]$ is one dimensional with basis t_α. This follows from the preceding and the fact that $\mathbf{g}_\alpha$ can not be perpendicular to $\mathbf{g}_{-\alpha}$ since otherwise it will be orthogonal to all of $\mathbf{g}$.

More facts about roots.

- $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$. Otherwise, choosing $x \in \mathbf{g}_\alpha, y \in \mathbf{g}_{-\alpha}$ with $\kappa(x, y) = 1$, we get

  $$[x, y] = t_\alpha, [t_\alpha, x] = [t_\alpha, y] = 0.$$

  So x, y, t_α span a solvable three dimensional algebra. Acting as ad on $\mathbf{g}$, it is superdiagonizable, by Lie's theorem, and hence ad t_α, which is in the commutator algebra of this subalgebra is nilpotent. Since it is ad semi-simple by definition of $\mathbf{h}$, it must lie in the center, which is impossible.

sl(2) subalgebras.

- Choose $e_\alpha \in \mathbf{g}_\alpha, f_\alpha \in \mathbf{g}_{-\alpha}$ with

  $$\kappa(e_\alpha, f_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}.$$

  Set

  $$h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha.$$

  Then $e_\alpha, f_\alpha, h_\alpha$ span a subalgebra isomorphic to $sl(2)$. Call it $sl(2)_\alpha$. We shall soon see that this notation is justified, i.e that $\mathbf{g}_\alpha$ is one dimensional and hence that $sl(2)_\alpha$ is well defined, independent of any "choices" of e_α, f_α but depends only on α.

The only multiples of a root which are roots are +(the root) or -(the root).

- Consider the action of $sl(2)_\alpha$ on the subalgebra $\mathbf{m} := \mathbf{h} \oplus \bigoplus \mathbf{g}_{n\alpha}$ where $n \in \mathbf{Z}$. The zero eigenvectors of h_α consist of $\mathbf{h} \subset \mathbf{m}$. One of these corresponds to the adjoint representation of $sl(2)_\alpha \subset \mathbf{h}$. The orthocomplement of $h_\alpha \in \mathbf{h}$ gives dim $\mathbf{h} - 1$ trivial representations of $sl(2)_\alpha$. This must exhaust all the even maximal weight representations, as we have accounted for all the zero weights of $sl(2)_\alpha$ acting on $\mathbf{g}$. In particular, dim $\mathbf{g}_\alpha = 1$ and no integer multiple of α other than $-\alpha$ is a root. Now consider the subalgebra $\mathbf{p} := \mathbf{h} \oplus \bigoplus \mathbf{g}_{c\alpha}$, $c \in \mathbf{C}$. This is a module for $sl(2)_\alpha$. Hence all such c's must be multiples of $1/2$. But $1/2$ can not occur, since the double of a root is not a root. Hence the $\pm\alpha$ are the only multiples of α which are roots.

Strings of roots.

Now consider $\beta \in \Phi,\ \beta \neq \pm\alpha$. Let

$$\mathbf{k} := \bigoplus \mathbf{g}_{\beta + j\alpha}.$$

Each non-zero summand is one dimensional, and $\mathbf{k}$ is an $sl(2)_\alpha$ module. Also $\beta + i\alpha \neq 0$ for any i, and evaluation on h_α gives $\beta(h_\alpha) + 2i$. All weights differ by multiples of 2 and so $\mathbf{k}$ is irreducible. Let q be the maximal integer so that $\beta + q\alpha \in \Phi$, and r the maximal integer so that $\beta - r\alpha \in \Phi$. Then the entire string

$$\beta - r\alpha, \beta - (r-1)\alpha, \ldots \beta + q\alpha$$

are roots, and

$$\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$$

The Cartan integers.

$$h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha.$$

$$\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$$

or

$$\beta(h_\alpha) = r - q \in \mathbf{Z}.$$

These integers are called the **Cartan integers.**

We can transfer the bilinear form κ from $\mathbf{h}$ to $\mathbf{h}^*$ by defining

$$(\gamma, \delta) = \kappa(t_\gamma, t_\delta).$$

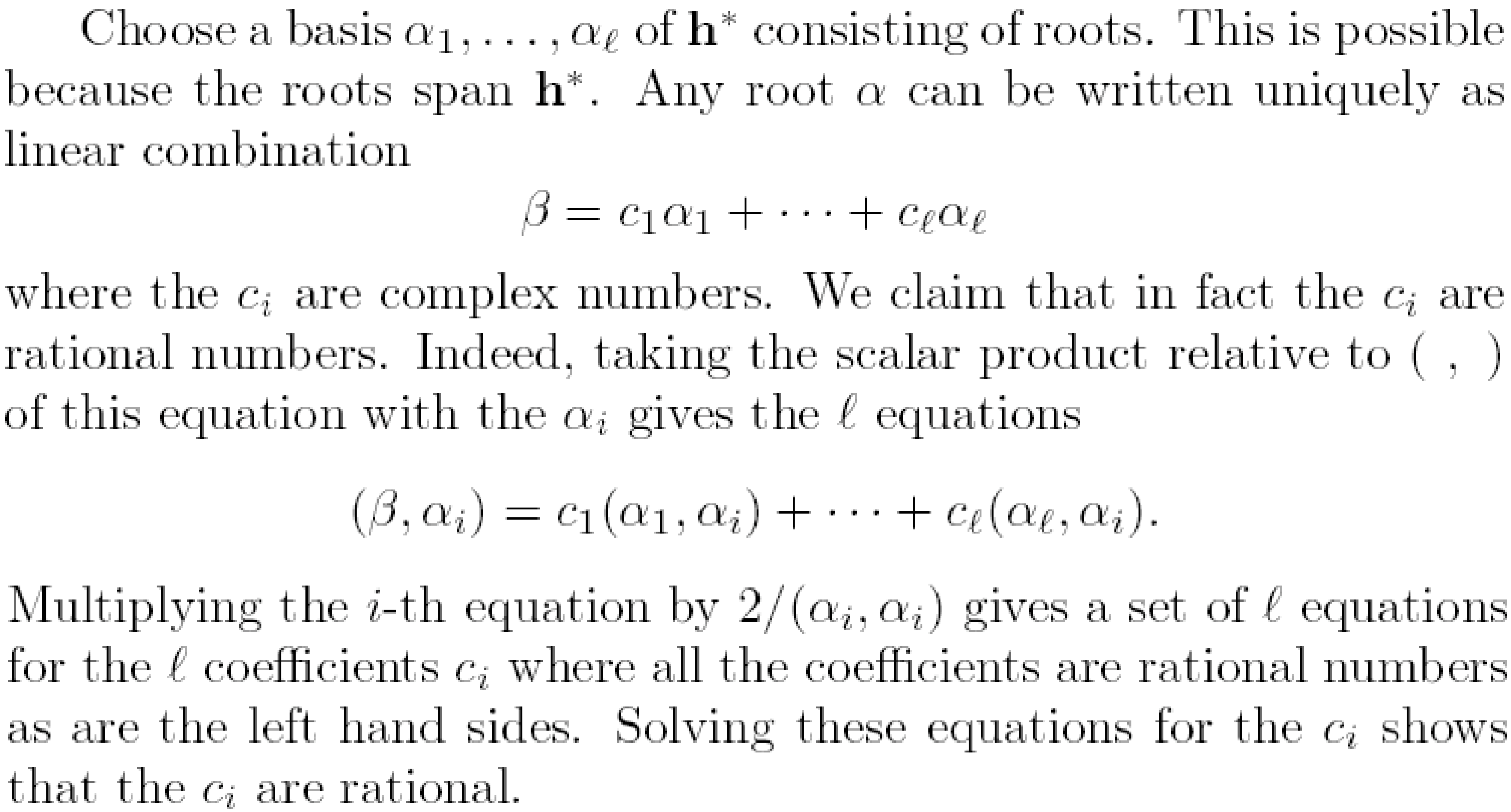
So

$$\begin{aligned}
\beta(h_\alpha) &= \kappa(t_\beta, h_\alpha) \\
&= \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} \\
&= \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.
\end{aligned}$$

so

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = r - q \in \mathbf{Z}.$$

Root bases.



Choose a basis $\alpha_1, \ldots, \alpha_\ell$ of $\mathbf{h}^*$ consisting of roots. This is possible because the roots span $\mathbf{h}^*$. Any root α can be written uniquely as linear combination

$$\beta = c_1\alpha_1 + \cdots + c_\ell\alpha_\ell$$

where the c_i are complex numbers. We claim that in fact the c_i are rational numbers. Indeed, taking the scalar product relative to $(\ ,\ )$ of this equation with the α_i gives the ℓ equations

$$(\beta, \alpha_i) = c_1(\alpha_1, \alpha_i) + \cdots + c_\ell(\alpha_\ell, \alpha_i).$$

Multiplying the i-th equation by $2/(\alpha_i, \alpha_i)$ gives a set of ℓ equations for the ℓ coefficients c_i where all the coefficients are rational numbers as are the left hand sides. Solving these equations for the c_i shows that the c_i are rational.

The space *E* .

Let E be the *real* vector space spanned by the $\alpha \in \Phi$. Then $(\ ,\ )$ restricts to a real scalar product on E. Also, for any $\lambda \neq 0 \in E$,

$$\begin{aligned}
(\lambda, \lambda) &:=: \kappa(t_\lambda, t_\lambda) \\
&:= \operatorname{tr}(\operatorname{ad} t_\lambda)^2 \\
&= \sum_{\alpha \in \Phi} \alpha(t_\lambda)^2 \\
&> 0.
\end{aligned}$$

So the scalar product $(\ ,\ )$ on E is positive definite. E is a Euclidean space.

Reflections in the hyperplane perpendicular to a root.

In the string of roots, β is q steps down from the top, so q steps up from the bottom is also a root, so

$$\beta-(r-q)\alpha$$

is a root, or

$$\beta-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha\in\Phi.$$

But

$$\beta-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha=s_\alpha(\beta)$$

where s_α denotes Euclidean reflection in the hyperplane perpendicular to α. In other words, for every $\alpha\in\Phi$

$$s_\alpha:\Phi\to\Phi. \tag{6}$$

The Weyl group.

for every $\alpha \in \Phi$

$$s_\alpha : \Phi \to \Phi.$$

The subgroup of the orthogonal group of E generated by these reflections is called the **Weyl group** and is denoted by W. We have thus associated to every semi-simple Lie algebra, and to every choice of Cartan subalgebra a finite subgroup of the orthogonal group generated by reflections. (This subgroup is finite, because all the generating reflections, s_α, and hence the group they generate, preserve the finite set of all roots, which span the space.) Once we will have completed the proof of the conjugacy theorem for Cartan subalgebras of a semi-simple algebra, then we will know that the Weyl group is determined, up to isomorphism, by the semi-simple algebra, and does not depend on the choice of Cartan subalgebra.

The Cartan integers again.

We define

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

So

$$\begin{aligned} \langle \beta, \alpha \rangle &= \beta(h_\alpha) && (7) \\ &= r - q \in \mathbf{Z} && (8) \end{aligned}$$

and

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle. \qquad (9)$$

The automorphism τ_α .

So far, we have defined the reflection s_α purely in terms of the root struction on E, which is the real subspace of $\mathbf{h}^*$ generated by the roots. But in fact, s_α, and hence the entire Weyl group arises as (an) automorphism(s) of $\mathbf{g}$ which preserve $\mathbf{h}$. Indeed, we know that $e_\alpha, f_\alpha, h_\alpha$ span a subalgebra $sl(2)_\alpha$ isomorphic to $sl(2)$. Now $\exp \operatorname{ad} e_\alpha$ and $\exp \operatorname{ad}(-f_\alpha)$ are elements of $\mathcal{E}(\mathbf{g})$. Consider

$$\tau_\alpha := (\exp \operatorname{ad} e_\alpha)(\exp \operatorname{ad}(-f_\alpha))(\exp \operatorname{ad} e_\alpha) \in \mathcal{E}(\mathbf{g}). \tag{10}$$

We claim that

Proposition 4 *The automorphism τ_α preserves $\mathbf{h}$ and on $\mathbf{h}$ it is given by*

$$\tau_\alpha(h) = h - \alpha(h) h_\alpha. \tag{11}$$

In particular, the transformation induced by τ_α on E is s_α.

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Proof. It suffices to prove (11). If $\alpha(h) = 0$, then both $\operatorname{ad} e_\alpha$ and $\operatorname{ad} f_\alpha$ vanish on h so $\tau_\alpha(h) = h$ and (11) is true. Now h_α and ker α span $\mathbf{h}$. So we need only check (11) for h_α where it says that $\tau(h_\alpha) = -h_\alpha$. But we have already verified this for the algebra $sl(2)$. QED

Direct proof.

$$\tau_\alpha := (\exp \operatorname{ad} e_\alpha)(\exp \operatorname{ad}(-f_\alpha))(\exp \operatorname{ad} e_\alpha) \in \mathcal{E}(\mathbf{g}).$$

$$\tau_\alpha(h) = h - \alpha(h)h_\alpha. \tag{11}$$

We can also verify (11) directly. We have

$$\exp(\operatorname{ad} e_\alpha)(h) = h - \alpha(h)e_\alpha$$

for any $h \in \mathbf{h}$. Now $[f_\alpha, e_\alpha] = -h_\alpha$ so

$$(\operatorname{ad} f_\alpha)^2(e_\alpha) = [f_\alpha, -h_\alpha] = [h_\alpha, f_\alpha] = -2f_\alpha.$$

So

$$\begin{aligned}
\exp(-\operatorname{ad} f_\alpha)(\exp \operatorname{ad} e_\alpha)h &= (\mathrm{id} \ - \operatorname{ad} f_\alpha + \frac{1}{2}(\operatorname{ad} f_\alpha)^2)(h - \alpha(h)e_\alpha) \\
&= h - \alpha(h)e_\alpha - \alpha(h)f_\alpha - \alpha(h)h_\alpha + \alpha(h)f \\
&= h - \alpha(h)h_\alpha - \alpha(h)e_\alpha.
\end{aligned}$$

If we now apply $\exp \operatorname{ad} e_\alpha$ to this last expression and use the fact that $\alpha(h_\alpha) = 2$, we get the right hand side of (11).

Bases.

$\Delta \subset \Phi$ is a called a **Base** if it is a basis of E (so $\#\Delta = \ell = \dim_{\mathbf{R}} E = \dim_{\mathbf{C}} \mathbf{h}$) and every $\beta \in \Phi$ can be written as $\sum_{\alpha \in \Delta} k_\alpha \alpha,\ k_\alpha \in \mathbf{Z}$ with either all the coefficients $k_\alpha \geq 0$ or all ≤ 0. Roots are accordingly called positive or negative and we define the height of a root by

$$\text{ht } \beta := \sum_\alpha k_\alpha.$$

Given a base, we get partial order on E by defining $\lambda \succ \mu$ iff $\lambda - \mu$ is a sum of positive roots or zero. We have

$$(\alpha, \beta) \leq 0, \quad \alpha, \beta \in \Delta \tag{12}$$

since otherwise $(\alpha, \beta) > 0$ and

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

is a root with the coefficient of $\beta = 1 > 0$ and the coefficient of $\alpha < 0$, contradicting the definition which says that roots must have all coefficients non-negative or non-positive.

Existence of bases.

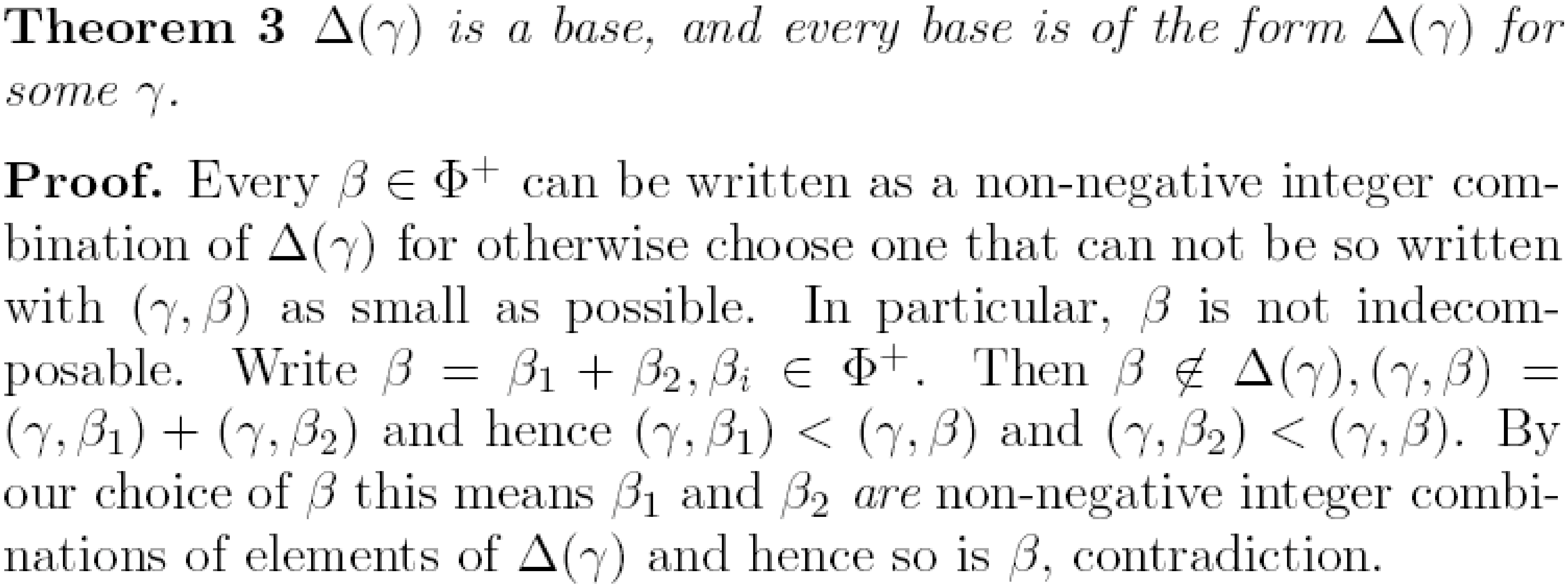
To construct a base, choose a $\gamma \in E, \mid (\gamma, \beta) \neq 0 \; \forall \; \beta \in \Phi$. Such an element is called **regular**. Then every root has positive or negative scalar product with γ, dividing the set of roots into two subsets:

$$\Phi = \Phi^+ \cup \Phi^-, \quad \Phi^- = -\Phi^+.$$

A root $\beta \in \Phi^+$ is called **decomposable** if $\beta = \beta_1 + \beta_2, \beta_1, \beta_2 \in \Phi^+$, indecomposable otherwise. Let $\Delta(\gamma)$ consist of the indecomposable elements of $\Phi^+(\gamma)$.

Theorem 3 $\Delta(\gamma)$ *is a base, and every base is of the form* $\Delta(\gamma)$ *for some* γ.

Existence of bases, continued.

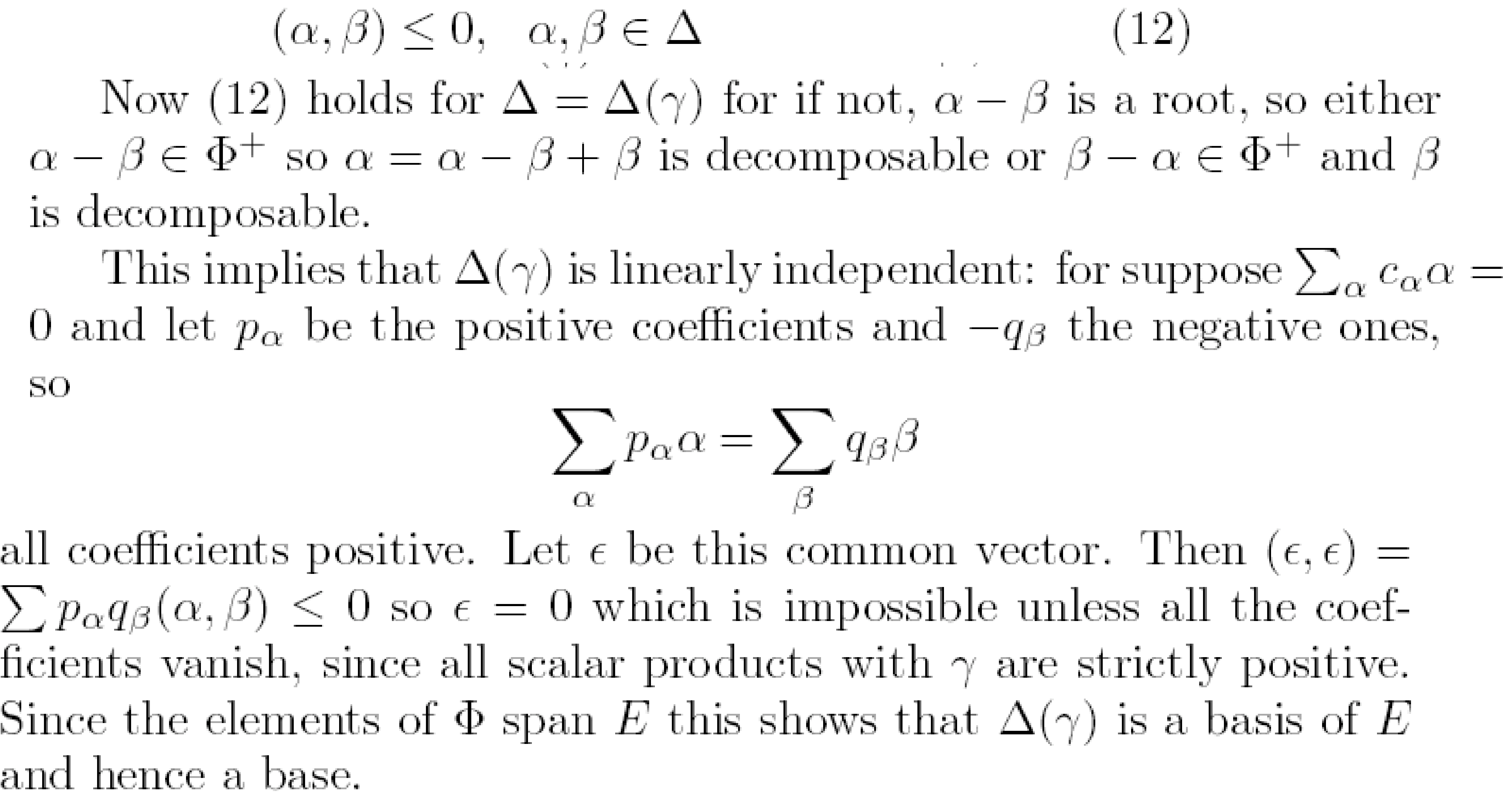


Theorem 3 *$\Delta(\gamma)$ is a base, and every base is of the form $\Delta(\gamma)$ for some γ.*

Proof. Every $\beta \in \Phi^+$ can be written as a non-negative integer combination of $\Delta(\gamma)$ for otherwise choose one that can not be so written with (γ, β) as small as possible. In particular, β is not indecomposable. Write $\beta = \beta_1 + \beta_2, \beta_i \in \Phi^+$. Then $\beta \notin \Delta(\gamma), (\gamma, \beta) = (\gamma, \beta_1) + (\gamma, \beta_2)$ and hence $(\gamma, \beta_1) < (\gamma, \beta)$ and $(\gamma, \beta_2) < (\gamma, \beta)$. By our choice of β this means β_1 and β_2 *are* non-negative integer combinations of elements of $\Delta(\gamma)$ and hence so is β, contradiction.

We must prove the linear independence of the $\Delta(\gamma)$.

Proof of linear indepedence.



$$(\alpha, \beta) \le 0, \quad \alpha, \beta \in \Delta \tag{12}$$

Now (12) holds for $\Delta = \Delta(\gamma)$ for if not, $\alpha - \beta$ is a root, so either $\alpha - \beta \in \Phi^+$ so $\alpha = \alpha - \beta + \beta$ is decomposable or $\beta - \alpha \in \Phi^+$ and β is decomposable.

This implies that $\Delta(\gamma)$ is linearly independent: for suppose $\sum_\alpha c_\alpha \alpha = 0$ and let p_α be the positive coefficients and $-q_\beta$ the negative ones, so

$$\sum_\alpha p_\alpha \alpha = \sum_\beta q_\beta \beta$$

all coefficients positive. Let ϵ be this common vector. Then $(\epsilon, \epsilon) = \sum p_\alpha q_\beta (\alpha, \beta) \le 0$ so $\epsilon = 0$ which is impossible unless all the coefficients vanish, since all scalar products with γ are strictly positive. Since the elements of Φ span E this shows that $\Delta(\gamma)$ is a basis of E and hence a base.

Now let us show that every base is of the desired form: For any base Δ, let $\Phi^+ = \Phi^+(\Delta)$ denote the set of those roots which are non-negative integral combinations of the elements of Δ and let $\Phi^- = \Phi^-(\Delta)$ denote the ones which are non-positive integral combinations of elements of Δ. Define $\delta_\alpha,\ \alpha \in \Delta$ to be the projection of α onto the orthogonal complement of the space spanned by the other elements of the base. Then

$$(\delta_\alpha, \alpha') = 0, \quad \alpha \neq \alpha',\ (\delta_\alpha, \alpha) = (\delta_\alpha, \delta_\alpha) > 0$$

so $\gamma = \sum r_\alpha \delta_\alpha,\ r_\alpha > 0$ satisfies

$$(\gamma, \alpha) > 0 \ \forall\ \alpha \in \Delta$$

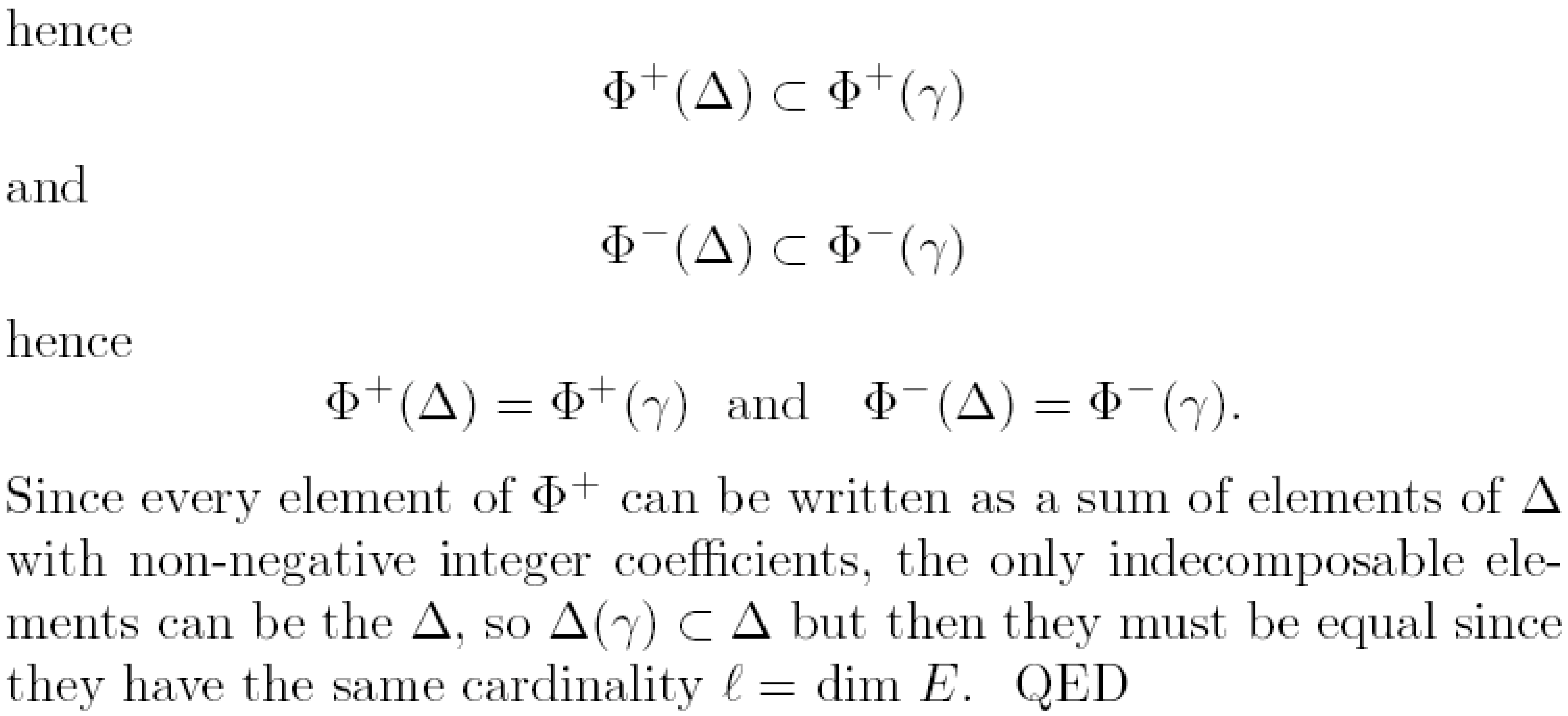
hence

$$\Phi^+(\Delta) \subset \Phi^+(\gamma)$$

and

$$\Phi^-(\Delta) \subset \Phi^-(\gamma)$$

Conclusion of the proof.



hence

$$\Phi^+(\Delta) \subset \Phi^+(\gamma)$$

and

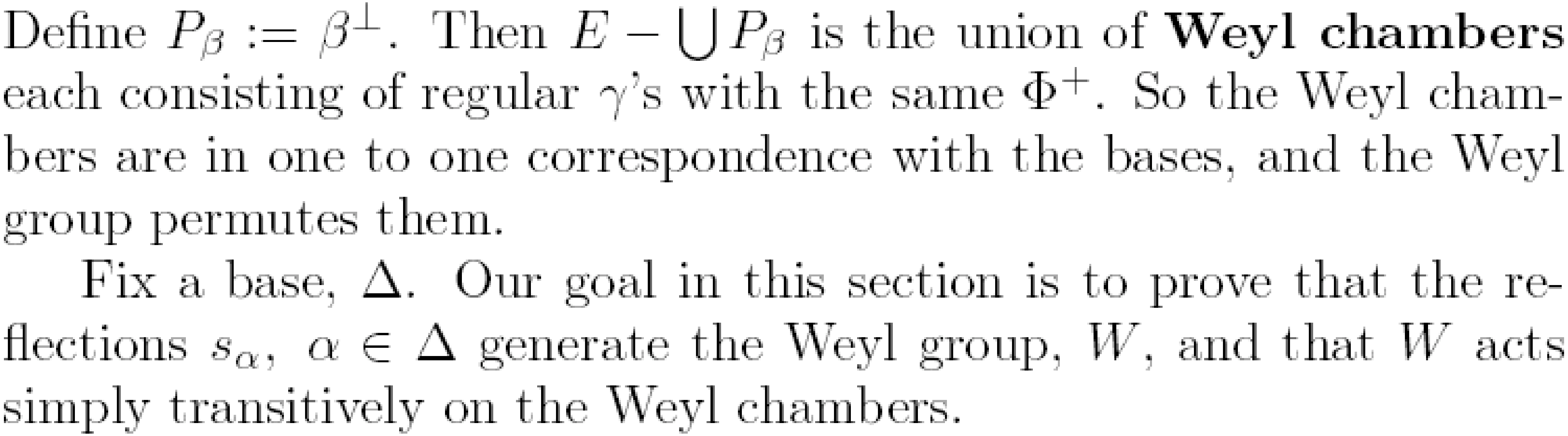
$$\Phi^-(\Delta) \subset \Phi^-(\gamma)$$

hence

$$\Phi^+(\Delta) = \Phi^+(\gamma) \;\text{ and }\; \Phi^-(\Delta) = \Phi^-(\gamma).$$

Since every element of Φ^+ can be written as a sum of elements of Δ with non-negative integer coefficients, the only indecomposable elements can be the Δ, so $\Delta(\gamma) \subset \Delta$ but then they must be equal since they have the same cardinality $\ell = \dim E$. QED

Weyl chambers.



Define $P_\beta := \beta^\perp$. Then $E - \bigcup P_\beta$ is the union of **Weyl chambers** each consisting of regular γ's with the same Φ^+. So the Weyl chambers are in one to one correspondence with the bases, and the Weyl group permutes them.

Fix a base, Δ. Our goal in this section is to prove that the reflections s_α, $\alpha \in \Delta$ generate the Weyl group, W, and that W acts simply transitively on the Weyl chambers.

Action of the basic reflections on the positive roots..

Each s_α, $\alpha \in \Delta$ sends $\alpha \mapsto -\alpha$. But acting on $\lambda = \sum c_\beta \beta$, the reflection s_α does not change the coefficient of any other element of the base. If $\lambda \in \Phi^+$ and $\lambda \neq \alpha$, we must have $c_\beta > 0$ for some $\beta \neq \alpha$ in the base Δ. Then the coefficient of β in the expansion of $s_\alpha(\lambda)$ is positive, and hence all its coefficients must be non-negative. So $s_\alpha(\lambda) \in \Phi^+$. In short, the only element of Φ^+ sent into Φ^- is α. So if

$$\delta := \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \text{ then } s_\alpha \delta = \delta - \alpha.$$

Positive roots and partial sums.

If $\beta \in \Phi^+,\ \beta \notin \Delta$, then we can not have $(\beta, \alpha) \leq 0\ \forall \alpha \in \Delta$ for then $\beta \cup \Delta$ would be linearly independent. So $\beta - \alpha$ is a root for some $\alpha \in \Delta$, and since we have changed only one coefficient, it must be a positive root. Hence any $\beta \in \Phi$ can be written as

$$\beta = \alpha_1 + \cdots + \alpha_p \quad \alpha_i \in \Delta$$

where all the partial sums are positive roots.

To be continued.