Math 128 lecture 9

Cartan's criterion for solvability and Weyl's theorem on the complete reducibility of semi-simple Lie algebras.

Review of facts from linear algebra.

Let *u* be a linear transformation on a finite dimensional complex vector space *V*. Then there are operators *s* and *n* which are polynomials in *u* where *s* is semi-simple and *n* is nilpotent and are characterized by $u = s + n$ and $ns = sn$.

Define

$$V_{p,q} := V \otimes V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

with p copies of V and q copies of V^*. Let $u \in \mathrm{End}(V)$ act on V^* by $-u^*$ and on V_{pq} by derivation, so , for example,

$$u_{12} = u \otimes 1 \otimes 1 - 1 \otimes u^* \otimes 1 - 1 \otimes 1 \otimes u^*.$$

Proposition 1 *If $u = s + n$ is the decomposition of u then $u_{pq} = s_{pq} + n_{pq}$ is the decomposition of u_{pq}.*

Review, continued.

If $\phi : k \to k$ is a map, we define $\phi(s)$ by $\phi(s)_{|V_i} = \phi(\lambda_i)$. If we choose a polynomial such that $P(0) = 0$, $P(\lambda_i) = \phi(\lambda_i)$ then $P(u) = \phi(s)$.

Proposition 2 *Suppose that ϕ is additive. Then*

$$(\phi(s))_{pq} = \phi(s_{pq}).$$

As an immediate consequence we obtain

Proposition 3 *Notation as above. If $A \subset B \subset V_{p,q}$ with $u_{pq}B \subset A$ then for any additive map, $\phi(s)_{pq}B \subset A$*

Proposition 4 (over $\mathbf{C}$) *Let $u = s + n$ as above. If $\operatorname{tr}(u\phi(s)) = 0$ for $\phi(s) = \overline{s}$ then u is nilpotent.*

Proof. $\operatorname{tr} u\phi(s) = \sum m_i \lambda_i \overline{\lambda_i} = \sum m_i |\lambda_i|^2$. So the condition implies that all the $\lambda_i = 0$. QED

Cartan's criterion.

Let $\mathbf{g} \subset \operatorname{End}(V)$ be a Lie subalgebra where V is finite dimensional vector space over $\mathbf{C}$. Then

$$\mathbf{g} \text{ is solvable} \Leftrightarrow \quad \operatorname{tr}(xy) = 0 \ \forall x \in \mathbf{g}, \quad y \in [\mathbf{g}, \mathbf{g}].$$

Proof. Suppose $\mathbf{g}$ is solvable. Choose a basis for which $\mathbf{g}$ is upper triangular. Then every $y \in [\mathbf{g}, \mathbf{g}]$ has zeros on the diagonal, Hence $\operatorname{tr}(xy) = 0$. For the reverse implication, it is enough to show that $[\mathbf{g}, \mathbf{g}]$ is nilpotent, and, by Engel, that each $u \in [\mathbf{g}, \mathbf{g}]$ is nilpotent. So it is enough to show that $\operatorname{tr} u\overline{s} = 0$, where s is the semisimple part of u, by Proposition 4 above. If it were true that $\overline{s} \in \mathbf{g}$ we would be done, but this need not be so. Write

$$u = \sum [x_i, y_i].$$

Proof of Cartan's criterion, continued.

$$u = \sum [x_i, y_i].$$

Now for $a, b, c \in \mathrm{End}(V)$

$$\begin{aligned}
\mathrm{tr}([a,b]c) &= \mathrm{tr}(abc - bac) \\
&= \mathrm{tr}(bca - bac) \\
&= \mathrm{tr}(b[c,a]) \quad \text{so} \\
\mathrm{tr}(u\overline{s}) &= \sum \mathrm{tr}([x_i, y_i]\overline{s}) \\
&= \sum \mathrm{tr}(y_i[\overline{s}, x_i]).
\end{aligned}$$

So it is enough to show that $\mathrm{ad}\,\overline{s} : \mathbf{g} \to [\mathbf{g}, \mathbf{g}]$. We know that $\mathrm{ad}\,u : \mathbf{g} \to [\mathbf{g}, \mathbf{g}]$, and we can, by Lagrange interpolation, find a polynomial P such that $P(u) = \overline{s}$. The result now follows from Prop. 3:

Since $\mathrm{End}(V) \sim V_{1,1}$, take $A = [\mathbf{g}, \mathbf{g}]$ and $B = \mathbf{g}$. Then $\mathrm{ad}\,u = u_{1,1}$ so $u_{1,1}\mathbf{g} \subset [\mathbf{g}, \mathbf{g}[$ and hence $\overline{s}_{1,1}\mathbf{g} \subset [\mathbf{g}, \mathbf{g}]$ or $[\overline{s}, x] \in [\mathbf{g}, \mathbf{g}]$ $\forall x \in \mathbf{g}$. QED

Solvable ideals.

If $\mathbf{i}$ is an ideal of $\mathbf{g}$ and $\mathbf{g}/\mathbf{i}$ is solvable, then $D^{(n)}(\mathbf{g}/\mathbf{i}) = 0$ implies that $D^{(n)}\mathbf{g} \subset \mathbf{i}$. If $\mathbf{i}$ itself is solvable with $D^{(m)}\mathbf{i} = 0$, then $D^{(m+n)}\mathbf{g} = 0$. So we have proved:

Proposition 5 *If* $\mathbf{i} \subset \mathbf{g}$ *is an ideal, and both* $\mathbf{i}$ *and* $\mathbf{g}/\mathbf{i}$ *are solvable, so is* $\mathbf{g}$.

The radical.

If $\mathbf{i}$ and $\mathbf{j}$ are solvable ideals, then $(\mathbf{i}+\mathbf{j})/\mathbf{j} \sim \mathbf{i}/(\mathbf{i}\cap\mathbf{j})$ is solvable, being the homomorphic image of a solvable algebra. So, by the previous proposition:

Proposition 6 *If* $\mathbf{i}$ *and* $\mathbf{j}$ *are solvable ideals in* $\mathbf{g}$ *so is* $\mathbf{i}+\mathbf{j}$. *In particular, every Lie algebra* $\mathbf{g}$ *has a largest solvable ideal which contains all other solvable ideals. It is denoted by* rad $\mathbf{g}$ *or simply by* $\mathbf{r}$ *when* $\mathbf{g}$ *is fixed.*

An algebra $\mathbf{g}$ is called **semi-simple** if rad $\mathbf{g} = 0$. Since $D\mathbf{i}$ is an ideal whenever $\mathbf{i}$ is (by Jacobi's identity), if $\mathbf{r} \neq 0$ then the last non-zero $D^{(n)}\mathbf{r}$ is an abelian ideal. So an equivalent definition is: $\mathbf{g}$ is semi-simple if it has no non-zero abelian ideals.

We shall call a Lie algebra **simple** if it is not abelian and if it has no proper ideals. We shall show in the next section that every semi-simple Lie algebra is the direct sum of simple Lie algebras in a unique way.

Invariant bilinear forms on a Lie algebra.

A bilinear form $(\, , \,) : \mathbf{g} \times \mathbf{g} \to k$ is called **invariant** if

$$([x,y],z) + (y,[x,z]) = 0 \quad \forall x,y,z \in \mathbf{g}. \tag{2}$$

Notice that if $(\, , \,)$ is an invariant form, and $\mathbf{i}$ is an ideal, then $\mathbf{i}^{\perp}$ is again an ideal.

Invariant bilinear forms from representations, the Killing form.

One way of producing invariant forms is from representations: if(ρ, V) is a representation of $\mathbf{g}$, then

is invariant. Indeed,

$$([x, y], z)_\rho + (y, [x, z])_\rho$$

$$\begin{aligned} &= \operatorname{tr}\{(\rho(x)\rho(y) - \rho(y)\rho(x))\rho(z)) + \rho(y)(\rho(x)\rho(z) - \rho(z)\rho(x))\} \\ &= \operatorname{tr}\{\rho(x)\rho(y)\rho(z) - \rho(y)\rho(z)\rho(x)\} \\ &= 0. \end{aligned}$$

In particular, if we take $\rho = \mathrm{ad},\ V = \mathbf{g}$ the corresponding bilinear form is called the **Killing form** and will be denoted by $(\ ,\ )_\kappa$. We will also sometimes write $\kappa(x, y)$ instead of $(x, y)_\kappa$.

Theorem 6 $\mathbf{g}$ *is semi-simple if and only if its Killing form is non-degenerate.*

Proof. Suppose $\mathbf{g}$ is not semi-simple and so has a non-zero abelian ideal, $\mathbf{a}$. We will show that $(x, y)_\kappa = 0 \; \forall x \in \mathbf{a}, y \in \mathbf{g}$. Indeed, let $\sigma = \operatorname{ad} x \operatorname{ad} y$. Then σ maps $\mathbf{g} \to \mathbf{a}$ and $\mathbf{a} \to 0$. Hence in terms of a basis starting with elements of $\mathbf{a}$ and extending, it (is upper triangular and) has 0 along the diagonal. Hence $\operatorname{tr} \sigma = 0$. Hence if $\mathbf{g}$ is *not* semisimple then its Killing form is degenerate.

Conversely, suppose that $\mathbf{g}$ is semi-simple. We wish to show that the Killing form is non-degenerate. So let $\mathbf{u} := \mathbf{g}^{\perp} = \{x | \operatorname{tr} \operatorname{ad} x \operatorname{ad} y = 0 \ \forall y \in \mathbf{g}\}$. If $x \in \mathbf{u}, z \in \mathbf{g}$ then

$$\begin{aligned}
\operatorname{tr}\{\operatorname{ad}[x,z] \operatorname{ad} y\} &= \operatorname{tr}\{\operatorname{ad} x \operatorname{ad} z \operatorname{ad} y - \operatorname{ad} z \operatorname{ad} x \operatorname{ad} y)\} \\
&= \operatorname{tr}\{\operatorname{ad} x (\operatorname{ad} z \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} z)\} \\
&= \operatorname{tr} \operatorname{ad} x \operatorname{ad}[z,y] \\
&= 0,
\end{aligned}$$

so $\mathbf{u}$ is an ideal. In particular, $\operatorname{tr}_{\mathbf{u}}(\operatorname{ad} x_{\mathbf{u}} \operatorname{ad} y_{\mathbf{u}}) = \operatorname{tr}_{\mathbf{g}}(\operatorname{ad}_{\mathbf{g}} x \operatorname{ad}_{\mathbf{g}} y)$ for $x, y \in \mathbf{u}$, as can be seen from a block decomposition starting with a basis of $\mathbf{u}$ and extending to $\mathbf{g}$.

If we take $y \in D\mathbf{u}$, we see that $\operatorname{tr} \operatorname{ad} \mathbf{u} D \operatorname{ad} \mathbf{u} = 0$, so $\operatorname{ad} \mathbf{u}$ is solvable by Cartan's criterion. But the kernel of the map $\mathbf{u} \to \operatorname{ad} \mathbf{u}$ is the center of $\mathbf{u}$. So if $\operatorname{ad} \mathbf{u}$ is solvable, so is $\mathbf{u}$. QED

The decomposition of a semi-simple Lie algebra.

Proposition 7 *Let* $\mathbf{g}$ *be a semisimple algebra,* $\mathbf{i}$ *any ideal of* $\mathbf{g}$*, and* $\mathbf{i}^\perp$ *its orthocomplement with respect to its Killing form. Then* $\mathbf{i} \cap \mathbf{i}^\perp = 0$.

Indeed, $\mathbf{i} \cap \mathbf{i}^\perp$ is an ideal on which $tr \operatorname{ad} x \operatorname{ad} y \equiv 0$ hence is solvable by Cartan's criterion. Since $\mathbf{g}$ is semi-simple, there are no non-trivial solvable ideals. QED

Therefore

Proposition 8 *Every semi-simple Lie algebra is the direct sum of simple Lie algebras.*

Proposition 9 $D\mathbf{g} = \mathbf{g}$ *for a semi-simple Lie algebra.*

(Since this is true for each simple component.)

The decomposition of a semi-simple Lie algebra, continued.

Proposition 10 *Let* $\phi : \mathbf{g} \to \mathbf{s}$ *be a surjective homomorphism of a semi-simple Lie algebra onto a simple Lie algebra. Then if* $\mathbf{g} = \bigoplus \mathbf{g}_i$ *is a decomposition of* $\mathbf{g}$ *into simple ideals, the restriction,* ϕ_i *of* ϕ *to each summand is zero, except for one summand where it is an isomorphism.*

Proof. Since $\mathbf{s}$ is simple, the image of every ϕ_i is 0 or all of $\mathbf{s}$. If ϕ_i is surjective for some i then it is an isomorphism since $\mathbf{g}_i$ is simple. There is at least one i for which it is surjective since ϕ is surjective. On the other hand, it can not be surjective for for two ideals, $\mathbf{g}_i, \mathbf{g}_j$ $i \neq j$ for then $\phi[\mathbf{g}_i, \mathbf{g}_j] = 0 \neq [\mathbf{s}, \mathbf{s}] = \mathbf{s}$. QED

Complete reducibility of semi-simple Lie algebras.

Theorem 7 **[Weyl.]** *Every finite dimensional representation of a semi-simple Lie algebra is completely reducible.*

Proof.

1. If $\rho : \mathbf{g} \to \operatorname{End} V$ is injective, then the form $(\ ,\ )_\rho$ is non-degenerate. Indeed, the ideal consisting of all x such that $(x, y)_\rho = 0\ \forall y \in \mathbf{g}$ is solvable by Cartan's criterion, hence 0.

The Casimir element.

The **Casimir operator.** Let (e_i) and (f_i) be bases of $\mathbf{g}$ which are dual with respect to some non-degenerate invariant bilinear form, $(,)$. So $(e_i, f_j) = \delta_{ij}$. As the form is non-degenerate and invariant, it defines a map of

$$\mathbf{g} \otimes \mathbf{g} \mapsto \operatorname{End} \mathbf{g}; \quad x \otimes y(w) = (y, w)x.$$

This map is an isomorphism and is a $\mathbf{g}$ morphism. Under this map,

$$\sum e_i \otimes f_i(w) = \sum (w, f_i) e_i = w$$

by the definition of dual bases. Hence under the inverse map

$$\operatorname{End} \mathbf{g} \mapsto \mathbf{g} \otimes \mathbf{g}$$

the identity element, id, corresponds to $\sum e_i \otimes f_i$ (and so this expression is independent of the choice of dual bases).

Set

$$C := \sum e_i \cdot f_i \in U(L).$$

under the inverse map $\quad \operatorname{End}\mathbf{g} \mapsto \mathbf{g} \otimes \mathbf{g}$

the identity element, id, corresponds to $\sum e_i \otimes f_i$ (and so this expression is independent of the choice of dual bases). Since id is annihilated by commutator by any element of $\operatorname{End}(\mathbf{g})$, we conclude that $\sum_i e_i \otimes f_i$ is annihilated by the action of all $(\operatorname{ad} x)_2 = \operatorname{ad} x \otimes 1 + 1 \otimes \operatorname{ad} x, \; x \in \mathbf{g}$. Indeed, for $x, e, f, y \in \mathbf{g}$ we have

$$\begin{aligned}
((\operatorname{ad} x)_2(e \otimes f))\, y &= (\operatorname{ad} xe \otimes f + e \otimes \operatorname{ad} xf)\, y \\
&= (f, y)[x, e] + ([x, f], y)e \\
&= (f, y)[x, e] - (f, [x, y])e \quad \text{by } (2) \\
&= ((\operatorname{ad} x)(e \otimes f) - (e \otimes f)(\operatorname{ad} x))\, y.
\end{aligned}$$

$$C := \sum e_i \cdot f_i \in U(L).$$

Thus C is the image of the element $\sum_i e_i \otimes f_i$ under the multiplication map $\mathbf{g} \otimes \mathbf{g} \mapsto U(\mathbf{g})$, and is independent of the choice of dual bases. Furthermore, C is annihilated by $\operatorname{ad} x$ acting on $U(\mathbf{g})$. In other words, it commutes with all elements of $\mathbf{g}$, and hence with all of $U(\mathbf{g})$; it is in the center of $U(\mathbf{g})$.

The Casimir element and the Casimir operator.

The C corresponding to the Killing form is called the **Casimir element**, its image in any representation is called the **Casimir operator**.

3. Suppose that $\rho : \mathbf{g} \to \operatorname{End} V$ is injective. The (image of the) central element corresponding to $(\ ,\ )_\rho$ defines an element of $\operatorname{End} V$ denoted by C_ρ and

$$\begin{aligned} \operatorname{tr} C_\rho &= \operatorname{tr} \rho(\sum e_i f_i) \\ &= \operatorname{tr} \sum \rho(e_i)\rho(f_i) \\ &= \sum_i (e_i, f_i) \\ &= \dim \mathbf{g} \end{aligned}$$

The key proposition.

Proposition 11 *Let $0 \to V \to W \to k \to 0$ be an exact sequence of* $\mathbf{g}$ *modules, where* $\mathbf{g}$ *is semi-simple, and the action of* $\mathbf{g}$ *on k is trivial (as it must be). Then this sequence splits, i.e. there is a line in W supplementary to V on which* $\mathbf{g}$ *acts trivially.*

The proof of the proposition and of the theorem is almost identical to the proof we gave above for the special case of $sl(2)$. We will need only one or two additional arguments. As in the case of $sl(2)$, the proposition is a special case of the theorem we want to prove. But we shall see that it is sufficient to prove the theorem.

Proof of proposition. It is enough to prove the proposition for the case that V is an irreducible module. Indeed, if V_1 is a submodule, then by induction on dim V we may assume the theorem is known for $0 \to V/V_1 \to W/V_1 \to k \to 0$ so that there is a one dimensional invariant subspace M in W/V_1 supplementary to V/V_1 on which the action is trivial. Let N be the inverse image of M in W. By another application of the proposition, this time to the sequence

$$0 \to V_1 \to N \to M \to 0$$

we find an invariant line, P, in N complementary to V_1. So $N = V_1 \oplus P$. Since $(W/V_1) = (V/V_1) \oplus M$ we must have $P \cap V = \{0\}$. But since dim $W =$ dim $V + 1$, we must have $W = V \oplus P$. In other words P is a one dimensional subspace of W which is complementary to V.

Proof of the proposition, concluded.

Next we can reduce to proving the proposition for the case that $\mathbf{g}$ acts faithfully on V. Indeed, let $\mathbf{i}$ = the kernel of the action on V. For all $x \in \mathbf{g}$ we have, by hypothesis, $xW \subset V$, and for $x \in \mathbf{i}$ we have $xV = 0$. Hence $D\mathbf{i}$ acts trivially on W. But $\mathbf{i} = D\mathbf{i}$ since $\mathbf{i}$ is semi-simple. Hence $\mathbf{i}$ acts trivially on W and we may pass to $\mathbf{g}/\mathbf{i}$.

This quotient is again semi-simple, since $\mathbf{i}$ is a sum of some of the simple ideals of $\mathbf{g}$.

So we are reduced to the case that V is irreducible and the action, ρ, of $\mathbf{g}$ on V is injective. Then we have an invariant element C_ρ whose image in $\operatorname{End} W$ must map $W \to V$ since every element of $\mathbf{g}$ does. (We may assume that $\mathbf{g} \neq 0$.) On the other hand, $C_\rho \neq 0$, indeed its trace is $\dim \mathbf{g}$. The restriction of C_ρ to V can not have a non-trivial kernel, since this would be an invariant subspace. Hence the restriction of C_ρ to V is an isomorphism. Hence $\ker C_\rho : W \to V$ is an invariant line supplementary to V. We have proved the proposition.

Proof of theorem from proposition. Let $0 \to E' \to E$ be an exact sequence of $\mathbf{g}$ modules, and we may assume that $E' \neq 0$. We want to find an invariant complement to E' in E. Define W to be the subspace of $\mathrm{Hom}_k(E, E')$ whose restriction to E' is a scalar times the identity, and let $V \subset W$ be the subspace consisting of those linear transformations whose restrictions to E' is zero. Each of these is a submodule of $\mathrm{End}(E)$. We get a sequence

$$0 \to V \to W \to k \to 0$$

and hence a complementary line of invariant elements in W. In particular, we can find an element, T which is invariant, maps $E \to E'$, and whose restriction to E' is non-zero. Then $\ker T$ is an invariant complementary subspace. QED

As an illustration of construction of the Casimir operator consider $\mathbf{g} = sl(2)$ with

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{tr}(\operatorname{ad} h)^2 &= 8 \\ \operatorname{tr}(\operatorname{ad} e)(\operatorname{ad} f) &= 4 \end{aligned}$$

so the dual basis to the basis h, e, f is $h/8, f/4, e/4$, or, if we divide the metric by 4, the dual basis is $h/2, f, e$ and so the Casimir operator C is

$$\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe.$$

This coincides with the C that we used in Chapter II.