Math 128 Lecture 1

The Campbell-Baker-Hausdorff formula.

A Lie algebra is:

A **Lie algebra** over a field $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ together with a bilinear map

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

where the image of the pair $A, B \in \mathfrak{g}$ is denoted by $[A, B]$ and where this "multiplication" is subject to the conditions:

- Anti-symmetry: $[B, A] = -[A, B]$ and
- Jacobi's identity: $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$.

In this course the field $\mathbb{K}$ will be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. During most of the course it will be the field of complex numbers. But at the beginning, for motivational purposes, we will consider the field of real numbers.

Lie algebras from associative algebras.

For example, if $\mathbb{A}$ is an associative algebra (for example the algebra of all $n \times n$ matrices over $\mathbb{K}$) we can define the commutator bracket

$$[A, B] := AB - BA.$$

This is clearly anti-symmetric in A and B. As to Jacobi,

$$\begin{aligned}
[A,[B,C]] &= ABC - ACB - BCA + CBA \\
[[A,B],C] &= ABC - BAC - CAB + CBA \\
[B,[A,C]] &= BAC - BCA - ACB + CAB
\end{aligned}$$

and the sum of the last two right hand sides equals the first right hand side.

We will study in detail the converse to this: "to what extend does a Lie algebra arise this way from an associative algebra" a bit later on in the course.

But the key importance of Lie algebras is their relation to Lie groups which some of you may or may not have seen.

Exponentials and logarithms.

Recall the power series:

$$\exp X = 1+X+\frac{1}{2}X^2+\frac{1}{3!}X^3+\cdots\ , \ \ \log(1+X) = X-\frac{1}{2}X^2+\frac{1}{3}X^3+\cdots\ .$$

We want to study these series in a ring where convergence makes sense; for example in the ring of $n \times n$ matrices. The exponential series converges everywhere, and the series for the logarithm converges in a small enough neighborhood of the origin. Of course,

$$\log(\exp X) = X; \quad \exp(\log(1+X)) = 1+X$$

where these series converge, or as formal power series.

The product of two exponentials.

In particular, if A and B are two elements which are close enough to 0 we can study the convergent series

$$\log[(\exp A)(\exp B)]$$

which will yield an element C such that $\exp C = (\exp A)(\exp B)$. The problem is that A and B need not commute. For example, if we retain only the linear and constant terms in the series we find

$$\log[(1 + A + \cdots)(1 + B + \cdots)] = \log(1 + A + B + \cdots) = A + B + \cdots .$$

The exponential of a sum is not the product of their exponentials in general.

On the other hand, if we go out to terms second order, the non-commutativity begins to enter:

$$\log[(1 + A + \frac{1}{2}A^2 + \cdots)(1 + B + \frac{1}{2}B^2 + \cdots)] =$$

$$A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 - \frac{1}{2}(A + B + \cdots)^2$$

$$= A + B + \frac{1}{2}[A, B] + \cdots$$

Let us look at the terms on third order:

We have

$$(1+A+\frac{1}{2}A^2+\frac{1}{3!}A^3)(1+B+\frac{1}{2}B^2+\frac{1}{3!}B^3) =$$

$$1+(A+B)+\frac{1}{2}(A^2+2AB+B^2)+\frac{1}{3!}(A^3+3A^2B+3AB^2+B^3).$$

We want to substitute this into the series for $\log(1+X) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 + ...$ and collect the third order terms. The X term yields

$$\frac{1}{3!}(A^3+3A^2B+3AB^2+B^3).$$

The $-\frac{1}{2}X^2$ term yields

$$-\frac{1}{4}\left[(A+B)(A^2+2AB+B^2)+(A^2+2AB+B^2)(A+B)\right].$$

The term $\frac{1}{3}X^3$ yields

$$\frac{1}{3}\left[A^3+A^2B+ABA+BA^2+AB^2+BAB+B^2A+B^3\right].$$

The third order terms.

We have $\frac{1}{6}(A^3 + 3A^2B + 3AB^2 + B^3)$

$$-\frac{1}{4}\left[(A+B)(A^2+2AB+B^2)+(A^2+2AB+B^2)(A+B)\right]$$

$$+\frac{1}{3}\left[A^3+A^2B+ABA+BA^2+AB^2+BAB+B^2A+B^3\right].$$

The coefficient of A^3 and of B^3 is $\frac{1}{6}-\frac{1}{2}+\frac{1}{3}=0$. We tabulate the remaining coefficients:

A^2B	$\frac{1}{2}-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}$	$=$	$\frac{1}{12}$
ABA	$-\frac{1}{2}+\frac{1}{3}$	$=$	$-\frac{2}{12}$
BA^2	$-\frac{1}{4}+\frac{1}{3}$	$=$	$\frac{1}{12}$
AB^2	$\frac{1}{2}-\frac{1}{4}-\frac{1}{2}+\frac{1}{3}$	$=$	$\frac{1}{12}$
BAB	$-\frac{1}{2}+\frac{1}{3}$	$=$	$-\frac{2}{12}$
B^2A	$-\frac{1}{4}+\frac{1}{3}$	$=$	$\frac{1}{12}$

The third order terms can be expressed in terms of the Lie bracket!

A^2B	$\frac{1}{2}-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}$	$=$	$\frac{1}{12}$
ABA	$-\frac{1}{2}+\frac{1}{3}$	$=$	$-\frac{2}{12}$
BA^2	$-\frac{1}{4}+\frac{1}{3}$	$=$	$\frac{1}{12}$
AB^2	$\frac{1}{2}-\frac{1}{4}-\frac{1}{2}+\frac{1}{3}$	$=$	$\frac{1}{12}$
BAB	$-\frac{1}{2}+\frac{1}{3}$	$=$	$-\frac{2}{12}$
B^2A	$-\frac{1}{4}+\frac{1}{3}$	$=$	$\frac{1}{12}$

We get

$$\frac{1}{12}\left(A^2B + AB^2 + B^2A + BA^2 - 2ABA - 2BAB]\right)$$

$$= \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]].$$

The Cambell-Baker-Hausdorff formula.

This suggests that the series for $\log[(\exp A)(\exp B)]$ can be expressed entirely in terms of successive Lie brackets of A and B. This is so, and is the content of the Campbell-Baker-Hausdorff formula.

One of the important consequences of the mere existence of this formula is the following. Suppose that $\mathbf{g}$ is the Lie algebra of a Lie group G. Then the *local* structure of G near the identity, i.e. the rule for the product of two elements of G sufficiently closed to the identity is determined by its Lie algebra $\mathbf{g}$. Indeed, the exponential map is locally a diffeomorphism from a neighborhood of the origin in $\mathbf{g}$ onto a neighborhood W of the identity, and if $U \subset W$ is a (possibly smaller) neighborhood of the identity such that $U \cdot U \subset W$, the the product of $a = \exp \xi$ and $b = \exp \eta$, with $a \in U$ and $b \in U$ is then completely expressed in terms of successive Lie brackets of ξ and η.

We will give two proofs.

We will give two proofs of this important theorem. One will be geometric - the explicit formula for the series for $\log[(\exp A)(\exp B)]$ will involve integration, and so makes sense over the real or complex numbers. We will derive the formula from the "Maurer-Cartan equations" which we will explain in the course of our discussion. Our second version will be more algebraic. It will involve such ideas as the universal enveloping algebra, comultiplication and the Poincaré-Birkhoff-Witt theorem. In both proofs, many of the key ideas are at least as important as the theorem itself.

Notation:

To state this formula we introduce some notation. Let ad A denote the operation of bracketing on the left by A, so

$$\mathrm{ad}A(B) := [A, B].$$

The function ψ .

Define the function ψ by

$$\psi(z) = \frac{z \log z}{z - 1}$$

which is defined as a convergent power series around the point $z = 1$ so

$$\psi(1+u) = (1+u)\frac{\log(1 + u)}{u} = (1+u)(1-\frac{u}{2}+\frac{u^2}{3}+\cdots) = 1+\frac{u}{2}-\frac{u^2}{6}+\cdots .$$

The formula.

$$\psi(1+u) = (1+u)\frac{\log(1+u)}{u} = (1+u)(1-\frac{u}{2}+\frac{u^2}{3}+\cdots) = 1+\frac{u}{2}-\frac{u^2}{6}+\cdots .$$

In fact, we will also take this as a *definition* of the formal power series for ψ in terms of u. The Campbell-Baker-Hausdorff formula says that

$$\log((\exp A)(\exp B)) = A + \int_0^1 \psi\left((\exp \operatorname{ad} A)(\exp t \operatorname{ad} B)\right) B dt. \quad (2)$$

Remarks about the formula.

$$\log((\exp A)(\exp B)) = A + \int_0^1 \psi\left((\exp \operatorname{ad} A)(\exp t \operatorname{ad} B)\right) B\,dt. \quad (2)$$

1. The formula says that we are to substitute

$$u = (\exp \operatorname{ad} A)(\exp t \operatorname{ad} B) - 1$$

into the definition of ψ, apply this operator to the element B and then integrate. In carrying out this computation we can ignore all terms in the expansion of ψ in terms of ad A and ad B where a factor of ad B occurs on the right, since $(\operatorname{ad} B)B = 0$. For example, to obtain the expansion through terms of degree three in the Campbell-Baker-Hausdorff formula, we need only retain quadratic and lower order terms in u, and so

$$\begin{aligned} u &= \operatorname{ad} A + \frac{1}{2}(\operatorname{ad} A)^2 + t \operatorname{ad} B + \frac{t^2}{2}(\operatorname{ad} B)^2 + \cdots \\ u^2 &= (\operatorname{ad} A)^2 + t(\operatorname{ad} B)(\operatorname{ad} A) + \cdots \\ \int_0^1 \left(1 + \frac{u}{2} - \frac{u^2}{6}\right) dt &= 1 + \frac{1}{2}\operatorname{ad} A + \frac{1}{12}(\operatorname{ad} A)^2 - \frac{1}{12}(\operatorname{ad} B)(\operatorname{ad} A) + \cdots, \end{aligned}$$

$$\log((\exp A)(\exp B)) = A + \int_0^1 \psi\left((\exp \mathrm{ad}\ A)(\exp t \mathrm{ad}\ B)\right) B dt. \quad (2)$$

$$\int_0^1 \left(1 + \frac{u}{2} - \frac{u^2}{6}\right) dt \quad = \quad 1 + \frac{1}{2}\mathrm{ad}\ A + \frac{1}{12}(\mathrm{ad}\ A)^2 - \frac{1}{12}(\mathrm{ad}\ B)(\mathrm{ad}\ A) + \cdots$$

where the dots indicate either higher order terms or terms with ad B occurring on the right. So up through degree three (2) gives

$$\log(\exp A)(\exp B) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \cdots$$

agreeing with our preceding computation.

$$\log((\exp A)(\exp B)) = A + \int_0^1 \psi\left((\exp \mathrm{ad}\ A)(\exp t \mathrm{ad}\ B)\right) B dt. \quad (2)$$

2. The meaning of the exponential function on the left hand side of the Campbell-Baker-Hausdorff formula differs from its meaning on the right. On the right hand side, exponentiation takes place in the algebra of endomorphisms of the ring in question. In fact, we will want to make a fundamental reinterpretation of the formula. We want to think of A, B, etc. as elements of a Lie algebra, $\mathbf{g}$. Then the exponentiations on the right hand side of (2) are still taking place in $\mathrm{End}(\mathbf{g})$. On the other hand, if $\mathbf{g}$ is the Lie algebra of a Lie group G, then there is an exponential map: $\exp\colon \mathbf{g} \to G$, and this is what is meant by the exponentials on the left of (2). This exponential map is a diffeomorphism on some neighborhood of the origin in $\mathbf{g}$, and its inverse, log, is defined in some neighborhood of the identity in G. This is the meaning we will attach to the logarithm occurring on the left in (2).

Remarks, continued.

$$\log((\exp A)(\exp B)) = A + \int_0^1 \psi\left((\exp \operatorname{ad} A)(\exp t \operatorname{ad} B)\right) B dt. \quad (2)$$

3. The most crucial consequence of the Campbell-Baker-Hausdorff formula is that it shows that the local structure of the Lie group G (the multiplication law for elements near the identity) is completely determined by its Lie algebra.

4. For example, we see from the right hand side of (2) that group multiplication and group inverse are analytic if we use exponential coordinates.

5. Consider the function τ defined by

$$\tau(w) := \frac{w}{1 - e^{-w}}. \quad (3)$$

This is a familiar function from analysis, as it enters into the Euler-Maclaurin formula, see below. (It is the exponential generating function of $(-1)^k b_k$ where the b_k are the Bernoulli numbers.) Then

$$\psi(z) = \tau(\log z).$$

Proper credit.

6. The formula is named after three mathematicians, Campbell, Baker, and Hausdorff. But this is a misnomer. Substantially earlier than the works of any of these three, there appeared a paper by Friedrich Schur, “Neue Begruendung der Theorie der endlichen Transformationsgruppen,” *Mathematische Annalen* **35** (1890), 161-197. Schur writes down, as convergent power series, the composition law for a Lie group in terms of ”canonical coordinates”, i.e., in terms of linear coordinates on the Lie algebra. He writes down recursive relations for the coefficients, obtaining a version of the formulas we will give below. I am indebted to Prof. Schmid for this reference.

Strategy of the proof.

$$\log((\exp A)(\exp B)) = A + \int_0^1 \psi\left((\exp \mathrm{ad}\ A)(\exp t\mathrm{ad}\ B)\right) B dt. \quad (2)$$

Our strategy for the proof of (2) will be to prove a differential version of it:

$$\frac{d}{dt} \log\left((\exp A)(\exp tB)\right) = \psi\left((\exp \mathrm{ad}\ A)(\exp t\ \mathrm{ad}\ B)\right) B. \quad (4)$$

Since $\log \exp A = A$ when $t = 0$, integrating (4) from 0 to 1 will prove (2). Let us define $\Gamma = \Gamma(t) = \Gamma(t, A, B)$ by

$$\Gamma = \log\left((\exp A)(\exp tB)\right). \quad (5)$$

Then

$$\exp \Gamma = \exp A \exp tB$$

$$\Gamma = \log\left((\exp A)(\exp tB)\right).$$

so

$$\exp\Gamma = \exp A \exp tB$$

and so

$$\begin{aligned}
\frac{d}{dt}\exp\Gamma(t) &= \exp A \frac{d}{dt}\exp tB \\
&= \exp A(\exp tB)B \\
&= (\exp\Gamma(t))B \quad \text{so} \\
(\exp -\Gamma(t))\frac{d}{dt}\exp\Gamma(t) &= B.
\end{aligned}$$

We will prove (4) by finding a general expression for

$$\exp(-C(t))\frac{d}{dt}\exp(C(t))$$

where $C = C(t)$ is a curve in the Lie algebra, $\mathbf{g}$, see (11) below.