Lie Algebras

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Chapter 3

The classical simple algebras

In this chapter we introduce the "classical" finite dimensional simple Lie algebras, which come in four families: the algebras $sl(n+1)$ consisting of all traceless $(n+1)\times(n+1)$ matrices, the orthogonal algebras, on even and odd dimensional spaces (the structure for the even and odd cases are different) and the symplectic algebras (whose definition we will give below). We will prove that they are indeed simple by a uniform method - the method that we used in the preceding chapter to prove that $sl(2)$ is simple. So we axiomatize this method.

3.1 Graded simplicity.

We introduce the following conditions on the Lie algebra $\mathbf{g}$:

$$\begin{aligned}
\mathbf{g} \quad &= \quad \bigoplus_{i=-1}^{\infty} \mathbf{g}_i && (3.1)\\
[\mathbf{g}_i, \mathbf{g}_j] \quad &\subset \quad \mathbf{g}_{i+j} && (3.2)\\
[\mathbf{g}_1, \mathbf{g}_{-1}] \quad &= \quad \mathbf{g}_0 && (3.3)\\
[\mathbf{g}_{-1}, x] = 0 \quad &\Rightarrow \quad x = 0,\ \forall\ x \in \mathbf{g}_i,\ \forall i \geq 0 && (3.4)\\
\text{There exists a } d \in \mathbf{g}_0 \quad &\text{satisfying} \quad [d, x] = kx,\ x \in \mathbf{g}_k,\ \forall k, && (3.5)
\end{aligned}$$

and

$$\mathbf{g}_{-1} \ \text{ is irreducible under the (adjoint) action of } \mathbf{g}_0. \qquad (3.6)$$

Condition (3.4) means that if $x \in \mathbf{g}_i, i \geq 0$ is such that $[y, x] = 0$ for all $y \in \mathbf{g}_{-1}$ then $x = 0$.

We wish to show that any non-zero $\mathbf{g}$ satisfying these six conditions is simple. We know that $\mathbf{g}_{-1}, \mathbf{g}_0$ and $\mathbf{g}_1$ are all non-zero, since $0 \neq d \in \mathbf{g}_0$ by (3.5) and

$[\mathbf{g}_{-1}, \mathbf{g}_1] = \mathbf{g}_0$ by (3.3). So $\mathbf{g}$ can not be the one dimensional commutative algebra, and hence what we must show is that any non-zero ideal I of $\mathbf{g}$ must be all of $\mathbf{g}$.

We first show that any ideal I must be a **graded ideal**, i.e. that

$$I = I_{-1} \oplus I_0 \oplus I_1 \oplus \cdots , \quad \text{where } I_j := I \cap \mathbf{g}_j .$$

Indeed, write any $x \in \mathbf{g}$ as $x = x_{-1} + x_0 + x_1 + \cdots + x_k$ and successively bracket by d to obtain

$$\begin{array}{rcl} x & = & x_{-1} + x_0 + x_1 + \cdots + x_k \\ [d, x] & = & -x_{-1} + 0 + x_1 + \cdots + kx_k \\ [d, [d, x]] & = & x_{-1} + 0 + x_1 + \cdots + k^2 x_k \\ \vdots & \vdots & \vdots \\ (\operatorname{ad} d)^k x & = & (-1)^k x_{-1} + 0 + x_1 + \cdots + k^k x_k \\ (\operatorname{ad} d)^{k+1} x & = & (-1)^{k+1} x_{-1} + 0 + x_1 + \cdots + k^{k+1} x_k . \end{array}$$

The matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & k \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (-1)^k & 0 & 1 & \cdots & k^k \\ (-1)^{k+1} & 0 & 1 & \cdots & k^{k+1} \end{pmatrix}$$

is non singular. Indeed, it is a van der Monde matrix, that is a matrix of the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t_1 & t_2 & 1 & \cdots & t_{k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ t_1^k & t_2^k & 1 & \cdots & t_{k+2}^k \\ t_1^{k+1} & t_2^{k+1} & 1 & \cdots & t_{k+2}^{k+1} \end{pmatrix}$$

whose determinant is

$$\prod_{i<j} (t_i - t_j)$$

and hence non-zero if all the t_j are distinct. Since $t_1 = -1, t_2 = 0, t_3 = 1$ etc. in our case, our matrix is invertible, and so we can solve for each of the components of x in terms of the $(\operatorname{ad} d)^j x$. In particular, if $x \in I$ then all the $(\operatorname{ad} d)^j x \in I$ since I is an ideal, and hence all the component x_j of x belong to I as claimed.

The subspace $I_{-1} \subset \mathbf{g}_{-1}$ is invariant under the adjoint action of $\mathbf{g}_0$ on $\mathbf{g}_{-1}$, and since we are assuming that this action is irreducible, there are two possibilities: $I_{-1} = 0$ or $I_{-1} = \mathbf{g}_{-1}$. We will show that in the first case $I = 0$ and in the second case that $I = \mathbf{g}$.

Indeed, if $I_{-1} = 0$ we will show inductively that $I_j = 0$ for all $j \geq 0$. Suppose $0 \neq y \in \mathbf{g}_0$. Since every element of $[I_{-1}, y]$ belongs to I and to $\mathbf{g}_{-1}$ we conclude that $[\mathbf{g}_{-1}, y] = 0$ and hence that $y = 0$ by (3.4). Thus $I_0 = 0$. Suppose that we know that $I_{j-1} = 0$. Then the same argument shows that any $y \in I_j$ satisfies $[\mathbf{g}_{-1}, y] = 0$ and hence $y = 0$. So $I_j = 0$ for all j, and since I is the sum of all the I_j we conclude that $I = 0$.

Now suppose that $I_{-1} = \mathbf{g}_{-1}$. Then $\mathbf{g}_0 = [\mathbf{g}_{-1}, \mathbf{g}_1] = [I_{-1}, \mathbf{g}_1] \subset I$. Furthermore, since $d \in \mathbf{g}_0 \subset I$ we conclude that $\mathbf{g}_k \subset I$ for all $k \neq 0$ since every element y of such a $\mathbf{g}_k$ can be written as $y = \frac{1}{k}[d, y] \in I$. Hence $I = \mathbf{g}$. QED

For example, the Lie algebra of all polynomial vector fields, where

$$g_k = \{\sum X^i \frac{\partial}{\partial x_i} \quad X^i \text{ homogenous polynomials of degree } k+1\}$$

is a simple Lie algebra. Here d is the Euler vector field

$$d = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}.$$

This algebra is infinite dimensional. We are primarily interested in the finite dimensional Lie algebras.

3.2 $sl(n+1)$

Write the most general matrix in $sl(n+1)$ as

$$\begin{pmatrix} -\operatorname{tr} A & w^* \\ v & A \end{pmatrix}$$

where A is an arbitrary $n \times n$ matrix, v is a column vector and $w^* = (w_1, \ldots, w_n)$ is a row vector. Let $\mathbf{g}_{-1}$ consist of matrices with just the top row, i.e. with $v = A = 0$. Let $\mathbf{g}_1$ consist of matrices with just the left column, i.e. with $A = w^* = 0$. Let $\mathbf{g}_0$ consist of matrices with just the central block, i.e. with $v = w^* = 0$. Let

$$d = \frac{1}{n+1} \begin{pmatrix} -n & 0 \\ 0 & I \end{pmatrix}$$

where I is the $n \times n$ identity matrix. Thus g_0 acts on g_{-1} as the algebra of all endomorphisms, and so g_{-1} is irreducible. We have

$$[\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \begin{pmatrix} 0 & w^* \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} -\langle w^*, v \rangle & 0 \\ 0 & v \otimes w^* \end{pmatrix},$$

where $\langle w^*, v \rangle$ denotes the value of the linear function w^* on the vector v, and this is precisely the trace of the rank one linear transformation $v \otimes w^*$. Thus all our axioms are satisfied. The algebra $sl(n+1)$ is simple.

3.3 The orthogonal algebras.

The algebra $o(2)$ is one dimensional and (hence) commutative. In our (real) Euclidean three dimensional space, the algebra $o(3)$ has a basis X, Y, Z (infinitesimal rotations about each of the axes) with bracket relations

$$[X,Y] = Z,\ [Y,Z] = X,\ [Z,X] = Y,$$

(the usual formulae for "vector product" in three dimensions". But we are over the complex numbers, so can consider the basis $X+iY, -X+iY, iZ$ and find that

$$[iZ, X+iY] = X+iY,\ \ [iZ, -X+iY] = -(-X+iY),\ \ [X+iY, -X+iY] = 2iZ.$$

These are the bracket relations for $sl(2)$ with $e = X+iY,\ f = -X+iY,\ h = iZ$. In other words, the complexification of our three dimensional world is the irreducible three dimensional representation of $sl(2)$ so $o(3) = sl(2)$ which is simple.

To study the higher dimensional orthogonal algebras it is useful to make two remarks:

If V is a vector space with a non-degenerate symmetric bilinear form $(\ ,\ )$, we get an isomorphism of V with its dual space V^* sending every $u \in V$ to the linear function ℓ_u where $\ell_u(v) = (v,u)$. This gives an identification of

$$\text{End}(V) = V \otimes V^* \ \text{ with } \ V \otimes V.$$

Under this identification, the elements of $o(V)$ become identified with the anti-symmetric two tensors, that is with elements of $\wedge^2(V)$. (In terms of an orthonormal basis, a matrix A belongs to $o(V)$ if and only if it is anti-symmetric.)

Explicitly, an element $u \wedge v$ becomes identified with the linear transformation $A_{u\wedge v}$ where

$$A_{u\wedge v} x = (x,v)u - (u,x)v.$$

This has the following consequence. Suppose that $z \in V$ with $(z,z) \neq 0$, and let w be any element of V. Then

$$A_{w\wedge z} z = (z,z)w - (z,w)z$$

and so $U(o(V))z = V$. On the other hand, suppose that $u \in V$ with $(u,u) = 0$. We can find $v \in V$ with $(v,v) = 0$ and $(v,u) = 1$. Now suppose in addition that dim $V \geq 3$. We can then find a $z \in V$ orthogonal to the plane spanned by u and v and with $(z,z) = 1$. Then

$$A_{z\wedge v} u = z,$$

so $z \in U(o(V))e$ and hence $U(o(V))u = V$. We have proved:

1 *If dim $V \geq 3$, then every non-zero vector in V is cyclic, i.e the representation of $o(V)$ on V is irreducible.*

(In two dimensions this is false - the line spanned by a vector e with $(e,e) = 0$ is a one dimensional invariant subspace.)

We now show that

2 *$o(V)$ is simple for dim $V \geq 5$.*

For this, begin by writing down the bracket relations for elements of $o(V)$ in terms of their parametrization by elements of $\wedge^2 V$. Direct computation shows that

$$[A_{u\wedge v}, A_{x\wedge y}] = (v,x)A_{u\wedge y} - (u,x)A_{v\wedge y} - (v,y)A_{u\wedge x} + (u,y)A_{v\wedge x}. \tag{3.7}$$

Now let $n = \dim V - 2$ and choose a basis

$$u, v, x_1, \ldots, x_n$$

of V where

$$(u,u) = (u,x_i) = (v,v) = (v,x_i) = 0 \ \forall i, \ (u,v) = 1, \ (x_i,x_j) = \delta_{ij}.$$

Let $\mathbf{g} := o(V)$ and write W for the subspace spanned by the x_i. Set

$$d := A_{u\wedge v}$$

and

$$\mathbf{g}_{-1} := \{A_{v\wedge x}, \ x \in W\}, \quad \mathbf{g}_0 := o(W) \oplus \mathbf{C}d, \ \mathbf{g}_1 := \{A_{u\wedge x}, \ x \in W\}.$$

It then follows from (3.7) that d satisfies (3.5). The spaces $\mathbf{g}_{-1}$ and $\mathbf{g}_1$ look like copies of W with the $o(W)$ part of $\mathbf{g}_0$ acting as $o(W)$, hence irreducibly since $\dim W \geq 3$. All our remaining axioms are easily verified. Hence $o(V)$ is simple for $\dim V \geq 5$.

We have seen that $o(3) = sl(2)$ is simple.

However $o(4)$ is not simple, being isomorphic to $sl(2) \oplus sl(2)$: Indeed, if Z_1 and Z_2 are vector spaces equipped with non-degenerate anti-symmetric bilinear forms $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ then $Z_1 \otimes Z_2$ has a non-degenerate symmetric bilinear form $(\ ,\ )$ determined by

$$(u_1 \otimes u_2, v_1 \otimes v_2) = \langle u_1, v_1\rangle_1 \langle u_2, v_2\rangle_2.$$

The algebra $sl(2)$ acting on its basic two dimensional representation infinitesimally preserves the antisymmetric form given by

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1y_2 - x_2y_1.$$

Hence, if we take $Z = Z_1 = Z_2$ to be this two dimensional space, we see that $sl(2) \oplus sl(2)$ acts as infinitesimal orthogonal transformations on $Z \otimes Z$ which is four dimensional. But $o(4)$ is six dimensional so the embedding of $sl(2) \oplus sl(2)$ in $o(4)$ is in fact an isomorphism since $3 + 3 = 6$.

3.4 The symplectic algebras.

We consider an even dimensional space with coordinates $q_1, q_2, \ldots, p_1, p_2, \ldots$. The polynomials have a Poisson bracket

$$\{f, g\} := \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \tag{3.8}$$

This is clearly anti-symmetric, and direct computation will show that the Jacobi identity is satisfied. Here is a more interesting proof of Jacobi's identity: Notice that if f is a constant, then $\{f, g\} = 0$ for all g. So in doing bracket computations we can ignore constants. On the other hand, if we take g to be successively $q_1, \ldots, q_n, p_1, \ldots, p_n$ in (3.8) we see that the partial derivatives of f are completely determined by how it brackets with all g, in fact with all linear g. If we fix f, the map

$$h \mapsto \{f, h\}$$

is a **derivation**, i.e. it is linear and satisfies

$$\{f, h_1 h_2\} = \{f, h_1\} h_2 + h_1 \{f, h_2\}.$$

This follows immediately from from the definition (3.8). Now Jacobi's identity amounts to the assertion that

$$\{\{f, g\}, h\} = \{f, \{g, h\}\} - \{g, \{f, h\}\},$$

i.e. that the derivation

$$h \mapsto \{\{f, g\}, h\}$$

is the commutator of the of the derivations

$$h \mapsto \{f, h\} \quad \text{and} \qquad h \mapsto \{g, h\}.$$

It is enough to check this on linear polynomials h, and hence on the polynomials q_j and p_k. If we take $h = q_j$ then

$$\{f, q_j\} = \frac{\partial f}{\partial p_j}, \quad \{g, q_j\} = \frac{\partial g}{\partial p_j}$$

so

$$\begin{aligned}
\{f, \{g, q_j\}\} &= \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial q_i \partial p_j} - \frac{\partial f}{\partial q_i} \frac{\partial^2 g}{\partial p_i \partial p_j} \right) \\
\{f, \{f, q_j\}\} &= \sum \left(\frac{\partial g}{\partial p_i} \frac{\partial^2 f}{\partial q_i \partial p_j} - \frac{\partial g}{\partial q_i} \frac{\partial^2 f}{\partial p_i \partial p_j} \right) \text{ so} \\
\{f, \{g, q_j\}\} - \{g, \{f, q_j\}\} &= \frac{\partial}{\partial p_j} \{f, g\} \\
&= \{\{f, g\}, q_j\}
\end{aligned}$$

as desired, with a similar computation for p_k.

The symplectic algebra $sp(2n)$ is defined to be the subalgebra consisting of all homogeneous quadratic polynomials. We divide these polynomials into three groups as follows: Let $\mathbf{g}_1$ consist of homogeneous polynomials in the q's alone, so $\mathbf{g}_1$ is spanned by the q_iq_j. Let $\mathbf{g}_{-1}$ be the quadratic polynomials in the p's alone, and let $\mathbf{g}_0$ be the mixed terms, so spanned by the q_ip_j. It is easy to see that $\mathbf{g}_0 \sim gl(n)$ and that $[\mathbf{g}_{-1}, \mathbf{g}_1] = \mathbf{g}_0$. To check that $\mathbf{g}_{-1}$ is irreducible under $\mathbf{g}_0$, observe that $[p_1q_j, p_kp_\ell] = 0$ if $j \neq k$ or ℓ, and $[p_1q_j, p_jp_\ell]$ is a multiple of p_1p_ℓ. So we can by one or two brackets carry any non-zero element of $\mathbf{g}_{-1}$ into a non-zero multiply of p_1^2, and then get any monomial from p_1^2 by bracketing with p_iq_1 appropriately. The element d is given by $\frac{1}{2}(p_1q_1 + \cdots + p_nq_n)$.

We have shown that the symplectic algebra is simple, but we haven't really explained what it is. Consider the space of V of homogenous linear polynomials, i.e all polynomials of the form

$$\ell = a_1q_1 + \cdots + a_nq_n + b_1p_q + \cdots + b_np_n.$$

Define an anti-symmetric bilinear form ω on V by setting

$$\omega(\ell, \ell') := \{\ell, \ell').$$

From the formula (3.8) it follows that the Poisson bracket of two linear functions is a constant, so ω does indeed define an antisymmetric bilinear form on V, and we know that this bilinear form is non-degenerate. Furthermore, if f is a homogenous quadratic polynomial, and ℓ is linear, then $\{f, \ell\}$ is again linear, and if we denote the map

$$\ell \mapsto \{f, \ell\}$$

by $A = A_f$, then Jacobi's identity translates into

$$\omega(A\ell, \ell') + \omega(\ell A\ell') = 0 \tag{3.9}$$

since $\{\ell, \ell'\}$ is a constant. Condition (3.9) can be interpreted as saying that A belongs to the Lie algebra of the group of all linear transformations R on V which preserve ω, i.e. which satisfy

$$\omega(R\ell, R\ell') = \omega(\ell, \ell').$$

This group is known as the symplectic group. The form ω induces an isomorphism of V with V^* and hence of $\mathrm{Hom}(V, V) = V \otimes V^*$ with $V \otimes V$, and this time the image of the set of A satisfying (3.9) consists of all symmetric tensors of degree two, i.e. of $S^2(V)$. (Just as in the orthogonal case we got the anti-symmetric tensors). But the space $S^2(V)$ is the same as the space of homogenous polynomials of degree two. In other words, the symplectic algebra as defined above is the same as the Lie algebra of the symplectic group.

It is an easy theorem in linear algebra, that if V is a vector space which carries a non-degenerate anti-symmetric bilinear form, then V must be even dimensional, and if dim $V = 2n$ then it is isomorphic to the space constructed above. We will not pause to prove this theorem.

3.5 The root structures.

We are going to choose a basis for each of the classical simple algebras which generalizes the basis e, f, h that we chose for $sl(2)$. Indeed, for each classical simple algebra $\mathbf{g}$ we will first choose a maximal commutative subalgebra $\mathbf{h}$ all of whose elements are semi-simple = diagonizable in the adjoint representation. Since the adjoint action of all the elements of $\mathbf{h}$ commute, this means that they can be simultaneously diagonalized. Thus we can decompose $\mathbf{g}$ into a direct sum of simultaneous eigenspaces

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha} \mathbf{g}_\alpha \tag{3.10}$$

where $0 \neq \alpha \in \mathbf{h}^*$ and

$$\mathbf{g}_\alpha := \{x \in \mathbf{g} | [h, x] = \alpha(h)x \ \forall \ h \in \mathbf{h}\}.$$

The linear functions α are called **roots** (originally because the $\alpha(h)$ are roots of the characteristic polynomial of $\mathrm{ad}(h)$). The simultaneous eigenspace $\mathbf{g}_\alpha$ is called the root space corresponding to α. The collection of all roots will usually be denoted by Φ.

Let us see how this works for each of the classical simple algebras.

3.5.1 $A_n = sl(n+1)$.

We choose $\mathbf{h}$ to consist of the diagonal matrices in the algebra $sl(n+1)$ of all $(n+1) \times (n+1)$ matrices with trace zero. As a basis of $\mathbf{h}$ we take

$$\begin{aligned}
h_1 &:= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
h_2 &:= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
\vdots &:= \vdots \\
h_n &:= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.
\end{aligned}$$

Let L_i denote the linear function which assigns to each diagonal matrix its i-th (diagonal) entry, Let E_{ij} denote the matrix with one in the i, j position and zero's elsewhere. Then

$$[h, E_{ij}] = (L_i(h) - L_j(h))E_{ij} \quad \forall \ \ h \in \mathbf{h}$$

so the linear functions of the form

$$L_i - L_j, \quad i \neq j$$

are the roots.

We may subdivide the set of roots into two classes: the **positive roots**

$$\Phi^+ := \{L_i - L_j;\ i < j\}$$

and the **negative roots**

$$\Phi^- := -\Phi^+ = \{L_j - L_i,\ i < j\}.$$

Every root is either positive or negative. If we define

$$\alpha_i := L_i - L_{i+1}$$

then every positive root can be written as a sum of the α_i:

$$L_i - L_j = \alpha_i + \cdots + \alpha_{j-1}.$$

We have

$$\alpha_i(h_i) = 2,$$

and for $i \neq j$

$$\alpha_i(h_{i\pm 1}) = -1, \quad \alpha_i(h_j) = 0,\ j \neq i \pm 1. \tag{3.11}$$

The elements

$$E_{i,i+1},\ h_i,\ E_{i+1,i}$$

form a subalgebra of $sl(n+1)$ isomorphic to $sl(2)$. We may call it $sl(2)_i$.

3.5.2 $C_n = sp(2n), n \geq 2$.

Let $\mathbf{h}$ consist of all linear combinations of $p_1q_1, \ldots, p_nq_n$ and let L_i be defined by

$$L_i\left(a_1p_1q_1 + \cdots + a_np_nq_n\right) = a_i$$

so $L_1, \ldots, L_n$ is the basis of $\mathbf{h}^*$ dual to the basis $p_1q_1, \ldots, p_nq_n$ of $\mathbf{h}$.

If $h = a_1p_1q_1 + \cdots + a_np_nq_n$ then

$$\begin{aligned} [h, q^iq^j] &= (a_i + a_j)q^iq^j \\ [h, q^ip^j] &= (a_i - a_j)q^ip^j \\ [h, p^ip^j] &= -(a_i + a_j)p^ip^j \end{aligned}$$

so the roots are

$$\pm(L_i + L_j) \text{ all } i, j \quad \text{and} \quad L_i - L_j \; i \neq j.$$

We can divide the roots Φ into positive and negative roots by setting

$$\Phi^+ = \{L_i + L_j\}_{\text{all } ij} \cup \{L_i - L_j\}_{i<j}.$$

If we set

$$\alpha_1 := L_1 - L_2, \dots, \alpha_{n-1} := L_{n-1} - L_n, \alpha_n := 2L_n$$

then every positive root is a sum of the α_i. Indeed, $L_{n-1} + L_n = \alpha_{n-1} + \alpha_n$ and $2L_{n-1} = 2\alpha_{n-1} + \alpha_n$ and so on. In particular $2\alpha_{n-1} + \alpha_n$ is a root.

If we set

$$h_1 := p_1q_1 - p_2q_2, \dots, h_{n-1} := p_{n-1}q_{n-1} - p_nq_n, \; h_n := p_nq_n$$

then

$$\alpha_i(h_i) = 2$$

while for $i \neq j$

$$\begin{aligned} \alpha_i(h_{i\pm1}) &= -1, \quad i = 1, \dots, n-1 \\ \alpha_i(h_j) &= 0, \; j \neq i \pm 1, i = 1, \dots, n \\ \alpha_n(h_{n-1}) &= -2. \end{aligned} \tag{3.12}$$

In particular, the elements $h_i, q_ip_{i+1}, q_{i+1}p_i$ for $i = 1, \dots, n-1$ form a subalgebra isomorphic to $sl(2)$ as do the elements $h_n, \frac{1}{2}q_n^2, -\frac{1}{2}p_n^2$. We call these subalgebras $sl(2)_i, \; i = 1, \dots, n$.

3.5.3 $D_n = o(2n), \; n \geq 3$.

We choose a basis $u_1, \dots, u_n, v_1, \dots, v_n$ of our orthogonal vector space V such that

$$(u_i, u_j) = (v_i, v_j) = 0, \forall \; i, j, \quad (u_i, v_j) = \delta_{ij}.$$

We let $\mathbf{h}$ be the subalgebra of $o(V)$ spanned by the $A_{u_iv_i}$, $i = 1, \dots, n$. Here we have written A_{xy} instead of $A_{x\wedge y}$ in order to save space. We take

$$A_{u_1v_1}, \dots, A_{u_nv_n}$$

as a basis of $\mathbf{h}$ and let $L_1, \dots, L_n$ be the dual basis. Then

$$\pm L_k \pm L_\ell \; k \neq \ell$$

are the roots since from (3.7) we have

$$\begin{aligned} [A_{u_iv_i}, A_{u_ku_\ell}] &= (\delta_{ik} + \delta_{i\ell})A_{u_ku_\ell} \\ [A_{u_iv_i}, A_{u_kv_\ell}] &= (\delta_{ik} - \delta_{i\ell})A_{u_kv_\ell} \\ [A_{u_iv_i}, A_{v_kv_\ell}] &= -(\delta_{ik} + \delta_{i\ell})A_{v_kv_\ell}. \end{aligned}$$

We can choose as positive roots the

$$L_k + L_\ell, L_k - L_\ell, \quad k < \ell$$

and set

$$\alpha_i := L_i - L_{i+1},\ i = 1, \ldots, n-1, \quad \alpha_n := L_{n-1} + L_n.$$

Every positive root is a sum of these simple roots. If we set

$$h_i := A_{u_i v_i} - A_{u_{i+1} v_{i+1}}, i = 1, \ldots n-1,$$

and

$$h_n = A_{u_{n-1} v_{n-1}} + A_{u_n v_n}$$

then

$$\alpha_i(h_i) = 2$$

and for $i \neq j$

$$\begin{array}{rcl} \alpha_i(h_j) & = & 0 \ \ j \neq i \pm 1,\ i = 1, \ldots n-2 \\ \alpha_i(h_{i\pm 1}) & = & -1 \ \ i = 1, \ldots, n-2 \\ \alpha_{n-1}(h_{n-2}) & = & -1 \\ \alpha_n(h_{n-2}) & = & -1 \\ \alpha_n(h_{n-1}) & = & 0. \end{array} \tag{3.13}$$

For $i = 1, \ldots, n-1$ the elements $h_i, A_{u_i v_{i+1}}, A_{u_{i+1} v_i}$ form a subalgebra isomorphic to $sl(2)$ as do $h_n, A_{u_{n-1} u_n}, A_{v_{n-1} v_n}$.

3.5.4 $B_n = o(2n+1)\ n \geq 2$.

We choose a basis $u_1, \ldots, u_n, v_1, \ldots, v_n, x$ of our orthogonal vector space V such that

$$(u_i, u_j) = (v_i, v_j) = 0, \forall\ i, j, \quad (u_i, v_j) = \delta_{ij},$$

and

$$(x, u_i) = (x, v_i) = 0\ \forall\ i, (x, x) = 1.$$

As in the even dimensional case we let **h** be the subalgebra of $o(V)$ spanned by the $A_{u_i v_i}$, $i = 1, \ldots, n$ and take

$$A_{u_1 v_1}, \ldots, A_{u_n v_n}$$

as a basis of **h** and let $L_1, \ldots, L_n$ be the dual basis. Then

$$\pm L_i \pm L_j\ i \neq j, \pm L_i$$

are roots. We take

$$L_i \pm L_j,\ 1 \leq i < j \leq n,\ \text{together with}\ L_i,\ i = 1, \ldots, n$$

to be the positive roots, and

$$\alpha_i := L_i - L_{i+1},\ i = 1, \ldots, n-1,\ \alpha_n := L_n$$

to be the simple roots. We let

$$h_i := A_{u_i v_i} - A_{u_{i+1} v_{i+1}}, i = 1, \ldots n-1,$$

as in the even case, but set

$$h_n := 2A_{u_n v_n}.$$

Then every positive root can be written as a sum of the simple roots,

$$\alpha_i(h_i) = 2,\ i = 1, \ldots n,$$

and for $i \neq j$

$$\begin{aligned} \alpha_i(h_j) &= 0 \quad j \neq i \pm 1,\ i = 1, \ldots n \\ \alpha_i(h_{i\pm1}) &= -1 \quad i = 1, \ldots, n-2, n \\ \alpha_{n-1}(h_n) &= -2 \end{aligned} \tag{3.14}$$

Notice that in this case $\alpha_{n-1} + 2\alpha_n = L_{n-1} + L_n$ is a root. Finally we can construct subalgebras isomorphic to $sl(2)$, with the first $n-1$ as in the even orthogonal case and the last $sl(2)$ spanned by $h_n, A_{u_n x}, -A_{v_n x}$.

3.5.5 Diagrammatic presentation.

The information of the last four subsections can be summarized in each of the following four diagrams:

The way to read this diagram is as follows: each node in the diagram stands for a simple root, reading from left to right, starting with α_1 at the left. (In the diagram D_ℓ the two rightmost nodes are $\alpha_{\ell-1}$ and α_ℓ, say the top $\alpha_{\ell-1}$ and the bottom α_ℓ.) Two nodes α_i and α_j are connected by (one or more) edges if and only if $\alpha_i(h_j) \neq 0$.

In all cases, the difference, $\alpha_i - \alpha_j$ is never a root, and, for $i \neq j,\ \alpha_i(h_j) \leq 0$ and is an integer. If, for $i \neq j$, $\alpha_i(h_j) < 0$ then $\alpha_i + \alpha_j$ is a root.

In two of the cases (B_ℓ and C_ℓ) it happens that $\alpha_i(h_j) = -2$. Then $\alpha_i + \alpha_j$ and $\alpha_i + 2\alpha_j$ are roots, and we draw a double bond with an arrow pointing towards α_j. In this case 2 is is the maximum integer such that $\alpha_i + k\alpha_j$ is a root. In all other cases, this maximum integer k is one if the nodes are connected (and zero it they are not).

3.6 Low dimensional coincidences.

We have already seen that $o(4) \sim sl(2) \oplus sl(2)$. We also have

$$o(6) \sim sl(4).$$

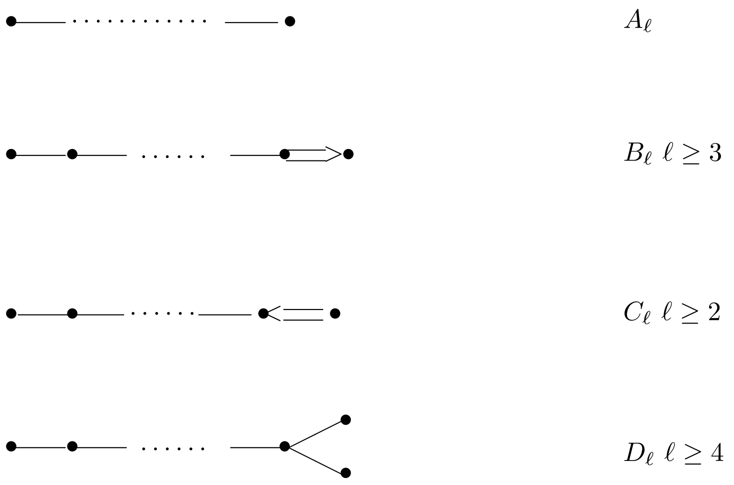

Figure 3.1: Dynkin diagrams of the classical simple algebras.

Both algebras are fifteen dimensional and both are simple. So to realize this isomorphism we need only find an orthogonal representation of $sl(4)$ on a six dimensional space. If we let $V = \mathbf{C}^4$ with the standard representation of $sl(4)$, we get a representation of $sl(4)$ on $\wedge^2(V)$ which is six dimensional. So we must describe a non-degenerate bilinear form on $\wedge^2 V$ which is invariant under the action of $sl(4)$. We have a map, wedge product, of

$$\wedge^2 V \times \wedge^2 V \rightarrow \wedge^4 V.$$

Furthermore this map is symmetric, and invariant under the action of $gl(4)$. However $sl(4)$ preserves a basis (a non-zero element) of $\wedge^4 V$ and so we may identify $\wedge^4 V$ with $\mathbf{C}$. It is easy to check that the bilinear form so obtained is non-degenerate

We also have the identification

$$sp(4) \sim o(5)$$

both algebras being ten dimensional. To see this let $V = \mathbf{C}^4$ with an antisymmetric form ω preserved by $Sp(4)$. Then $\omega \otimes \omega$ induces a symmetric bilinear form on $V \otimes V$ as we have seen. Sitting inside $V \otimes V$ as an invariant subspace is $\wedge^2 V$ as we have seen, which is six dimensional. But $\wedge^2 V$ is not irreducible as a representation of $sp(4)$. Indeed, $\omega \in \wedge^2 V^*$ is invariant, and hence its kernel is a five dimensional subspace of $\wedge^2 V$ which is invariant under $sp(4)$. We thus get a non-zero homomorphism $sp(4) \rightarrow o(5)$ which must be an isomorphism since $sp(4)$ is simple.

These coincidences can be seen in the diagrams. If we were to allow $\ell = 2$ in the diagram for B_ℓ it would be indistinguishable from C_2. If we were to allow $\ell = 3$ in the diagram for D_ℓ it would be indistinguishable from A_3.

3.7 Extended diagrams.

It follows from Jacobi's identity that in the decomposition (3.10), we have

$$[\mathbf{g}_\alpha, \mathbf{g}_{\alpha'}] \subset \mathbf{g}_{\alpha+\alpha'} \tag{3.15}$$

with the understanding that the right hand side is zero if $\alpha + \alpha'$ is not a root. In each of the cases examined above, every positive root is a linear combination of the simple roots with non-negative integer coefficients. Since the algebra is finite, there must be a **maximal** positive root β in the sense that $\beta + \alpha_i$ is not a root for any simple root. For example, in the case of $A_n = sl(n+1)$, the root $\beta := L_1 - L_{n+1}$ is maximal. The corresponding $\mathbf{g}_\beta$ consists of all $(n+1)\times(n+1)$ matrices with zeros everywhere except in the upper right hand corner. We can also consider the **minimal root** which is the negative of the maximal root, so

$$\alpha_0 := -\beta = L_{n+1} - L_1$$

in the case of A_n. Continuing to study this case, let

$$h_0 := h_{n+1} - h_1.$$

Then we have

$$\alpha_i(h_i) = 2,\ i = 0, \dots n$$

and

$$\alpha_0(h_1) = \alpha_0(h_n) = -1, \quad \alpha_0(h_i) = 0, \quad i \neq 0, 1, n.$$

This means that if we write out the $(n+1)\times(n+1)$ matrix whose entries are $\alpha_i(h_j),\ i, j = 0, \dots n$ we obtain a matrix of the form

$$2I - M$$

where $M_{ij} = 1$ if and only if $j = \pm 1$ with the understanding that $n + 1 = 0$ and $-1 = n$, i.e we do the subscript arithmetic mod n. In other words, M is the adjacency matrix of the cyclic graph with $n + 1$ vertices labeled $0, \dots n$. Also, we have

$$h_0 + h_1 + \cdots + h_n = 0.$$

If we apply α_i to this equation for $i = 0, \dots n$ we obtain

$$(2I - M)\mathbf{1} = 0,$$

where $\mathbf{1}$ is the column vector all of whose entries are 1. We can write this equation as

$$M\mathbf{1} = \mathbf{2}\mathbf{1}.$$

In other words, **1** is an eigenvector of M with eigenvalue 2.

In the chapters that follow we shall see that any finite dimensional simple Lie algebra has roots, simple roots, maximal roots etc. giving rise to a matrix M with integer entries which is irreducible (in the sense of non-negative matrices - definition later on) and which has an eigenvector with positive (integer) entries with eigenvalue 2. This will allow us to classify the simple (finite dimensional) Lie algebras.