Lie Algebras

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Chapter 10

Kostant's Dirac operator.

Let $\mathbf{p}$ be a vector space with a non-degenerate symmetric bilinear form. We have the Clifford algebra $C(\mathbf{p})$ and the identification of $o(\mathbf{p}) = \wedge^2(\mathbf{p})$ inside $C(\mathbf{p})$.

10.1 Antisymmetric trilinear forms.

Let ϕ be an antisymmetric trilinear form on $\mathbf{p}$. Then ϕ defines an antisymmetric map

$$b = b_\phi : \mathbf{p} \otimes \mathbf{p} \to \mathbf{p}$$

by the formula

$$(b(y, y'), y'') = \phi(y, y', y'') \ \ \forall\, y, y', y'' \in \mathbf{p}.$$

This bilinear map "leaves (,) invariant" in the sense that

$$(b(y, y'), y'') = (y, b(y', y'')).$$

Conversely, any antisymmetric map $b : \mathbf{p} \otimes \mathbf{p} \to \mathbf{p}$ satisfying this condition defines an antisymmetric form ϕ. Finally either of these two objects defines an element $v \in \wedge^3\mathbf{p}$ by

$$-2(v, y \wedge y' \wedge y'') = (b(y, y'), y'') = \phi(y, y', y''). \tag{10.1}$$

We can write this relation in several alternative ways: Since

$$-2(v, y \wedge y' \wedge y'') = -2(\iota(y')\iota(y)v, y'') = 2(\iota(y)\iota(y')v, y'')$$

we have

$$b(y, y') = 2\iota(y)\iota(y')v. \tag{10.2}$$

Also, $\iota(y)v \in \wedge^2\mathbf{p}$ and so is identified with an element of $o(\mathbf{p})$ by commutator in the Cliford algebra:

$$\mathrm{ad}(\iota(y)v)(y') = [\iota(y)v, y'] = -2\iota(y')\iota(y)v$$

so

$$\mathrm{ad}(\iota(y)v)(y') = [\iota(y)v, y'] = b(y, y'). \tag{10.3}$$

10.2 Jacobi and Clifford.

Given an antisymmetric bilinear map $b : \mathbf{p} \otimes \mathbf{p} \to \mathbf{p}$ we may define

$$\mathrm{Jac}(b) : \mathbf{p} \otimes \mathbf{p} \otimes \mathbf{p} \to \mathbf{p}$$

by

$$\mathrm{Jac}(b)(y, y', y'') = b(b(y, y'), y'') + b(b(y', y''), y) + b(b(y'', y), y')$$

so that the vanishing of $\mathrm{Jac}(b)$ is the usual Jacobi identity. It is easy to check that $\mathrm{Jac}(b)$ is antisymmetric and that if b satisfies $(b(y, y'), y'') = (y, b(y', y''))$ then the four form

$$y, y', y'', y''' \mapsto (\mathrm{Jac}(b)(y, y', y''), y''')$$

is antisymmetric. We claim that if $v \in \wedge^3\mathbf{p}$ as in the preceding subsection, then

$$\iota(y'')\iota(y')\iota(y)v^2 = \frac{1}{2}\,\mathrm{Jac}(b)(y, y', y''). \tag{10.4}$$

To prove this observe that

$$\begin{aligned}
\iota(y)v^2 &= (\iota(y)v)v - v(\iota(y)v)\\
\iota(y')\iota(y)v^2 &= (\iota(y')\iota(y)v)v + (\iota(y)v)(\iota(y')v) - (\iota(y')v)(\iota(y)v) + v(\iota(y')\iota(y)v)\\
\iota(y'')\iota(y')\iota(y)v^2 &= -(\iota(y')\iota(y)v)\iota(y'')v + (\iota(y'')v)(\iota(y')\iota(y)v) + (\iota(y'')\iota(y)v)\iota(y')v\\
&\quad +(\iota(y)v)(\iota(y'')\iota(y')v - (\iota(y'')\iota(y')v)(\iota(y)v) - (\iota(y')v)(\iota(y'')\iota(y)v))\\
&= [\iota(y'')v, \iota(y')\iota(y)v] + [\iota(y')v, \iota(y)\iota(y'')v] + [\iota(y)v, \iota(y')\iota(y'')v]\\
&= \frac{1}{2}\,\mathrm{Jac}(b)(y, y', y'')
\end{aligned}$$

by (10.2) and (10.3).

Equation (10.4) describes the degree four component of v^2 in terms of $\mathrm{Jac}(b)$. We can be explicit about the degree zero component of v^2. We claim that

$$(v^2)_0 = \frac{1}{24}\,\mathrm{tr}\sum_{j=1}^{n}[y \to \epsilon_j b(y_j, b(y_j, y))], \quad \epsilon_j := (y_j, y_j). \tag{10.5}$$

Indeed, by (**??**) we know that $(v^2)_0 = -(v, v)$ and since $y_i \wedge y_j \wedge y_k,\ i < j < k$

form an "orthonormal" basis of $\wedge^3\mathbf{p}$ we have

$$\begin{aligned}
-(v,v) &= -\sum_{1\le i<j<k\le n} \pm(v, y_i\wedge y_j\wedge y_k)^2,\ \pm=\epsilon_i\epsilon_j\epsilon_k\\
&= -\frac{1}{6}\sum_{i=1,j=1,k=1}^{n,n,n} \pm(v, y_i\wedge y_j\wedge y_k)^2\\
&= -\frac{1}{6}\sum_{i=1,j=1,k=1}^{n,n,n} \pm(\iota(y_k)\iota(y_j)v, y_i)^2\\
&= -\frac{1}{24}\sum_{i=1,j=1,k=1}^{n,n,n} \pm(b(y_j,y_k), y_i)^2\\
&= -\frac{1}{24}\sum_{j=1,k=1}^{n,n} \epsilon_j\epsilon_k(b(y_j,y_k), b(y_j,y_k))\\
&= \frac{1}{24}\sum_{j=1,k=1}^{n,n} \epsilon_j\epsilon_k(b(y_j,b(y_j,y_k)), y_k)
\end{aligned}$$

proving (10.5).

10.3 Orthogonal extension of a Lie algebra.

Let us get back to the general case of a Lie algebra $\mathbf{r}$ acting as infinitesimal orthogonal transformations on $\mathbf{p}$ and the map $\nu:\mathbf{r}\to\wedge^2\mathbf{p}$ given by (**??**). Suppose that the Lie algebra $\mathbf{r}$ has a non-degenerate invariant symmetric bilinear form $(\ ,\ )_{\mathbf{r}}$. We have the transpose map

$$\nu^\dagger:\wedge^2\mathbf{p}\to\mathbf{r}$$

since both $\mathbf{r}$ and $\wedge^2\mathbf{p}$ have non-degenerate symmetric bilinear forms. For y and y' in $\mathbf{p}$, let us define

$$[y,y']_{\mathbf{r}} := -2\nu^\dagger(y\wedge y'),.$$

This map is an $\mathbf{r}$ morphism which says that

$$[x,[y,y']_{\mathbf{r}}] = [x\cdot y, y']_{\mathbf{r}} + [y, x\cdot y']_{\mathbf{r}}, \tag{10.6}$$

where the bracket on the left denotes the Lie bracket on $\mathbf{r}$. Also, we have

$$\begin{aligned}
(x,[y,y']_{\mathbf{r}})_{\mathbf{r}} &= -2(x,\nu^\dagger(y\wedge y'))_{\mathbf{r}}\\
&= -2(\nu(x), y\wedge y')_{\mathbf{p}}\\
&= -2(\iota(y)\nu(x), y')_{\mathbf{p}}\\
&= (x\cdot y, y')_{\mathbf{p}}.
\end{aligned}$$

So we have proved

$$(x,[y,y']_{\mathbf{r}})_{\mathbf{r}} = (x\cdot y,y')_{\mathbf{p}}. \tag{10.7}$$

This has the following significance: Suppose that we want to make $\mathbf{r}\oplus\mathbf{p}$ into a Lie algebra with an invariant symmetric bilinear form $(\ ,\ )$ such that

- $\mathbf{r}$ and $\mathbf{p}$ are orthogonal under $(\ ,\ )$,
- the restriction of $(\ ,\ )$ to $\mathbf{r}$ is $(\ ,\ )_{\mathbf{r}}$ and the restriction of $(\ ,\ )$ to $\mathbf{p}$ is $(\ ,\ )_{\mathbf{p}}$, and
- $[\mathbf{r},\mathbf{p}]\subset\mathbf{p}$ and the bracket of an element of $\mathbf{r}$ with an element of $\mathbf{p}$ is given by $[x,y]=x\cdot y$.

Then

$$\text{the } \mathbf{r} \text{ component of } [y,y'] \text{ must be given by } [y,y']_{\mathbf{r}}.$$

Thus to define a Lie algebra structure on $\mathbf{r}\oplus\mathbf{p}$ we must specify the $\mathbf{p}$ component of the bracket of two elements of $\mathbf{p}$. This amounts to specifying a $v\in\wedge^3\mathbf{p}$ as we have seen, and the condition that the Jacobi identity hold for x,y,y' with $x\in\mathbf{r}$ and $y,y'\in\mathbf{p}$ amounts to the condition that $v\in\wedge^3\mathbf{p}$ be invariant under the action of $\mathbf{r}$. It then follows that if we try to define $[\ ,\ ]=[\ ,\ ]_v$ by

$$[y,y'] = [y,y']_{\mathbf{r}} + 2\iota(y)\iota(y')v$$

then

$$([z,z'],z'') = (z,[z',z''])$$

for any three elements of $\mathbf{g}:=\mathbf{r}\oplus\mathbf{p}$, and the Jacobi identity is satisfied if at least one of these elements belongs to $\mathbf{r}$. Furthermore, for any $x\in\mathbf{r}$ we have

$$\begin{aligned}
([[y,y'],y''],x) &= ([[y,y'],y''],x)_{\mathbf{r}}\\
&= ([y,y'],[y'',x])_{\mathbf{p}} \quad \text{by (10.7)}\\
&= ([x,[y,y']],y'')_{\mathbf{p}}\\
&= ([[x,y],y'],y'')_{\mathbf{p}} + ([y,[x,y']],y'')_{\mathbf{p}}\\
&= ([x,y],[y',y''])_{\mathbf{p}} + ([x,y'],[y'',y])_{\mathbf{p}}\\
&= (x,[y,[y',y'']])_{\mathbf{r}} + (x,[y',[y'',y]])_{\mathbf{r}}
\end{aligned}$$

or

$$([[y,y'],y'']+[[y',y''],y]+[[y'',y],y'],x)=0.$$

In other words, the $\mathbf{r}$ component of the Jacobi identity holds for three elements of $\mathbf{p}$.

So what remains to be checked is the $\mathbf{p}$ component of the Jacobi identity for three elements of $\mathbf{p}$. This is the sum

$$\mathrm{Jac}(b)(y,y',y'') + [y,y']_{\mathbf{r}}\cdot y'' + [y',y'']_{\mathbf{r}}\cdot y + [y'',y]_{\mathbf{r}}\cdot y'.$$

Let us choose an"orthonormal" basis $\{x_i\},\ i=1,\dots,r$ of $\mathbf{r}$ and write

$$[y,y']_{\mathbf{r}}=\sum_i \epsilon_i([y,y'],x_i)x_i,\quad \epsilon_i:=(x_i,x_i)_{\mathbf{r}}=\pm 1$$

so

$$[y,y']_{\mathbf{r}}\cdot y''=\sum_i \epsilon_i([y,y']_{\mathbf{r}},x_i)x_i\cdot y''.$$

Then by (**??**) and (10.4) we see that the Jacobi identity is

$$\iota(y)\iota(y')\iota(y'')\left(v^2+\nu(\mathrm{Cas}_{\mathbf{r}})\right)=0$$

where

$$\mathrm{Cas}_{\mathbf{r}}:=\sum_i \epsilon_i x_i^2\in U(\mathbf{r})$$

does not depend on the choice of basis, and $\nu: U(\mathbf{r})\to C(\mathbf{p})$ is the extension of the homomorphism $\nu:\mathbf{r}\to C(\mathbf{p})$. In particular, we have proved that v defines an extension of the Lie algebra structure satisfying our condition if and only if

$$v^2+\nu(\mathrm{Cas}_{\mathbf{r}})\in\mathbf{C} \tag{10.8}$$

i.e. has no component of degree four.

Suppose that this condition holds. We then have defined a Lie algebra structure on

$$\mathbf{g}=\mathbf{r}\oplus\mathbf{p}.$$

We let $P_{\mathbf{r}}$ and $P_{\mathbf{p}}$ denote projections onto the first and second components of our decomposition. Our Lie bracket on $\mathbf{g}$, denoted simply by [,] satisfies

$$\begin{aligned}
[x,x'] &= [x,x']_{\mathbf{r}},\quad x,x'\in\mathbf{r} &(10.9)\\
[x,y] &= x\cdot y,\quad x\in\mathbf{r},y\in\mathbf{p} &(10.10)\\
P_{\mathbf{r}}[y,y'] &= [y,y']_{\mathbf{r}}=-2\nu^\dagger(y\wedge y')\quad y,y'\in\mathbf{p} &(10.11)\\
P_{\mathbf{p}}[y,y'] &= b(y,y')=2\iota(y)\iota(y')v,\quad y,y'\in\mathbf{p}. &(10.12)
\end{aligned}$$

From now on we will assume that we are over the complex numbers or that we are over the reals and the symmetric bilinear forms are positive definite. This is not for any mathematical reasons but because the formulas become a bit complicated if we put in all the signs. We leave the general case to the reader.

10.4 The value of $[v^2+\nu(\mathrm{Cas}_{\mathbf{r}})]_0$.

Condition (10.8) says that the degree four component of $v^2+\nu(\mathrm{Cas}_{\mathbf{r}})$ vanishes. Assume that this holds, so we have constructed a Lie algebra. We will now compute the degree zero component of $v^2+\nu(\mathrm{Cas}_{\mathbf{r}})$. The answer will be given in equation (10.13) below.

We can write (10.5) as

$$(v^2)_0 = \frac{1}{24}\operatorname{tr} P_{\mathbf{p}} \sum_{j=1}^{n} \operatorname{ad}_{\mathbf{g}}(y_j) P_{\mathbf{p}} \operatorname{ad}_{\mathbf{g}}(y_j) P_{\mathbf{p}}$$

$$= \frac{1}{24}\operatorname{tr} P_{\mathbf{p}} \sum_{j=1}^{n} \operatorname{ad}_{\mathbf{g}}(y_j) P_{\mathbf{p}} \operatorname{ad}_{\mathbf{g}}(y_j)$$

in view of (10.12) where $\operatorname{ad}_{\mathbf{g}}$ denotes the adjoint action on all of $\mathbf{g}$. On the other hand, we have from (**??**) that

$$\nu(\mathrm{Cas}_{\mathbf{r}})_0 = \frac{1}{8}\operatorname{tr}\sum_i (\operatorname{ad} x_i)^2 P_{\mathbf{p}} = \frac{1}{8}\sum_{i=1}^{r}\sum_{j=1}^{n}([x_i,[x_i,y_j]],y_j).$$

We can rewrite this sum as

$$\frac{1}{8}\sum_{i=1,j=1}^{r,n}([x_i,y_j],[y_j,x_i])$$

which equals $\frac{1}{8}\sum_{i=1,j=1,k=1}^{r,n,n}([x_i,y_j],y_k)(y_k,[y_j,x_i])$. But this equals

$$\begin{aligned} & \frac{1}{8}\sum_{i=1,j=1,k=1}^{r,n,n}(x_i,[y_j,y_k])([y_k,y_j],x_i) \\ = \quad & \frac{1}{8}\sum_{j=1,k=1}(P_{\mathbf{r}}[y_j,y_k],P_{\mathbf{r}}[y_k,y_j]) \\ = \quad & \frac{1}{8}\sum_{j=1,k=1}^{n,n}(P_{\mathbf{r}}[y_j,y_k],[y_k,y_j]) \\ = \quad & \frac{1}{8}\sum_{j=1,k=1}^{n,n}([y_j,P_{\mathbf{r}}[y_j,y_k]],y_k) \\ = \quad & \frac{1}{8}\operatorname{tr} P_{\mathbf{p}}\sum_{j=1}^{n}\operatorname{ad}_{\mathbf{g}}(y_j)P_{\mathbf{r}}\operatorname{ad}_{\mathbf{g}}(y_j)P_{\mathbf{p}}. \end{aligned}$$

In other words

$$\nu(\mathrm{Cas}_{\mathbf{r}})_0 = \frac{1}{8}\operatorname{tr} P_{\mathbf{p}}\sum_{j=1}^{n}\operatorname{ad}_{\mathbf{g}}(y_j)P_{\mathbf{r}}\operatorname{ad}_{\mathbf{g}}(y_j)P_{\mathbf{p}} = \frac{1}{8}\operatorname{tr}\sum_{j=1}^{n}\operatorname{ad}_{\mathbf{g}}(y_j)P_{\mathbf{r}}\operatorname{ad}_{\mathbf{g}}(y_j)P_{\mathbf{p}}.$$

Multiplying this equation by $1/3$ and adding it to the above expression for $(v^2)_0$ gives

$$\frac{1}{3}\nu(\mathrm{Cas}_{\mathbf{r}})_0 + (v^2)_0 = \frac{1}{24}\operatorname{tr}\sum_{j=1}^{n}(\operatorname{ad}_{\mathbf{g}} y_j)^2 P_{\mathbf{p}}.$$

We can write

$$\nu(\mathrm{Cas}_{\mathbf{r}})_0 = \frac{1}{8}\sum_{i=1,j=1}^{r,n}([x_i,y_j],[y_j,x_i]) = \frac{1}{8}\sum_{i=1,j=1}^{r,n}(x_i,[y_j,[y_j,x_i]])$$

$$= \frac{1}{8}\operatorname{tr} P_{\mathbf{r}}\sum_{j=1}^{n}(\mathrm{ad}_{\mathbf{g}}\, y_j)^2 P_{\mathbf{r}} = \frac{1}{8}\operatorname{tr}\sum_{j=1}^{n}(\mathrm{ad}_{\mathbf{g}}\, y_j)^2 P_{\mathbf{r}}.$$

Multiplying by 1/3 and adding to the preceding equation gives

$$\frac{2}{3}\nu(\mathrm{Cas}_{\mathbf{r}})_0 + (v^2)_0 = \frac{1}{24}\operatorname{tr}\sum_{j=1}^{n}(\mathrm{ad}_{\mathbf{g}}\, y_j)^2.$$

On the other hand

$$\nu(\mathrm{Cas}_{\mathbf{r}})_0 = \frac{1}{8}\sum_{i=1,j=1}^{r,n}([x_i,[x_i,y_j]],y_j) = \frac{1}{8}\operatorname{tr}\mathrm{ad}_{\mathbf{p}}(\mathrm{Cas}_{\mathbf{r}})$$

$$= \frac{1}{8}\left(\operatorname{tr}\mathrm{ad}_{\mathbf{g}}(\mathrm{Cas}_{\mathbf{r}}) - \operatorname{tr}\mathrm{ad}_{\mathbf{r}}(\mathrm{Cas}_{\mathbf{r}})\right).$$

Multiplying by 1/3 and adding to the preceding equation, and using the fact that $\mathrm{Cas}_{\mathbf{g}} = \mathrm{Cas}_{\mathbf{r}} + \sum y_j^2$ gives

$$\nu(\mathrm{Cas}_{\mathbf{r}}) + v^2 = \frac{1}{24}\left(\operatorname{tr}\mathrm{ad}_{\mathbf{g}}(\mathrm{Cas}_{\mathbf{g}}) - \operatorname{tr}\mathrm{ad}_{\mathbf{r}}(\mathrm{Cas}_{\mathbf{r}})\right) \tag{10.13}$$

when (10.8) holds.

Suppose now that the Lie algebra **r** is reductive and that the Lie algebra **g** we created out of **r** and **p** using a $v \in \wedge^3\mathbf{p}$ satisfying (10.8) is also reductive. Using (**??**) for **g** and for **r** in the right hand side of (10.13) yields

$$\nu(\mathrm{Cas}_{\mathbf{r}}) + v^2 = \left((\rho_{\mathbf{g}},\rho_{\mathbf{g}}) - (\rho_{\mathbf{r}},\rho_{\mathbf{r}})\right). \tag{10.14}$$

10.5 Kostant's Dirac Operator.

Suppose that we have constructed our Lie algebra $\mathbf{g} = \mathbf{r} + \mathbf{p}$ from a $v \in \wedge^3\mathbf{p}$ satisfying (10.8). We are going to define

$$\not{K} \in U(\mathbf{g}) \otimes C(\mathbf{p})$$

as follows: Let $y_1,\ldots,y_n$ be an orthonormal basis of **p**. Then

$$\not{K} := \sum_i y_i \otimes y_i + 1 \otimes v. \tag{10.15}$$

(On the left of the tensor product sign the $y_i \in \mathbf{p}$ is considered as an element of $U(\mathbf{g})$ via the canonical injection of **p** in $U(\mathbf{p}) \subset U(\mathbf{g})$ and on the right of the tensor product sign it lies in $C(\mathbf{p})$ via the canonical injection of **p** into $C(\mathbf{p})$.)

We have a homomorphism $U(\mathbf{r}) \to U(\mathbf{g})$, in particular a Lie algebra injection $\mathbf{r} \to U(\mathbf{g})$. We also have a Lie algebra homomorphism $\nu : \mathbf{r} \to C(\mathbf{p})$. In particular, we have the diagonal Lie algebra map

$$\operatorname{diag} : \mathbf{r} \to U(\mathbf{g}) \otimes C(\mathbf{p}), \quad \operatorname{diag}(x) = x \otimes 1 + 1 \otimes \nu(x)$$

and this extends to an algebra map

$$\operatorname{diag} : U(\mathbf{r}) \to U(\mathbf{g}) \otimes C(\mathbf{p}).$$

For example,

$$\operatorname{diag}(\mathrm{Cas}_{\mathbf{r}}) = \sum_i (x_i \otimes 1 + 1 \otimes \nu(x_i))^2$$

where $x_1, \ldots, x_r$ is an orthonormal basis of $\mathbf{r}$. In other words

$$\operatorname{diag}(\mathrm{Cas}_{\mathbf{r}}) = \sum_{i=1}^{r} x_i^2 \otimes 1 + 2\sum_{i=1}^{r} x_i \otimes \nu(x_i) + \sum_{i=1}^{r} 1 \otimes \nu(x_i)^2. \tag{10.16}$$

We claim that

$$\not{K}^2 = \mathrm{Cas}_{\mathbf{g}} \otimes 1 - \operatorname{diag}(\mathrm{Cas}_{\mathbf{r}}) + \frac{1}{24}\left(\operatorname{tr}\operatorname{ad}_{\mathbf{g}}(\mathrm{Cas}_{\mathbf{g}}) - \operatorname{tr}\operatorname{ad}_{\mathbf{r}}(\mathrm{Cas}_{\mathbf{r}})\right) 1 \otimes 1. \tag{10.17}$$

To prove this, let us write (10.15) as

$$\not{K} = \not{K}' + \not{K}''.$$

So

$$(\not{K}'')^2 = 1 \otimes v^2$$

and hence

$$(\not{K}'')^2 + \sum_{j=1}^{r} 1 \otimes \nu(x_i)^2 = \frac{1}{24}\left(\operatorname{tr}\operatorname{ad}_{\mathbf{g}}(\mathrm{Cas}_{\mathbf{g}}) - \operatorname{tr}\operatorname{ad}_{\mathbf{r}}(\mathrm{Cas}_{\mathbf{r}})\right) 1 \otimes 1$$

by (10.13). We have

$$\begin{aligned}
(\not{K}')^2 &= \sum_{ij} y_i y_j \otimes y_i y_j \\
&= \sum_i y_i^2 \otimes 1 + \sum_{i \neq j} y_i y_j \otimes y_i y_j \\
&= \sum_i y_i^2 \otimes 1 + \sum_{i<j} (y_i y_j - y_j y_i) \otimes y_i y_j \\
&= \sum_i y_i^2 \otimes 1 + \sum_{i<j} [y_i, y_j] \otimes y_i y_j \\
&= \sum_i y_i^2 \otimes 1 - 2\sum_{i<j} \nu^\dagger(y_i \wedge y_j) \otimes y_i \wedge y_j + 2\sum_{i<j} \iota(y_i)\iota(y_j)v \otimes y_i \wedge y_j,
\end{aligned}$$

where we have used the decomposition of $[y_i, y_j]$ into its $\mathbf{r}$ and $\mathbf{p}$ components to get to the last expression. We can write the middle term in the last expression as

$$\begin{aligned}
-2\sum_{i<j}\nu^\dagger(y_i\wedge y_j)\otimes y_i\wedge y_j &= -2\sum_{k=1}^{r}\sum_{i<j}(\nu^\dagger(y_i\wedge y_j),x_k)x_k\otimes y_i\wedge y_j\\
&= -2\sum_{k=1}^{r}\sum_{i<j}(y_i\wedge y_j,\nu(x_k))x_k\otimes y_i\wedge y_j\\
&= -2\sum_{k=1}^{r}\sum_{i<j}x_k\otimes(y_i\wedge y_j,\nu(x_k))y_i\wedge y_j\\
&= -2\sum_{i}x_i\otimes\nu(x_i).
\end{aligned}$$

Since $\sum x_i^2\otimes 1+\sum y_j^2\otimes 1=\mathrm{Cas}_{\mathbf{g}}\otimes 1$ we conclude that

$$\begin{gathered}
(\not{K}')^2+(\not{K}'')^2+\mathrm{diag}(\mathrm{Cas}_{\mathbf{r}})\\
=\mathrm{Cas}_{\mathbf{g}}\otimes 1+\frac{1}{24}\left(\mathrm{tr}\,\mathrm{ad}_{\mathbf{g}}(\mathrm{Cas}_{\mathbf{g}})-\mathrm{tr}\,\mathrm{ad}_{\mathbf{r}}(\mathrm{Cas}_{\mathbf{r}})\right)1\otimes 1+2\sum_{i<j}\iota(y_i)\iota(y_j)v\otimes y_i\wedge y_j.
\end{gathered}$$

To complete the proof of (10.17) we must show that

$$\not{K}'\not{K}''+\not{K}''\not{K}'=-2\sum_{i<j}\iota(y_i)\iota(y_j)v\otimes y_i\wedge y_j.$$

But

$$\begin{aligned}
\not{K}'\not{K}''+\not{K}''\not{K}' &= \sum y_j\otimes[y_j,v]\\
&= 2\sum y_j\otimes\iota(y_j)v\\
&= 2\sum_{i<k}\sum_{j}y_j\otimes(\iota(y_j)v,y_i\wedge y_k)y_i\wedge y_k\\
&= \sum_{i<k}\sum_{j}y_j\otimes(2v,y_i\wedge y_k\wedge y_j)y_i\wedge y_k\\
&= \sum_{i<k}\sum_{j}y_j\otimes(2\iota(y_k)\iota(y_i)v,y_j)y_i\wedge y_j\\
&= \sum_{i<k}\sum_{j}(2\iota(y_k)\iota(y_i)v,y_j)y_j\otimes y_i\wedge y_j\\
&= \sum_{i<k}2\iota(y_k)\iota(y_i)v\otimes y_i\wedge y_k,
\end{aligned}$$

completing the proof.

In the case where $\mathbf{r}$ and $\mathbf{g}$ are reductive we have the alternative formula

$$\not{K}^2=\mathrm{Cas}_{\mathbf{g}}\otimes 1-\mathrm{diag}(\mathrm{Cas}_{\mathbf{r}})+\left((\rho_{\mathbf{g}},\rho_{\mathbf{g}})-(\rho_{\mathbf{r}},\rho_{\mathbf{r}})\right)1\otimes 1. \tag{10.18}$$

Suppose that λ is the highest weight of a finite dimensional irreducible representation V_λ of $\mathbf{g}$ so that we get a surjective homomorphism

$$U(\mathbf{g}) \to \mathrm{End}(V_\lambda)$$

and hence a corresponding homomorphism

$$U(\mathbf{g}) \otimes C(\mathbf{p}) \to \mathrm{End}(V_\lambda) \otimes C(\mathbf{p}).$$

Also let diag_λ denote the composition of this homomorphism with

$$\mathrm{diag} : U(\mathbf{r}) \to \mathbf{U}(\mathbf{g}) \otimes \mathbf{C}(\mathbf{p}).$$

Then from the value of the Casimir (**??**) and (10.18) we get

$$\not{K}_\lambda^2 = ((\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) - (\rho_{\mathbf{r}}, \rho_{\mathbf{r}}))\, 1 \otimes 1 - \mathrm{diag}_\lambda(\mathrm{Cas}_{\mathbf{r}}). \tag{10.19}$$

10.6 Eigenvalues of the Dirac operator.

We consider the situation where $\mathbf{g} = \mathbf{r} \oplus \mathbf{p}$ is a Lie algebra with invariant symmetric bilinear form, where $\mathbf{r}$ has the same rank as $\mathbf{g}$, and where we have chosen a common Cartan subalgebra

$$\mathbf{h} \subset \mathbf{r} \subset \mathbf{g}.$$

We let ℓ denote the dimension of $\mathbf{h}$, i.e. the common rank of $\mathbf{r}$ and $\mathbf{g}$. We let $\Phi = \Phi_{\mathbf{g}}$ denote the set of roots of $\mathbf{g}$, let $W = W_{\mathbf{g}}$ denote the Weyl group of $\mathbf{g}$, and let $W_{\mathbf{r}}$ denote the Weyl group of $\mathbf{r}$ so that

$$W_{\mathbf{r}} \subset W$$

and we let c denote the index of $W_{\mathbf{r}}$ in W.

A choice of positive roots Φ^+ for $\mathbf{g}$ amounts to a choice of a Borel subalgebra $\mathbf{b}$ of $\mathbf{g}$ and then $\mathbf{b} \cap \mathbf{r}$ is a Borel subalgebra of $\mathbf{r}$, which picks out a system of positive roots $\Phi_{\mathbf{r}}^+$ for $\mathbf{r}$ and then

$$\Phi_{\mathbf{r}}^+ \subset \Phi^+.$$

The corresponding Weyl chambers are

$$D = D_{\mathbf{g}} = \{\lambda \in \mathbf{h}_{\mathbf{R}}^* | (\lambda, \phi) \geq 0 \ \ \forall \phi \in \Phi^+\}$$

and

$$D_{\mathbf{r}} = \{\lambda \in \mathbf{h}_{\mathbf{R}}^* | (\lambda, \phi) \geq 0 \ \ \forall \phi \in \Phi_{\mathbf{r}}^+\}$$

so

$$D \subset D_{\mathbf{r}}$$

and we have chosen a cross-section C of $W_{\mathbf{r}}$ in W as

$$C = \{w \in W | wD \subset D_{\mathbf{r}}\},$$

so

$$W = W_{\mathbf{r}} \cdot C, \quad D_{\mathbf{r}} = \bigcup_{w \in C} wD.$$

We let $\mathbf{L} = \mathbf{L_g} \subset \mathbf{h}^*_{\mathbf{R}}$ denote the lattice of $\mathbf{g}$ integral linear forms on $\mathbf{h}$, i.e.

$$\mathbf{L} = \{\mu \in \mathbf{h}^* | 2\frac{(\mu, \phi)}{(\phi, \phi)} \in \mathbf{Z}\ \forall \phi \in \Delta\}.$$

We let

$$\rho = \rho_{\mathbf{g}} = \frac{1}{2} \sum_{\phi \in \Delta^+} \phi$$

and

$$\rho_{\mathbf{r}} = \frac{1}{2} \sum_{\phi \in \Delta^+_{\mathbf{r}}} \phi.$$

We set

$$\mathbf{L_r} = \text{ the lattice spanned by } \mathbf{L} \text{ and } \rho_{\mathbf{r}},$$

and

$$\Lambda := \mathbf{L} \cap D, \quad \Lambda_{\mathbf{r}} := \mathbf{L_r} \cap D_{\mathbf{r}}.$$

For any $\mathbf{r}$ module Z we let $\Gamma(Z)$ denote its set of weights, and we shall assume that

$$\Gamma(Z) \subset \mathbf{L_r}.$$

For such a representation define

$$m_Z := \max_{\gamma \in \Gamma(Z)} (\gamma + \rho_{\mathbf{r}}, \gamma + \rho_{\mathbf{r}}). \tag{10.20}$$

For any $\mu \in \Lambda_{\mathbf{r}}$ we let Z_μ denote the irreducible module with highest weight μ.

Proposition 1 *Let*

$$\Gamma_{\max}(Z) := \{\mu \in \Gamma(Z) | (\mu + \rho_{\mathbf{r}}, \mu + \rho_{\mathbf{r}}) = m_Z\}.$$

Let $\mu \in \Gamma_{\max}(Z)$. *Then*

1. $\mu \in \Lambda_{\mathbf{r}}$.
2. *If* $z \neq 0$ *is a weight vector with weight* μ *then* z *is a highest weight vector, and hence the submodule* $U(\mathbf{r})z$ *is irreducible and equivalent to* Z_μ.
3. *Let*

$$Y_{\max} := \sum_{\mu \in \Gamma_{\max}(Z)} Z_\mu$$

and

$$Y := U(\mathbf{r}) Y_{\max}.$$

Then $m_Z - (\rho_{\mathbf{r}}, \rho_{\mathbf{r}})$ *is the maximal eigenvalue of* $\mathrm{Cas}_{\mathbf{r}}$ *on* Z *and* Y *is the corresponding eigenspace.*

Proof. We first show that

$$\mu \in \Gamma_{\max} \Rightarrow \ \mu + \rho_{\mathbf{r}} \in \Lambda_{\mathbf{r}}.$$

Suppose not, so there exists a $w \neq 1, \ w \in W_{\mathbf{r}}$ such that

$$w\mu + w\rho_{\mathbf{r}} \in \Lambda_{\mathbf{r}}.$$

But w changes the sign of some of the positive roots (the number of such changes being equal the length of w in terms of the generating reflections), and so $\rho_{\mathbf{r}} - w\rho_{\mathbf{r}}$ is a non-trivial sum of positive roots. Therefore

$$(w\mu + w\rho_{\mathbf{r}}, \rho_{\mathbf{r}} - w\rho_{\mathbf{r}}) \geq 0, \quad (\rho_{\mathbf{r}} - w\rho_{\mathbf{r}}, \rho_{\mathbf{r}} - w\rho_{\mathbf{r}}) > 0$$

and

$$w\mu + \rho_{\mathbf{r}} = (w\mu + w\rho_{\mathbf{r}}) + (\rho_{\mathbf{r}} - w\rho_{\mathbf{r}})$$

satisfies

$$(w\mu + \rho_{\mathbf{r}}, w\mu + \rho_{\mathbf{r}}) > (w\mu + w\rho_{\mathbf{r}}, w\mu + w\rho_{\mathbf{r}}) = (\mu + \rho_{\mathbf{r}}, \mu + \rho) = m_Z$$

contradicting the definition of m_Z. Now suppose that z is a weight vector with weight μ which is not a highest weight vector. Then there will be some irreducible component of Z containing z and having some weight μ' such that $\mu' - \mu$ is a non trivial sum of positive roots. We have

$$\mu' + \rho_{\mathbf{r}} = (\mu' - \mu) + (\mu + \rho_{\mathbf{r}})$$

so by the same argument we conclude that

$$(\mu' + \rho_{\mathbf{r}}, \mu' + \rho_{\mathbf{r}}) > m_Z$$

since $\mu + \rho_{\mathbf{r}} \in \Lambda_{\mathbf{r}}$, and again this is impossible. Hence z is a highest weight vector implying that $\mu \in \Lambda_{\mathbf{r}}$. This proves 1) and 2).

We have already verified that the eigenvalue of the Casimir $\mathrm{Cas}_{\mathbf{r}}$ on any Z_γ is $(\gamma + \rho_{\mathbf{r}}, \gamma + \rho_{\mathbf{r}}) - (\rho_{\mathbf{r}}, \rho_{\mathbf{r}})$. This proves 3).

Consider the irreducible representation V_ρ of $\mathbf{g}$ corresponding to $\rho = \rho_{\mathbf{g}}$. By the same arguments, any weight $\gamma \neq \rho$ of V_ρ lying in D must satisfy $(\gamma, \gamma) < (\rho, \rho)$ and hence any weight γ of V_ρ satisfying $(\gamma, \gamma) = (\rho, \rho)$ must be of the form

$$\gamma = w\rho$$

for a unique $w \in W$. But

$$w\rho = \rho - \sum_{\phi \in J_w} \phi = \rho - \phi_J$$

where

$$J_w := w(-\Phi^+) \cap \Phi^+.$$

We know that all the weights of V_ρ are of the form $\rho - \phi_J$ as J ranges over all subsets of Φ^+. So

$$(\rho, \rho) \geq (\rho - \phi_J, \rho - \phi_J) \tag{10.21}$$

where we have strict inequality unless $J = J_w$ for some $w \in W$.

Now let $\lambda \in \Lambda$, let V_λ be the corresponding irreducible module with highest weight λ and let γ be a weight of V_λ. As usual, let J denote a subset of the positive roots, $J \subset \Phi^+$. We claim that

Proposition 2 *We have*

$$(\lambda + \rho, \lambda + \rho) \geq (\gamma + \rho - \phi_J, \gamma + \rho - \phi_J) \tag{10.22}$$

with strict inequality unless there exists a $w \in W$ such that

$$\gamma = w\lambda, \quad \textit{and} \;\; J = J_w$$

in which case the w is unique.

Proof. Choose w such that

$$w^{-1}(\gamma + \rho - \phi_J) \in \Lambda.$$

Since $w^{-1}(\gamma)$ is a weight of V_λ, $\lambda - w^{-1}(\gamma)$ is a sum (possibly empty) of positive roots. Also $w^{-1}(\rho - \phi_J)$ is a weight of V_ρ and hence $\rho - w^{-1}(\rho - \phi_J)$ is a sum (possibly empty) of positive roots. Since

$$\lambda + \rho = (\lambda - w^{-1}\gamma) + (\rho - w^{-1}(\rho - \phi_J) + w^{-1}(\gamma + \rho - \phi_J)),$$

we conclude that

$$(\lambda + \rho, \lambda + \rho) \geq (w^{-1}(\gamma + \rho - \phi_J), w^{-1}(\gamma + \rho - \phi_J)) = (\gamma + \rho - \phi_J, \gamma + \rho - \phi_J)$$

with strict inequality unless $\lambda - w^{-1}(\gamma) = 0 = \rho - w^{-1}(\rho - \phi_J)$, and this last equality implies that $J = J_w$. QED

We have the spin representation Spin ν where $\nu : \mathbf{r} \rightarrow C(\mathbf{p})$. Call this module S. Consider

$$V_\lambda \otimes S$$

as a $\mathbf{r}$ module. Then, letting γ denote a weight of V_λ, we have

$$\Gamma(V_\lambda \otimes S) = \{\mu = \gamma + \rho_{\mathbf{p}} - \phi_J\} \tag{10.23}$$

where

$$\rho_{\mathbf{p}} = \frac{1}{2} \sum_{J \in \Phi_{\mathbf{p}}^+} \phi, \quad \Phi_{\mathbf{p}}^+ := \Phi^+ / \Phi_{\mathbf{r}}^+.$$

In other words, $\Phi_{\mathbf{p}}$ are the roots of $\mathbf{g}$ which are not roots of $\mathbf{r}$, or, put another way, they are the weights of $\mathbf{p}$ considered as a $\mathbf{r}$ module. (Our equal rank assumption says that 0 does not occur as one of these weights.) For the weights μ of $V_\lambda \otimes S$ the form (10.23) gives

$$\mu + \rho_{\mathbf{r}} = \gamma + \rho - \phi_J, \quad J \subset \Delta^+_{\mathbf{p}}.$$

So if we set $Z = V_\lambda \otimes S$ as a $\mathbf{r}$ module, (10.22) says that

$$(\lambda + \rho, \lambda + \rho) \geq m_Z.$$

But we may take $J = \emptyset$ as one of our weights showing that

$$m_Z = (\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}). \tag{10.24}$$

To determine $\Gamma_{\max}(Z)$ as in Prop. 1 we again use Prop.2 and (10.23): A $\mu = \gamma + \rho_{\mathbf{p}} - \phi_J$ belongs to $\Gamma_{\max}(Z)$ if and only if $\gamma = w\lambda$ and $J = J_w$. But then

$$\rho_{\mathbf{g}} - \phi_J = w\rho_{\mathbf{g}}.$$

Since $\rho_{\mathbf{g}} = \rho_{\mathbf{r}} + \rho_{\mathbf{p}}$ we see from the form (10.23) that

$$\mu + \rho_{\mathbf{r}} = w(\lambda + \rho_{\mathbf{g}}) \tag{10.25}$$

where w is unique, and

$$J_w \subset \Phi^+_{\mathbf{p}}.$$

We claim that this condition is the same as the condition $w(D) \subset D_{\mathbf{r}}$ defining our cross-section, C. Indeed, $w \in C$ if and only if $(\phi, w\rho_{\mathbf{g}}) > 0, \ \forall\, \phi \in \Phi^+_{\mathbf{r}}$. But $(\phi, w\rho) = (w^{-1}\phi, \rho) > 0$ if and only if $\phi \in w(\Phi^+)$. Since $J_w = w(-\Phi^+) \cap \Phi^+$, we see that $J_w \subset \Phi^+_{\mathbf{p}}$ is equivalent to the condition $w \in C$.

Now for $\mu \in \Gamma_{\max}(Z)$ we have

$$\mu = w(\lambda + \rho) - \rho_{\mathbf{r}} =: w \bullet \lambda \tag{10.26}$$

where $\gamma = w(\lambda)$ and so has multiplicity one in V_λ.

Furthermore, we claim that the weight $\rho_{\mathbf{p}} - \phi_{J_w}$ has multiplicity one in S. Indeed, consider the representation

$$Z_{\rho_{\mathbf{r}}} \otimes S$$

of $\mathbf{r}$. It has the weight $\rho = \rho_{\mathbf{r}} + \rho_{\mathbf{p}}$ as a highest weight, and in fact, all of the weights of $V_{\rho_{\mathbf{g}}}$ occur among its weights. Hence, on dimensional grounds, say from the Weyl character formula, we conclude that it coincides, as a representation of $\mathbf{r}$, with the restriction of the representation $V_{\rho_{\mathbf{g}}}$ to $\mathbf{r}$. But since $\rho_{\mathbf{g}} - \phi_{J_w} = w\rho_{\mathbf{g}}$ has multiplicity one in $V_{\rho_{\mathbf{g}}}$, we conclude that $\rho_{\mathbf{p}} - \phi_{J_w}$ has multiplicity one in S.

We have proved that each of the $w \bullet \lambda$ have multiplicity one in $V_\lambda \otimes S$ with corresponding weight vectors

$$z_{w\bullet\lambda} := v_{w\lambda} \otimes e_-^{J_w} e^+.$$

So each of the submodules

$$Z_{w\bullet\lambda} := U(\mathbf{r})z_{w\bullet\lambda} \tag{10.27}$$

occurs with multiplicity one in $V_\lambda \otimes S$. The length of $w \in C$ (in terms of the simple reflections of W determined by Δ) is the number of positive roots changed into negative roots, i.e. the cardinality of J_w. This cardinality is the sign of $\det w$ and also determines whether $e_-^J e_+$ belongs to S_+ or to S_-. From Prop.2 and equation (10.24) we know that the maximum eigenvalue of $\mathrm{Cas}_\mathbf{r}$ on $V_\lambda \otimes S$ is

$$(\lambda + \rho_\mathbf{g}, \lambda + \rho_\mathbf{g}) - (\rho_\mathbf{r}, \rho_\mathbf{r}).$$

Now $\not{K}_\lambda \in \mathrm{End}(V_\lambda \otimes S)$ commutes with the action of $\mathbf{r}$ with

$$\not{K}_\lambda : \begin{array}{c} V_\lambda \otimes S_+ \to V_\lambda \otimes S_- \\ \\ V_\lambda \otimes S_- \to V_\lambda \otimes S_+. \end{array}$$

Furthermore, by (10.19), the kernel of $\not{K}_\lambda^2$ is the eigenspace of $\mathrm{Cas}_\mathbf{r}$ corresponding to the eigenvalue $(\lambda + \rho, \lambda + \rho) - (\rho_\mathbf{r}, \rho_\mathbf{r})$. Thus

$$\mathrm{Ker}(\not{K}_\lambda^2) = \sum_{w \in C} Z_{w\bullet\lambda}.$$

Each of these modules lies either in $V \otimes S_+$ or $V \otimes S_-$, one or the other but not both. Hence

$$\mathrm{Ker}(\not{K}_\lambda^2) = \mathrm{Ker}(\not{K}_\lambda)$$

and so

$$\mathrm{Ker}(\not{K}_\lambda)|_{V_\lambda \otimes S_+} = \sum_{w \in C,\ \det w = 1} Z_{w\bullet\lambda} \tag{10.28}$$

and

$$\mathrm{Ker}(\not{K}_\lambda)|_{V_\lambda \otimes S_-} = \sum_{w \in C,\ \det w = -1} Z_{w\bullet\lambda} \tag{10.29}$$

Let

$$K_\pm := \sum_{w \in C,\ \det w = \pm 1} Z_{w\bullet\lambda}. \tag{10.30}$$

It follows from (10.28) that $\not{K}_\lambda$ induces an injection of

$$(V_\lambda \otimes S_+)/K_+ \to V \otimes S_-$$

which we can follow by the projection

$$V_\lambda \otimes S_- \to (V_\lambda \otimes S_-)/K_-.$$

Hence $\not{K}_\lambda$ induces a bijection

$$\tilde{\not{K}}_\lambda : (V \otimes S_+)/K_+ \to (V_\lambda \otimes S_-)/K_-. \tag{10.31}$$

In short, we have proved that the sequence

$$0 \to K_+ \to V_\lambda \otimes S_+ \to V_\lambda \otimes S_- \to K_- \to 0 \tag{10.32}$$

is exact in a very precise sense, where the middle map is the Kostant Dirac operator: each summand of K_+ occurs exactly once in $V_\lambda \otimes S_+$ and similarly for K_-. This gives a much more precise statement of Theorem **??** and a completely different proof.

10.7 The geometric index theorem.

Let r be the representation of G on the space $\mathcal{F}(G)$ of smooth or on $L^2(G)$ of L^2 functions on G coming from right multiplication. Thus

$$[r(g)f](a) = f(ag).$$

Then $\not{K}$ acts on $\mathcal{F}(G) \otimes S$ or on $L^2(G) \otimes S$ and centralizes the action of $\operatorname{diag} \mathbf{r}$. If U is a module for R, we may consider $\mathcal{F}(G) \otimes S \otimes U$ or $L^2(G) \otimes S \otimes U$, and $\not{K} \otimes 1$ commutes with $\operatorname{diag} \mathbf{r} \otimes 1$ and with the action ρ of R on U, i.e with $1 \otimes 1 \otimes \rho$. If R is connected, this implies that $\not{K}$ commutes with the diagonal action of $\tilde{R}$, the universal cover of R, on $\mathcal{F} \otimes S \otimes U$ or $L^2(G) \otimes S \otimes U$ given by

$$k \mapsto r(k) \otimes \operatorname{Spin}(k) \otimes \rho(k), \qquad k \in R$$

where $\operatorname{Spin} : \tilde{R} \to \operatorname{Spin}(\mathbf{p})$ is the group homomorphism corresponding to the Lie algebra homomorphism ν. If G/R is a spin manifold, the invariants under this R action correspond to smooth or L^2 sections of $\mathbf{S} \otimes \mathcal{U}$ where $\mathbf{S}$ is the spin bundle of G/R and $\mathcal{U}$ is the vector bundle on G/R corresponding to U. Thus $\not{K}$ descends (by restriction) to a differential operator $\not{\partial}$ on G/R and we shall compute its G-index for irreducible U. The key result, due to Landweber, asserts that if U belongs to a multiplet coming from an irreducible V of G, then this index is, up to a sign, equal to V. If U does not belong to a multiplet, then this index is zero. We begin with some preliminary results due to Bott.

10.7.1 The index of equivariant Fredholm maps.

Let E and F be Hilbert spaces which are unitary modules for the compact Lie group G. Suppose that

$$E = \widehat{\bigoplus_n} E_n, \quad F = \widehat{\bigoplus_n} F_n$$

are completed direct sum decompositions into subspaces which are G-invariant and finite dimensional, and that

$$T : E \to F$$

is a Fredholm map (finite dimensional kernel and cokernel) such that

$$T(E_n) \subset F_n.$$

We write

$$\mathrm{Index}_G\, T = \mathrm{Ker}\, T - \mathrm{Coker}\, T$$

as an element of R(G), the ring of virtual representations of G. Thus $\mathrm{R}(G)$ is the space of finite linear combinations $\sum_\lambda a_\lambda V_\lambda, \quad a_\lambda \in \mathbf{Z}$ as V_λ ranges over the irreducible representations of G. (Here, and in what follows, we are regarding any finite dimensional representation of G as an element of $\mathrm{R}(G)$ by its decomposition into irreducibles, and similarly the difference of any two finite dimensional representations is an element of $\mathrm{R}(G)$.)

If we denote the restriction of T to E_n by T_n, then

$$\mathrm{Index}_G\, T = \sum \mathrm{Index}_G\, T_n$$

where all but a finite number of terms on the right vanish. For each n we have the exact sequence

$$0 \to \mathrm{Ker}\, T_n \to E_n \to F_n \to \mathrm{Coker}\, T_n \to 0.$$

Thus

$$\mathrm{Index}_G\, T_n = E_n - F_n$$

as elements of $\mathrm{R}(G)$. Therefore we can write

$$\mathrm{Index}_G\, T = \sum (E_n - F_n) \tag{10.33}$$

in $\mathrm{R}(G)$, where all but a finite number of terms on the right vanish. We shall refer to this as Bott's equation.

10.7.2 Induced representations and Bott's theorem.

Let R be a closed subgroup of G. Given any R-action ρ on a vector space U, we consider the associated vector bundle $G \times_R V$ over the homogeneous space G/R. The sections of this bundle are then equivariant U-valued functions on G satisfying $s(gk) = \rho(k)^{-1} s(g)$ for all $k \in R$. Applying the Peter-Weyl theorem, we can decompose the space of L^2 maps from G to U into a sum over the irreducible representations V_λ of G,

$$L^2(G) \otimes U \cong \widehat{\bigoplus}_\lambda V_\lambda \otimes V_\lambda^* \otimes U,$$

with respect to the $G \times G \times R$ action $l \otimes r \otimes \rho$. The R-equivariance condition is equivalent to requiring that the functions be invariant under the diagonal R-action $k \mapsto r(k) \otimes \rho(k)$. Restricting the Peter-Weyl decomposition above to the R invariant subspace, we obtain

$$\begin{aligned} L^2(G \times_R U) \quad &\cong \widehat{\bigoplus}_\lambda V_\lambda \otimes (V_\lambda^* \otimes U)^R \\ &\cong \widehat{\bigoplus}_\lambda V_\lambda \otimes \mathrm{Hom}_R(V_\lambda, U). \end{aligned} \tag{10.34}$$

The Lie group G acts on the space of sections by $l(g)$, the left action of G on functions, which is preserved by this construction. The space $L^2(G \times_H U)$ is thus an infinite dimensional representation of G.

The intertwining number of two representations gives us an inner product

$$\langle V, W\rangle_G = \dim_{\mathbf{C}} \mathrm{Hom}_G(V, W)$$

on $\mathrm{R}(G)$, with respect to which the irreducible representations of G form an orthonormal basis. Taking the formal completion of $\mathrm{R}(G)$, we define $\hat{\mathrm{R}}(G)$ to be the space of possibly infinite formal sums $\sum_\lambda a_\lambda V_\lambda$. The intertwining number then extends to a pairing $\mathrm{R}(G) \times \hat{\mathrm{R}}(G) \to \mathbf{Z}$.

If R is a subgroup of G, every representation of G automatically restricts to a representation of R. This gives us a pullback map $i^* : \mathrm{R}(G) \to \mathrm{R}(R)$, corresponding to the inclusion $i : R \hookrightarrow G$. The map $U \mapsto L^2(G \times_H U)$ discussed above assigns to each R-representation an induced infinite dimensional G-representation. Expressed in terms of our representation ring notation, this induction map becomes the homomorphism $i_* : \mathrm{R}(R) \to \hat{\mathrm{R}}(G)$ given by

$$i_* U = \sum_\lambda \langle i^* V_\lambda, U\rangle_R V_\lambda,$$

the formal adjoint to the pullback i^*. This is the content of the Frobenius reciprocity theorem.

A homogeneous differential operator on G/R is a differential operator $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ between two homogeneous vector bundles $\mathcal{E}$ and $\mathcal{F}$ that commutes with the left action of G on sections. If the operator is elliptic, then its kernel and cokernel are both finite dimensional representations of G, and thus its G-index is a virtual representation in $R(G)$. In this case, the index takes a particularly elegant form.

Theorem 1 (Bott) *If $D : \Gamma(G \times_H U_0) \to \Gamma(G \times_H U_1)$ is an elliptic homogeneous differential operator, then the G-equivariant index of D is given by*

$$\mathrm{Index}_G\, D = i_*(U_0 - U_1),$$

where $i_*(U_0 - U_1)$ is a finite element in $\hat{\mathrm{R}}(G)$, i.e. belongs to $\mathrm{R}(G)$.

In particular, note that the index of a homogeneous differential operator depends only on the vector bundles involved and not on the operator itself! To prove the theorem, just use Bott's formula (10.33), where the subscript n is replaced by λ labeling the the G-irreducibles.

10.7.3 Landweber's index theorem.

Suppose that G is semi-simple and simply connected and R is a reductive subgroup of maximal rank. Suppose further that G/R is a spin manifold, then we can compose the spin representation $S = S_+ \oplus S_-$ of $\mathrm{Spin}(\mathbf{p})$ with the lifted map Spin : $\tilde{R} \to \mathrm{Spin}(\mathbf{p})$ to obtain a homogeneous vector bundle, the spin bundle $\mathbf{S}$ over G/R. For any representation of R on U the Kostant Dirac operator descends to an operator

$$\partial\!\!\!/_U : \Gamma(\mathbf{S}_\pm \otimes \mathcal{U}) \to \Gamma(\mathbf{S}_\mp \otimes \mathcal{U}).$$

(This operator has the same symbol as the Dirac operator arising from the Levi-Civita connection on G/R twisted by $\mathcal{U}$, and has the same index by Bott's theorem. For the precise relation between this Dirac operator coming from $K\!\!\!/$ and the dirac operator coming from the Levi- civita connection we refer to Landweber's thesis.)

The following theorem of Landweber gives an expression for the index of this Kostant Dirac operator. In particular, if we consider G/T, where T is a maximal torus (which is always a spin manifold), this theorem becomes a version of the Borel-Weil-Bott theorem expressed in terms of spinors and the Dirac operator, instead of in its customary form involving holomorphic sections and Dolbeault cohomology.

Theorem 2 (Landweber) *Let G/R be a spin manifold, and let U_μ be an irreducible representation U_μ of R with highest weight μ. The G-equivariant index of the Dirac operator $\partial\!\!\!/_U$ is the virtual G-representation*

$$\mathrm{Index}_G\, \partial\!\!\!/_{U_\mu} = (-1)^{\dim \mathbf{p}/2}\ (-1)^w V_{w(\mu+\rho_H)-\rho_G} \tag{10.35}$$

if there exists an element $w \in W_G$ in the Weyl group of G such that the weight $w(\mu+\rho_H)-\rho_G$ is dominant for G. If no such w exists, then $\mathrm{Index}_G\, \partial\!\!\!/_{U_\mu} = 0$.

Proof. For any irreducible representation V_λ of G with highest weight λ we have

$$V_\lambda \otimes (S_+ - S_-) = \sum_{w\in C} (-1)^w U_{w\bullet\lambda}$$

by [GKRS]. Hence

$$\mathrm{Hom}_R(V_\lambda \otimes (S_+ - S_-), U_\mu) = 0$$

if $\mu \neq w \bullet \lambda$ for some $w \in C$ while

$$\mathrm{Hom}_R(V_\lambda \otimes (S_+ - S_-), U_\mu) = (-1)^w$$

if $\mu = w \bullet \lambda$. But, by (10.33) and Theorem 1 we have

$$\begin{aligned} \mathrm{Index}_G\, \partial\!\!\!/_U &= \widehat{\bigoplus}_\lambda V_\lambda \otimes (V_\lambda^* \otimes (S_+ - S_-) \otimes U_\mu)^R \\ &= \widehat{\bigoplus} \mathrm{Hom}_R(V_\lambda \otimes (S_+ - S_-)^*, U_\mu). \end{aligned}$$

Now $(S_+ - S_-)^* = S_+ - S_-$ if $\dim \mathbf{p} \cong 0 \mod(4)$ while $(S_+ - S_-)^* = S_- - S_+$ if $\dim \mathbf{p} \cong 2 \mod(4)$. Hence

$$\mathrm{Index}_G\, \partial\!\!\!/_{U_\mu} = (-1)^{\dim \mathbf{p}/2} \widehat{\bigoplus} \mathrm{Hom}_R(V_\lambda \otimes (S_+ - S_-), U_\mu). \tag{10.36}$$

The right hand side of (10.36) vanishes if μ does not belong to a multiplet, i.e is not of the form

$$w \bullet \lambda = w(\lambda + \rho_{\mathbf{g}}) - \rho_{\mathbf{r}}$$

for some λ. The condition $w \bullet \lambda = \mu$ can thus be written as

$$w^{-1}(\mu + \rho_{\mathbf{r}}) - \rho_{\mathbf{g}} = \lambda.$$

If this equation does hold, then we get the formula in the theorem (with w replaced by w^{-1} which has the same determinant). QED

In general, if G/R is not a spin manifold, then in order to obtain a similar result we must instead consider the operator

$$\partial\!\!\!/_{U_\mu} : \left(L^2(G) \otimes (S_\pm) \otimes U_\mu\right)^{\mathbf{r}} \to \left(L^2(G) \otimes (S_\mp) \otimes U_\mu\right)^{\mathbf{r}}$$

viewed as an operator on G, restricted to the space of $(S \otimes U_\mu)$-valued functions on G that are invariant under the diagonal $\mathbf{r}$-action $\varrho(Z) = \mathrm{diag}(Z) + \sigma(Z)$, where σ is the $\mathbf{r}$-action on U_μ. Note that if $S \otimes U_\mu$ is induced by a representation of the Lie group R, then this operator descends to a well-defined operator on G/R as before. In general, the G-equivariant index of this operator $\partial\!\!\!/_{U_\mu}$ is once again given by (10.35). To prove this, we note that Bott's identity (10.33) and his theorem continue to hold for the induction map $i_* : \mathrm{R}(\mathbf{r}) \to \hat{\mathrm{R}}(\mathbf{g})$ using the representation rings for the Lie algebras instead of the Lie groups. Working in the Lie algebra context, we no longer need concern ourselves with the topological obstructions occurring in the global Lie group picture. The rest of the proof of Theorem 2 continues unchanged.