Lie Algebras

Shlomo Sternberg

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Chapter 6

Classification of the simple finite dimensional Lie algebras.

In this chapter we classify all possible root systems of simple Lie algebras. A consequence, as we shall see, is the classification of the simple Lie algebras themselves. The amazing result - due to Killing with some repair work by Élie Cartan - is that with only five exceptions, the root systems of the classical algebras that we studied in Chapter III exhaust all possibilities.

The logical structure of this chapter is as follows: We first show that the root system of a simple Lie algebra is irreducible (definition below). We then develop some properties of the of the root structure of an irreducible root system, in particular we will introduce its extended Cartan matrix. We then use the Perron-Frobenius theorem to classify all possible such matrices. (For the expert, this means that we first classify the Dynkin diagrams of the affine algebras of the simple Lie algebras. Surprisingly, this is simpler and more efficient than the classification of the diagrams of the finite dimensional simple Lie algebras themselves.) From the extended diagrams it is an easy matter to get all possible bases of irreducible root systems. We then develop a few more facts about root systems which allow us to conclude that an isomorphism of irreducible root systems implies an isomorphism of the corresponding Lie algebras. We postpone the the proof of the existence of the exceptional Lie algebras until Chapter VIII, where we prove Serre's theorem which gives a unified presentation of all the simple Lie algebras in terms of generators and relations derived directly from the Cartan integers of the simple root system.

Throughout this chapter we will be dealing with semi-simple Lie algebras over the complex numbers.

6.1 Simple Lie algebras and irreducible root systems.

We choose a Cartan subalgebra $\mathbf{h}$ of a semi-simple Lie algebra $\mathbf{g}$, so we have the corresponding set Φ of roots and the real (Euclidean) space E that they span. We say that Φ is **irreducible** if Φ can *not* be partitioned into two disjoint subsets

$$\Phi = \Phi_1 \cup \Phi_2$$

such that every element of Φ_1 is orthogonal to every element of Φ_2.

Proposition 1 *If* $\mathbf{g}$ *is simple then* Φ *is irreducible.*

Proof. Suppose that Φ is not irreducible, so we have a decomposition as above. If $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ then

$$(\alpha+\beta,\alpha) = (\alpha,\alpha) > 0 \quad \text{and} \quad (\alpha+\beta,\beta) = (\beta,\beta) > 0$$

which means that $\alpha+\beta$ can not belong to either Φ_1 or Φ_2 and so is not a root. This means that

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = 0.$$

In other words, the subalgebra $\mathbf{g}_1$ of $\mathbf{g}$ generated by all the $\mathbf{g}_\alpha$, $\alpha \in \Phi_1$ is centralized by all the $\mathbf{g}_\beta$, so $\mathbf{g}_1$ is a proper subalgebra of $\mathbf{g}$, since if $\mathbf{g}_1 = \mathbf{g}$ this would say that $\mathbf{g}$ has a non-zero center, which is not true for any semi-simple Lie algebra. The above equation also implies that the normalizer of $\mathbf{g}_1$ contains all the $\mathbf{g}_\gamma$ where γ ranges over all the roots. But these $\mathbf{g}_\gamma$ generate $\mathbf{g}$. So $\mathbf{g_1}$ is a proper ideal in $\mathbf{g}$, contradicting the assumption that $\mathbf{g}$ is simple. QED

Let us choose a base Δ for the root system Φ of a semi-simple Lie algebra. We say that Δ is irreducible if we can not partition Δ into two non-empty mutually orthogonal sets as in the definition of irreducibility of Φ as above.

Proposition 2 Φ *is irreducible if and only if* Δ *is irreducible.*

Proof. Suppose that Φ is not irreducible, so has a decomposition as above. This induces a partition of Δ which is non-trivial unless Δ is wholly contained in Φ_1 or Φ_2. If $\Delta \subset \Phi_1$ say, then since E is spanned by Δ, this means that all the elements of Φ_2 are orthogonal to E which is impossible. So if Δ is irreducible so is Φ. Conversely, suppose that

$$\Delta = \Delta_1 \cup \Delta_2$$

is a partition of Δ into two non-empty mutually orthogonal subsets. We have proved that every root is conjugate to a simple root by an element of the Weyl group W which is generated by the simple reflections. Let Φ_1 consist of those roots which are conjugate to an element of Δ_1 and Φ_2 consist of those roots which are conjugate to an element of Δ_2. The reflections $s_\beta, \beta \in \Delta_2$ commute with the reflections s_α, $\alpha \in \Delta_1$, and furthermore

$$s_\beta(\alpha) = \alpha$$

since $(\alpha,\beta)=0$. So any element of Φ_1 is conjugate to an element of Δ_1 by an element of the subgroup W_1 generated by the $s_\alpha,\ \alpha\in\Delta_1$. But each such reflection adds or subtracts α. So Φ_1 is in the subspace E_1 of E spanned by Δ_1 and so is orthogonal to all the elements of Φ_2. So if Φ_1 is irreducible so is Δ. QED

We are now into the business of classifying irreducible bases.

6.2 The maximal root and the minimal root.

Suppose that Φ is an irreducible root system and Δ a base, so irreducible. Recall that once we have chosen Δ, every root β is an integer combination of the elements of Δ with all coefficients non-negative, or with all coefficients non-positive. We write $\beta\succ 0$ in the first case, and $\beta\prec 0$ in the second case. This defines a partial order on the elements of E by

$$\mu\prec\lambda \text{ if and only if } \lambda-\mu=\sum_{\alpha\in\Delta}k_\alpha\alpha, \tag{6.1}$$

where the k_α are non-negative integers. This partial order will prove very important to us in representation theory.

Also, for any $\beta=\sum k_\alpha\alpha\in\Phi^+$ we define its **height** by

$$\text{ht }\beta=\sum_\alpha k_\alpha. \tag{6.2}$$

Proposition 3 *Suppose that Φ is an irreducible root system and Δ a base. Then*

- *There exists a unique $\beta\in\Phi^+$ which is maximal relative to the ordering $\prec$.*
- *This $\beta=\sum k_\alpha\alpha$ where all the k_α are positive.*
- *$(\beta,\alpha)\geq 0$ for all $\alpha\in\Delta$ and $(\beta,\alpha)>0$ for at least one $\alpha\in\Delta$.*

Proof. Choose a $\beta=\sum k_\alpha\alpha$ which is maximal relative to the ordering. At least one of the $k_\alpha>0$. We claim that *all* the $k_\alpha>0$. Indeed, suppose not. This partitions Δ into Δ_1, the set of α for which $k_\alpha>0$ and Δ_2, the set of α for which $k_\alpha=0$. Now the scalar product of any two distinct simple roots is ≤ 0. (Recall that this followed from the fact that if $(\alpha_1,\alpha_2)>0$, then $s_2(\alpha_1)=\alpha_1-\langle\alpha_1,\alpha_2\rangle\alpha_2$ would be a root whose α_1 coefficient is positive and whose α_2 coefficient is negative which is impossible.) In particular, all the $(\alpha_1,\alpha_2)\leq 0,\ \alpha_1\in\Delta_1,\ \alpha_2\in\Delta_2$ and so

$$(\beta,\alpha_2)\leq 0,\quad\forall\ \alpha_2\in\Delta_2.$$

The irreducibility of Δ implies that $(\alpha_1,\alpha_2)\neq 0$ for at least one pair $\alpha_1\in\Delta_1,\ \alpha_2\in\Delta_2$. But this scalar product must then be negative. So

$$(\beta,\alpha_2)<0$$

and hence

$$s_{\alpha_2}\beta = \beta - \langle\beta, \alpha_2\rangle\alpha_2$$

is a root with

$$s_{\alpha_2}\beta - \beta \succ 0$$

contradicting the maximality of β. So we have proved that all the k_α are positive. Furthermore, this same argument shows that $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$. Since the elements of Δ form a basis of E, at least one of the scalar products must not vanish, and so be positive. We have established the second and third items in the proposition for any maximal β. We will now show that this maximal weight is unique.

Suppose there were two, β and β'. Write $\beta' = \sum k'_\alpha \alpha$ where all the $k'_\alpha > 0$. Then $(\beta, \beta') > 0$ since $(\beta, \alpha) \geq 0$ for all α and > 0 for at least one. Since $s_\beta\beta'$ is a root, this would imply that $\beta - \beta'$ is a root, unless $\beta = \beta'$. But if $\beta - \beta'$ is a root, it is either positive or negative, contradicting the maximality of one or the other. QED

Let us label the elements of Δ as $\alpha_1, \ldots, \alpha_\ell$, and let us set

$$\alpha_0 := -\beta$$

so that α_0 is the minimal root. From the second and third items in the proposition we know that

$$\alpha_0 + k_1\alpha_1 + \cdots + k_\ell\alpha_\ell = 0 \tag{6.3}$$

and that

$$\langle\alpha_0, \alpha_i\rangle \leq 0$$

for all i and < 0 for some i.

Let us take the left hand side (call it γ) of (6.3) and successively compute $\langle\gamma, \alpha_i\rangle,\ i = 0, 1, \ldots, \ell$. We obtain

$$\begin{pmatrix} 2 & \langle\alpha_1, \alpha_0\rangle & \cdots & \langle\alpha_\ell, \alpha_0\rangle \\ \langle\alpha_0, \alpha_1\rangle & 2 & \cdots & \langle\alpha_\ell, \alpha_1\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle\alpha_0, \alpha_\ell\rangle & \cdots & \langle\alpha_{\ell-1}, \alpha_\ell\rangle & 2 \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \\ \vdots \\ k_\ell \end{pmatrix} = 0.$$

This means that if we write the matrix on the left of this equation as $2I - A$, then A is a matrix with 0 on the diagonal and whose i, j entry is $-\langle\alpha_j, \alpha_i\rangle$.

So A a non-negative matrix with integer entries with the properties

- if $A_{ij} \neq 0$ then $A_{ji} \neq 0$,
- The diagonal entries of A are 0,
- A is irreducible in the sense that we can not partition the indices into two non-empty subsets I and J such that $A_{ij} = 0\ \forall\ i \in I,\ j \in J$ and
- A has an eigenvector of eigenvalue 2 with all its entries positive.

We will show that the Perron-Frobenius theorem allows us to classify all such matrices. From here it is an easy matter to classify all irreducible root systems and then all simple Lie algebras. For this it is convenient to introduce the language of graph theory.

6.3 Graphs.

An **undirected graph** $\Gamma = (N, E)$ consists of a set N (for us finite) and a subset E of the set of subsets of N of cardinality two. We call elements of N "nodes" or "vertices" and the elements of E "edges". If $e = \{i, j\} \in E$ we say that the "edge" e joins the vertices i and j or that "i and j are adjacent". Notice that in this definition our edges are "undirected": $\{i, j\} = \{j, i\}$, and we do not allow self-loops. An example of a graph is the "cycle" $A_\ell^{(1)}$ with $\ell + 1$ vertices, so $N = \{0, 1, 2, \dots, \ell\}$ with 0 adjacent to ℓ and to 1, with 1 adjacent to 0 and to 2 etc.

The **adjacency matrix** A of a graph Γ is the (symmetric) $0 - 1$ matrix whose rows and columns are indexed by the elements of N and whose i, j-th entry $A_{ij} = 1$ if i is adjacent to j and zero otherwise.

For example, the adjacency matrix of the graph $A_3^{(1)}$ is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

We can think of A as follows: Let V be the vector space with basis given by the nodes, so we can think of the i-th coordinate of a vector $x \in V$ as assigning the value x_i to the node i. Then $y = Ax$ assigns to i the sum of the values x_j summed over all nodes j adjacent to i.

A **path** of length r is a sequence of nodes $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ where each node is adjacent to the next. So, for example, the number of paths of length 2 joining i to j is the i, j-th entry in A^2 and similarly, the number of paths of length r joining i to j is the i, j-th entry in A^r. The graph is said to be **connected** if there is a path (of some length) joining every pair of vertices. In terms of the adjacency matrix, this means that for every i and j there is some r such that the i, j entry of A^r is non-zero. In terms of the theory of non-negative matrices (see below) this says that the matrix A is *irreducible.*

Notice that if $\mathbf{1}$ denotes the column vector all of whose entries are 1, then $\mathbf{1}$ is an eigenvector of the adjacency matrix of $A_\ell^{(1)}$, with eigenvalue 2, and all the entries of $\mathbf{1}$ are positive. In view of the Perron-Frobenius theorem to be stated below, this implies that 2 is the maximum eigenvalue of this matrix.

We modify the notion of the adjacency matrix as follows: We start with a connected graph Γ as before, but modify its adjacency matrix by replacing some of the ones that occur by positive integers a_{ij}. If, in this replacement $a_{ij} > 1$, we redraw the graph so that there is an arrow with a_{ij} lines pointing towards the node i. For example, the graph labeled $A_1^{(1)}$ in Table **Aff 1** corresponds to the matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

which clearly has $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as an positive eigenvector with eigenvalue 2.

Similarly, diagram $A_2^{(2)}$ in Table **Aff 2** corresponds to the matrix

$$\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

which has $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as eigenvector with eigenvalue 2. In the diagrams, the coefficient next to a node gives the coordinates of the eigenvector with eigenvalue 2, and it is immediate to check from the diagram that this is indeed an eigenvector with eigenvalue 2. For example, the 2 next to a node with an arrow pointing toward it in $C_\ell^{(1)}$ satisfies $2 \cdot 2 = 2 \cdot 1 + 2$ etc.

It will follow from the Perron Frobenius theorem to be stated and proved below, that these are the only possible connected diagrams with maximal eigenvector two.

All the graphs so far have zeros along the diagonal. If we relax this condition, and allow for any non-negative integer on the diagonal, then the only new possibilities are those given in Figure 4.

Let us call a matrix *symmetrizable* if $A_{ij} \neq 0 \Rightarrow A_{ji} \neq 0$. The main result of this chapter will be to show that the lists in the Figures 1-4 exhaust all irreducible matrices with non-negative integer matrices, which are symmetrizable and have maximum eigenvalue 2.

6.4 Perron-Frobenius.

We say that a real matrix T is **non-negative** (or **positive**) if all the entries of T are non-negative (or positive). We write $T \geq 0$ or $T > 0$. We will use these definitions primarily for square $(n \times n)$ matrices and for column vectors $= (n \times 1)$ matrices. We let

$$Q := \{x \in \mathbf{R}^n : x \geq 0, \quad x \neq 0\}$$

so Q is the non-negative "orthant" excluding the origin. Also let

$$C := \{x \geq 0 : \|x\| = 1\}.$$

So C is the intersection of the orthant with the unit sphere.

A non-negative matrix square T is called **primitive** if there is a k such that all the entries of T^k are positive. It is called **irreducible** if for any i, j there is a $k = k(i, j)$ such that $(T^k)_{ij} > 0$. For example, as mentioned above, the adjacency matrix of a connected graph is irreducible.

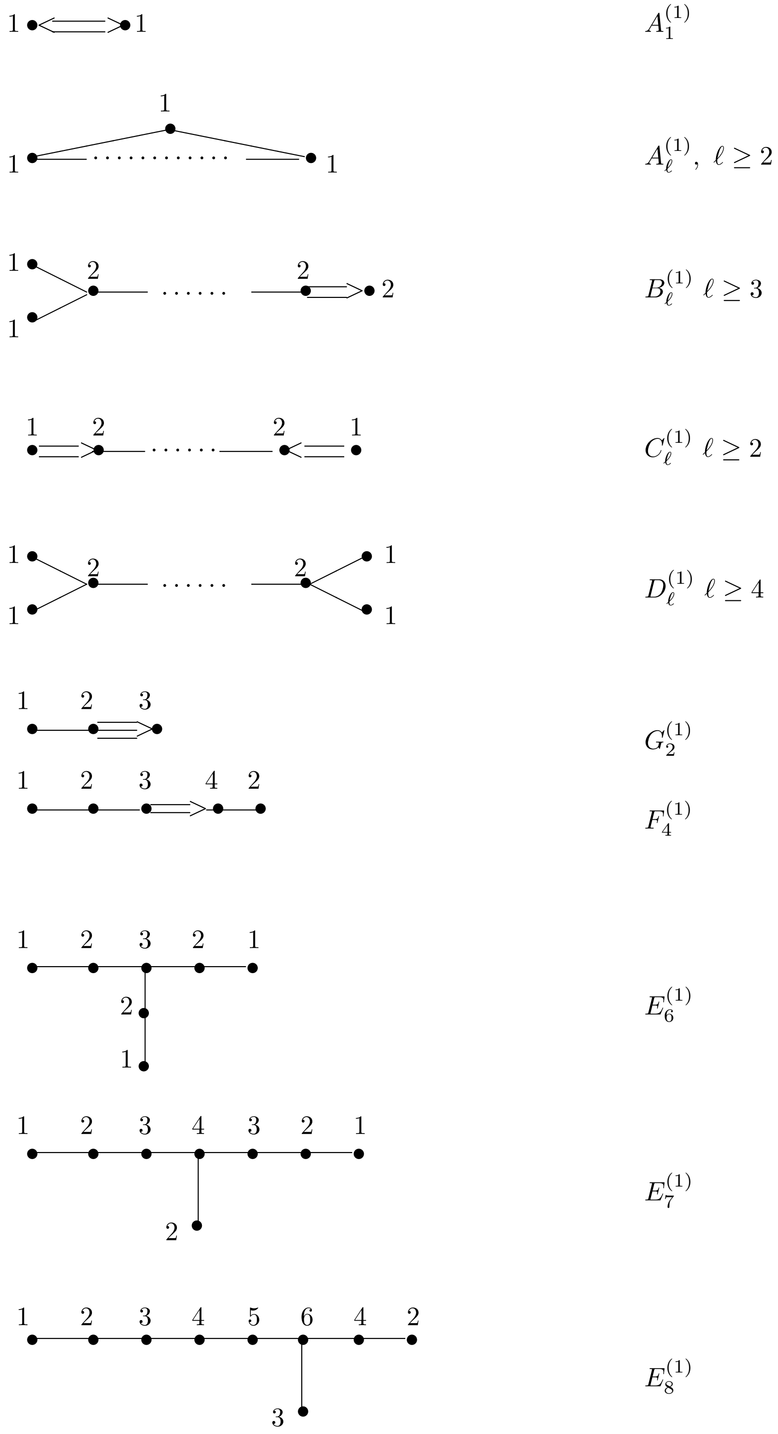

Figure 6.1: Aff 1.

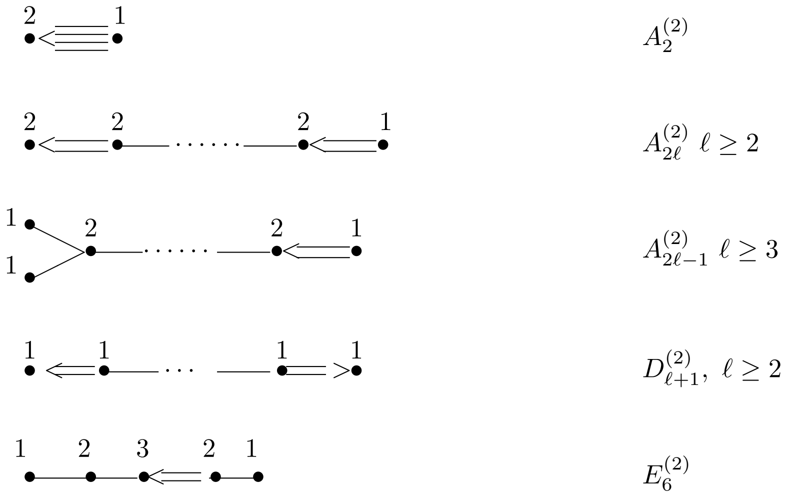

Figure 6.2: Aff 2

Figure 6.3: Aff 3

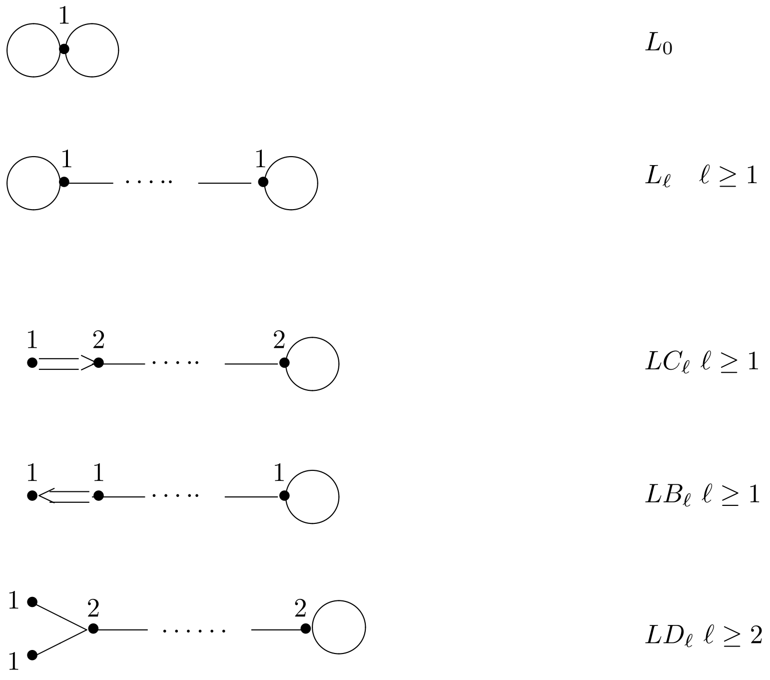

Figure 6.4: Loops allowed

If T is irreducible then $I+T$ is primitive.

In this section we will assume that T is non-negative and irreducible.

Theorem 1 Perron-Frobenius.

1. *T has a positive (real) eigenvalue $\lambda_{\max}$ such that all other eigenvalues of T satisfy*
$$|\lambda| \leq \lambda_{\max}.$$
2. *Furthermore $\lambda_{\max}$ has algebraic and geometric multiplicity one, and has an eigenvector x with $x > 0$.*
3. *Any non-negative eigenvector is a multiple of x.*
4. *More generally, if $y \geq 0,\ y \neq 0$ is a vector and μ is a number such that*
$$Ty \leq \mu y$$
then
$$y > 0, \quad \textit{and} \quad \mu \geq \lambda_{\max}$$
with $\mu = \lambda_{\max}$ if and only if y is a multiple of x.
5. *If $0 \leq S \leq T,\ S \neq T$ then every eigenvalue σ of S satisfies $|\sigma| < \lambda_{\max}$.*
6. *In particular, all the diagonal minors $T_{(i)}$ obtained from T by deleting the i-th row and column have eigenvalues all of which have absolute value $< \lambda_{\max}$.*

We will present a proof of this theorem after first showing how it classifies the possible connected diagrams with maximal eigenvalue two. But first let us clarify the meaning of the last two assertions of the theorem. The matrix $T_{(i)}$ is usually thought of as an $(n-1)\times(n-1)$ matrix obtained by "striking out" the i-th row and column. But we can also consider the matrix T_i obtained from T by replacing the i-th row and column by all zeros. If x is an n-vector which is an eigenvector of T_i, then the $n-1$ vector y obtained from x by omitting the (0) i-th entry of x is then an eigenvector of $T_{(i)}$ with the same eigenvalue (unless the vector x only had non-zero entries in the i-th position). Conversely, if y is an eigenvector of $T_{(i)}$ then inserting 0 at the i-th position will give an n-vector which is an eigenvector of T_i with with the same eigenvalue as that of y.

More generally, suppose that S is obtained from T by replacing a certain number of rows and the corresponding columns by all zeros. Then we may apply item 5) of the theorem to this $n \times n$ matrix, S, or the "compressed version" of S obtained by eliminating all these rows and columns.

We will want to apply this to the following special case. A subgraph Γ' of a graph Γ is the graph obtained by eliminating some nodes, and all edges emanating from these nodes. Thus, if A is the adjacency matrix of Γ and A' is the adjacency matrix of A, then A' is obtained from A by striking out some rows and their corresponding columns. Thus if Γ is irreducible, so that we may apply the Perron Frobenius theorem to A, and if Γ' is a proper subgraph (so we have actually deleted some rows and columns of A to obtain A'), then the maximum eigenvalue of A' is strictly less than the maximum eigenvalue of A' is strictly less than the maximum eigenvalue of A. Similarly, if an entry A_{ij} is > 1, the matrix A' obtained from A by decreasing this entry while still keeping it positive will have a strictly smaller maximal eigenvalue.

We now apply this theorem to conclude that the diagrams listed in Figures Aff 1, Aff2, and Aff 3 are all possible connected diagrams with maximal eigenvalue two. A direct check shows that the vector whose coordinate at each node is the integer attached to that node given in the figure is an eigenvector with eigenvalue 2. Perron-Frobenius then guarantees 2 is the maximal eigenvalue. But now that we have shown that for each of these diagrams the maximal eigenvalue is two, any "larger" diagram must have maximal eigenvalue strictly greater than two and any "smaller" diagram must have maximal eigenvalue strictly less than two.

To get started, this argument shows that $A_1^{(1)}$ is the only diagram for which there is an i, j for which *both* a_{ij} and a_{ji} are > 1. Indeed, if A were such a matrix, by striking out all but the i and j rows and columns, we would obtain a two by two matrix whose off diagonal entries are both ≥ 2. If there were strict inequality, the maximum eigenvalue of this matrix would have to be bigger than 2 (and hence also the original diagram) by Perron Frobenius.

So other than $A_1^{(1)}$, we may assume that if $a_{ij} > 1$ then $a_{ji} = 1$.

Since any diagram with some entry $a_{ij} \geq 4$ must contain $A_2^{(2)}$ we see that this is the only diagram with this property and with maximum eigenvalue 2.

So other than this case, all $a_{ij} \leq 3$.

Diagram $G_2^{(1)}$ shows that a diagram with only two vertices and a triple bond has maximum eigenvalue strictly less than 2, since it is contained in $G_2^{(1)}$ as a subdiagram. So any diagram with a triple bond must have at least three vertices. But then it must "contain" either $G_2^{(1)}$ or $D_4^{(3)}$. But as both of these have maximal eigenvalue 2, it can not strictly contain either. So $G_2^{(1)}$ and $D_4^{(3)}$.are the only possibilities with a triple bond.

Since $A_\ell^{(1)}, \ell \geq 2$ is a cycle with maximum eigenvalue 2, no graph can contain a cycle without actually being a cycle, i.e. being $A_\ell^{(1)}$. On the other hand, a simple chain with only single bonds is contained in $A_\ell^{(1)}$, and so must have maximum eigenvalue strictly less than 2, So other than $A_\ell^{(1)}$, every candidate must contain at least one branch point or one double bond.

If the graph contains two double bonds, there are three possibilities as to the mutual orientation of the arrows, they could point toward one another as in $C_\ell^{(1)}$, away from one another as in $D_{\ell+1}^{(2)}$ or in the same direction as in $A_{2\ell}^{(2)}$. But then these are the only possibilities for diagrams with two double bonds, as no diagram can strictly contain any of them.

Also, striking off one end vertex of $C_\ell^{(1)}$ yields a graph with one extreme vertex with a double bound, with the arrow pointing away from the vertex, and no branch points. Striking out one of the two vertices at the end opposite the double bond in $B_\ell^{(1)}$ yields a graph with one extreme vertex with with a double bound and with the arrow pointing toward this vertex. So either diagram must have maximum eigenvalue < 2.

Thus if there are no branch points, there must be at least one double bond and at least two vertices on either side of the double bond. The graph with exactly two vertices on either side is strictly contained in $F_4^{(1)}$ and so is excluded. So there must be at least three vertices on one side and two on the other of the double bond. But then $F_4^{(1)}$ and $E_6^{(2)}$ exhaust the possibilities for one double bond and no branch points.

If there is a double bond and a branch point then either the double bond points toward the branch, as in $A_{2\ell-1}^{(2)}$ or away from the branch as in $B_\ell^{(1)}$. But then these exhaust the possibilities for a diagram containing both a double bond and a branch point.

If there are two branch points, the diagram must contain $D_\ell^{(1)}$ and hence must coincide with $D_\ell^{(1)}$.

So we are left with the task of analyzing the possibilities for diagrams with no double bonds and a single branch point. Let m denote the minimum number of vertices on some leg of a branch (excluding the branch point itself). If $m \geq 2$, then the diagram contains $E_6^{(1)}$ and hence must coincide with $E_6^{(1)}$. So we may assume that $m = 1$. If two branches have only one vertex emanating, then the diagram is strictly contained in $D_\ell^{(1)}$ and hence excluded. So each of the two other legs have at least two or more vertices. If both legs have more than two vertices on them, the graph must contain, and hence coincide with $E_7^{(1)}$. We are left with the sole possibility that one of the legs emanating from the branch point has one vertex and a second leg has two vertices. But then either the graph contains or is contained in $E_8^{(1)}$ so $E_8^{(1)}$ is the only such possibility.

We have completed the proof that the diagrams listed in Aff 1, Aff 2 and Aff 3 are the only diagrams without loops with maximum eigenvalue 2.

If we allow loops, an easy extension of the above argument shows that the only new diagrams are the ones in the table "Loops allowed".

6.5 Classification of the irreducible Δ.

Notice that if we remove a vertex labeled 1 (and the bonds emanating from it) from any of the diagrams in **Aff 2** or **Aff 3** we obtain a diagram which can also be obtained by removing a vertex labeled 1 from one of the diagrams in**Aff 1**. (In the diagram so obtained we ignore the remaining labels.) Indeed, removing the right hand vertex labeled 1 from $D_4^{(3)}$ yields A_2 which is obtained from $A_2^{(1)}$ by removing a vertex. Removing the left vertex marked 1 gives G_2, the diagram obtained from $G_2^{(1)}$ by removing the vertex marked 1.

Removing a vertex from $A_2^{(2)}$ gives A_1. Removing the vertex labeled 1 from $A_{2\ell}^{(2)}$ yields $B_{2\ell}$, obtained by removing one of the vertices labeled 1 from $B_\ell^{(1)}$.

Removing a vertex labeled 1 from $A^{(2)}_{2\ell-1}$ yields $D_{2\ell}$ or $C_{2\ell}$, removing a vertex labeled 1 from $D^{(2)}_{\ell+1}$ yields $B_{\ell+1}$ and removing a vertex labeled 1 from $E^{(2)}_{6}$ yields F_4 or C_4.

Thus all irreducible Δ correspond to graphs obtained by removing a vertex labeled 1 from the table**Aff 1**. So we have classified all possible Dynkin diagrams of all irreducible Δ . They are given in the table labeled Dynkin diagrams.

6.6 Classification of the irreducible root systems.

It is useful to introduce here some notation due to Bourbaki: A subset Φ of a Euclidean space E is called a **root system** if the following axioms hold:

- Φ is finite, spans E and does not contain 0.
- If $\alpha \in \Phi$ then the only multiples of α which are in Φ are $\pm\alpha$.
- If $\alpha \in \Phi$ then the reflection s_α in the hyperplane orthogonal to α sends Φ into itself.
- If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbf{Z}$,

Recall that

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

so that the reflection s_α is given by

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

We have shown that each semi-simple Lie algebra gives rise to a root system, and derived properties of the root system. If we go back to the various arguments, we will find that most of them apply to a "general" root system according to the above definition. The one place where we used Lie algebra arguments directly, was in showing that if $\beta \neq \pm\alpha$ is a root then the collection of j such that $\beta + j\alpha$ is a root forms an unbroken chain going from $-r$ to q where $r - q = \langle \beta, \alpha \rangle$. For this we used the representation theory of $sl(2)$. So we now pause to give an alternative proof of this fact based solely on that axioms above, and in the process derive some additional useful information about roots.

For any two non-zero vectors α and β in E, the cosine of the angle between them is given by

$$\|\alpha\| \|\beta\| \cos\theta = (\alpha, \beta).$$

So

$$\langle \beta, \alpha \rangle = 2\frac{\|\beta\|}{\|\alpha\|} \cos\theta.$$

Interchanging the role of α and β and multiplying gives

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta.$$

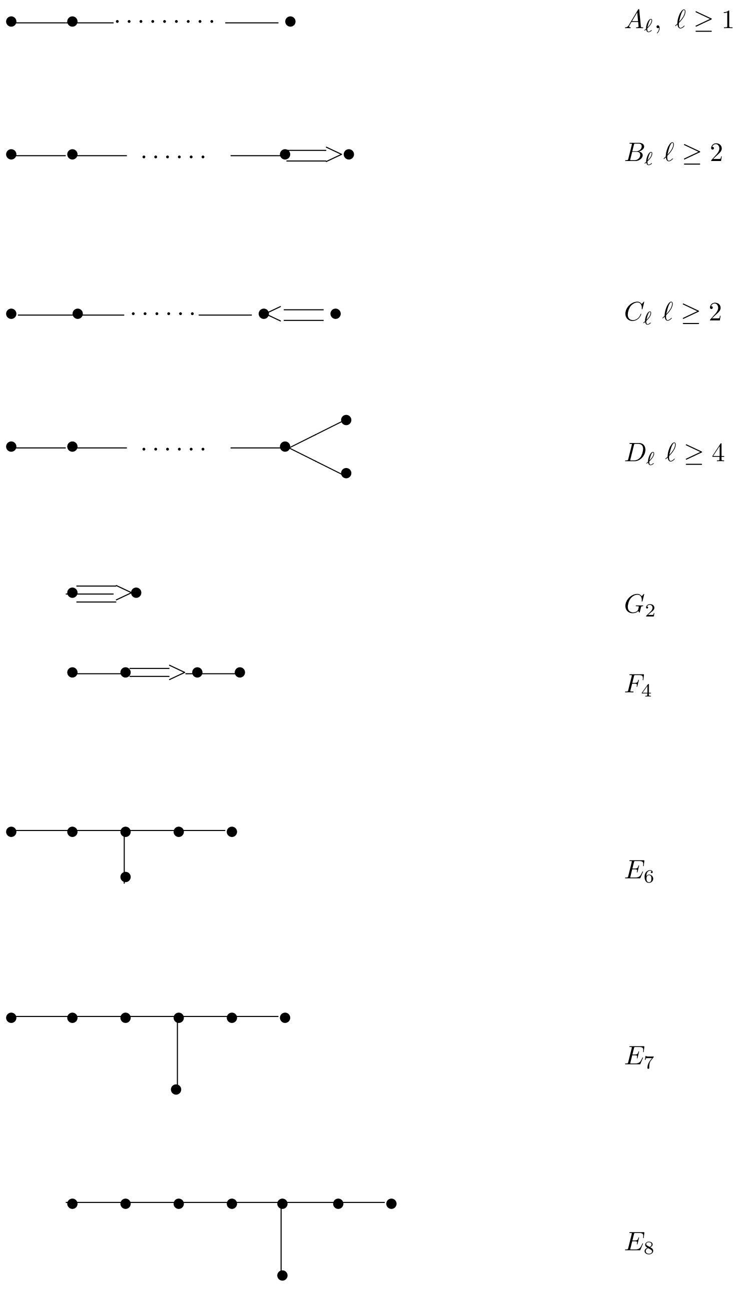

Figure 6.5: Dynkin diagrams.

The right hand side is a non-negative integer between 0 and 4. So assuming that $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$ The possibilities are listed in the following table:

$\langle\alpha, \beta\rangle$	$\langle\beta, \alpha\rangle$	$0 \leq \theta \leq \pi$	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-1	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Proposition 4 *If $\alpha \neq \pm\beta$ and if $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.*

Proof. The second assertion follows from the first by replacing β by $-\beta$. So we need to prove the first assertion. From the table, one or the other of $\langle\beta, \alpha\rangle$ or $\langle\alpha, \beta\rangle$ equals one. So either $s_\alpha\beta = \beta - \alpha$ is a root or $s_\beta\alpha = \alpha - \beta$ is a root. But roots occur along with their negatives so in either event $\alpha - \beta$ is a root. QED

Proposition 5 *Suppose that $\alpha \neq \pm\beta$ are roots. Let r be the largest integer such that $\beta - r\alpha$ is a root, and let q be the largest integer such that $\beta + q\alpha$ is a root. Then $\beta + i\alpha$ is a root for all $-r \leq i \leq q$. Furthermore $r - q = \langle\beta, \alpha\rangle$ so in particular $|q - r| \leq 3$.*

Proof. Suppose not. Then we can find a p and an s such that $-r \leq p < s \leq q$ such that $\beta + p\alpha$ is a root, but $\beta + (p+1)\alpha$ is not a root, and $\beta + s\alpha$ is a root but $\beta + (s-1)\alpha$ is not. The preceding proposition then implies that

$$\langle\beta + p\alpha, \alpha\rangle \geq 0 \quad \text{while} \quad \langle\beta + s\alpha, \alpha\rangle \leq 0$$

which is impossible since $\langle\alpha, \alpha\rangle = 2 > 0$.

Now s_α adds a multiple of α to any root, and so preserves the string of roots $\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta + q\alpha$. Furthermore

$$s_\alpha(\beta + i\alpha) = \beta - (\langle\beta, \alpha\rangle + i)\alpha$$

so s_α reverses the order of the string. In particular

$$s_\alpha(\beta + q\alpha) = \beta - r\alpha.$$

The left hand side is $\beta - (\langle\beta, \alpha\rangle + q)\alpha$ so $r - q = \langle\beta, \alpha\rangle$ as stated in the proposition. QED

We can now apply all the preceding definitions and arguments to conclude that the Dynkin diagrams above classify all the irreducible bases Δ of root systems.

Since every root is conjugate to a simple root, we can use the Dynkin diagrams to conclude that in an irreducible root system, either all roots have the same length (cases A, D, E) or there are two root lengths - the remaining cases. Furthermore, if β denotes a long root and α a short root, the ratios $\|\beta\|^2/\|\alpha\|^2$ are 2 in the cases B, C, and F_4, and 3 for the case G_2.

Proposition 6 *In an irreducible root system, the Weyl group W acts irreducibly on E. In particular, the W-orbit of any root spans E.*

Proof. Let E' be a proper invariant subspace. Let E'' denote its orthogonal complement, so

$$E = E' \oplus E''.$$

For any root α, If $e \in E'$ then $s_\alpha e = e - \langle e, \alpha\rangle\alpha \in E'$. So either $(e, \alpha) = 0$ for all e, and so $\alpha \in E''$ or $\alpha \in E'$. Since the roots span, they can't all belong to the same subspace. This contradicts the irreducibility. QED

Proposition 7 *If there are two distinct root lengths in an irreducible root system, then all roots of the same length are conjugate under the Weyl group. Also, the maximal weight is long.*

Proof. Suppose that α and β have the same length. We can find a Weyl group element W such that $w\beta$ is not orthogonal to α by the preceding proposition. So we may assume that $\langle\beta, \alpha\rangle \neq 0$. Since α and β have the same length, by the table above we have $\langle\beta, \alpha\rangle = \pm 1$. Replacing β by $-\beta = s_\beta\beta$ we may assume that $\langle\beta, \alpha\rangle = 1$. Then

$$\begin{aligned}(s_\beta s_\alpha s_\beta)(\alpha) &= (s_\beta s_\alpha)(\alpha - \beta)\\ &= s_\beta(-\alpha - \beta + \alpha)\\ &= s_\beta(-\beta)\\ &= \beta.\ \text{QED}\end{aligned}$$

Let (E, Φ) and (E', Φ') be two root systems. We say that a linear map $f : E \to E'$ is an **isomorphism** from the root system (E, Φ) to the root system (E', Φ') if f is a linear isomorphism of E onto E' with $f(\Phi) = \Phi'$ and

$$\langle f(\beta), f(\alpha)\rangle = \langle\beta, \alpha\rangle$$

for all $\alpha, \beta \in \Phi$.

Theorem 2 *Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a base of Φ. Suppose that (E', Φ') is a second root system with base $\Delta' = \{\alpha'_1, \ldots, \alpha'_\ell\}$ and that*

$$\langle\alpha_i, \alpha_j\rangle = \langle\alpha'_i, \alpha'_j\rangle, \quad \forall\ 1 \leq i, j \leq \ell.$$

Then the bijection

$$\alpha_i \mapsto \alpha'_i$$

extends to a unique isomorphism $f : (E, \Phi) \to (E', \Phi')$*. In other words, the Cartan matrix* A *of* Δ *determines* Φ *up to isomorphism. In particular, The Dynkin diagrams characterize all possible irreducible root systems.*

Proof. Since Δ is a basis of E and Δ' is a basis of E', the map $\alpha_i \mapsto \alpha_i'$ extends to a unique linear isomorphism of E onto E'. The equality in the theorem implies that for $\alpha, \beta \in \Delta$ we have

$$s_{f(\alpha)} f(\beta) = f(\beta) - \langle f(\beta), f(\alpha)\rangle f(\alpha) = f(s_\alpha \beta).$$

Since the Weyl groups are generated by these simple reflections, this implies that the map

$$w \mapsto f \circ w \circ f^{-1}$$

is an isomorphism of W onto W'. Every $\beta \in \Phi$ is of the form $w(\alpha)$ where $w \in W$ and α is a simple root. Thus

$$f(\beta) = f \circ w \circ f^{-1} f(\alpha) \in \Phi'$$

so $f(\Phi) = \Phi'$. Since $s_\alpha(\beta) = \beta - \langle \beta, \alpha\rangle \alpha$, the number $\langle \beta, \alpha\rangle$ is determined by the reflection s_α acting on β. But then the corresponding formula for Φ' together with the fact that

$$s_{f(\alpha)} = f \circ s_\alpha \circ f^{-1}$$

implies that

$$\langle f(\beta), f(\alpha)\rangle = \langle \beta, \alpha\rangle.$$

QED

6.7 The classification of the possible simple Lie algebras.

Suppose that $\mathbf{g}, \mathbf{h}$, is a pair consisting of a semi-simple Lie algebra $\mathbf{g}$, and a Cartan subalgebra $\mathbf{h}$. This determines the corresponding Euclidean space E and root system Φ. Suppose we have a second such pair $(\mathbf{g}', \mathbf{h}')$. We would like to show that an isomorphism of (E, Φ) with (E', Φ') determines a Lie algebra isomorphism of $\mathbf{g}$ with $\mathbf{g}'$. This would then imply that the Dynkin diagrams classify all possible simple Lie algebras. We would still be left with the problem of showing that the exceptional Lie algebras exist. We will defer this until Chapter VIII where we prove Serre's theorem with gives a direct construction of all the simple Lie algebras in terms of generators and relations determined by the Cartan matrix.

We need a few preliminaries.

Proposition 8 *Every positive root can be written as a sum of simple roots*

$$\alpha_{i_1} + \cdots \alpha_{i_k}$$

in such a way that every partial sum is again a root.

Proof. By induction (on say the height) it is enough to prove that for every positive root β which is not simple, there is a simple root α such that $\beta - \alpha$ is a root. We can not have $(\beta, \alpha) \leq 0$ for all $\alpha \in \Delta$ for this would imply that the set $\{\beta\} \cup \Delta$ is independent (by the same method that we used to prove that Δ was independent). So $(\beta, \alpha) > 0$ for some $\alpha \in \Delta$ and so $\beta - \alpha$ is a root. Since β is not simple, its height is at least two, and so subtracting α will not be zero or a negative root, hence positive. QED

Proposition 9 *Let* $\mathbf{g}, \mathbf{h}$ *be a semi-simple Lie algebra with a choice of Cartan subalgebra, Let* Φ *be the corresponding root system, and let* Δ *be a base. Then* $\mathbf{g}$ *is generated as a Lie algebra by the subspaces* $\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}, \alpha \in \Delta$*.*

From the representation theory of $sl(2)_\alpha$ we know that $[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{g}_{\alpha+\beta}$ if $\alpha + \beta$ is a root. Thus from the preceding proposition, we can successively obtain all the $\mathbf{g}_\beta$ for β positive by bracketing the $\mathbf{g}_\alpha$, $\alpha \in \Delta$. Similarly we can get all the $\mathbf{g}_\beta$ for β negative from the $\mathbf{g}_{-\alpha}$. So we can get all the root spaces. But $[\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}] = \mathbf{C}h_\alpha$ so we can get all of $\mathbf{h}$. The decomposition

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\gamma \in \Phi} \mathbf{g}_\gamma$$

then shows that we have generated all of $\mathbf{g}$.

Here is the big theorem:

Theorem 3 *Let* $\mathbf{g}, \mathbf{h}$ *and* $\mathbf{g}', \mathbf{h}'$ *be simple Lie algebras with choices of Cartan subalgebras, and let* Φ, Φ' *be the corresponding root systems. Suppose there is an isomorphism*

$$f : (E, \Phi) \to (E', \Phi')$$

which is an isometry of Euclidean spaces. Extend f *to an isomorphism of*

$$\mathbf{h}^* \to \mathbf{h}'^*$$

via complexification. Let $f : \mathbf{h} \to \mathbf{h}'$ *denote the corresponding isomorphism on the Cartan subalgebras obtained by identifying* $\mathbf{h}$ *and* $\mathbf{h}'$ *with their duals using the Killing form.*

Fix a base Δ *of* Φ *and* Δ' *of* Φ'*. Choose* $0 \neq x_\alpha \in \mathbf{g}_\alpha, \alpha \in \Delta$ *and* $0 \neq x'_{\alpha'} \in \mathbf{g}'_{\alpha'}$*. Extend* f *to a linear map*

$$f : \mathbf{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbf{g}_\alpha \to \mathbf{h}' \oplus \bigoplus_{\alpha' \in \Delta'} \mathbf{g}_{\alpha'}$$

by

$$f(x_\alpha) = x'_{\alpha'}.$$

Then f *extends to a unique isomorphism of* $\mathbf{g} \to \mathbf{g}'$*.*

Proof. The uniqueness is easy. Given x_α there is a unique $y_\alpha \in \mathbf{g}_{-\alpha}$ for which $[x_\alpha, y_\alpha] = h_\alpha$ so f, if it exists, is determined on the y_α and hence on all of $\mathbf{g}$ since the x_α and y_α generate $\mathbf{g}$ be the preceding proposition.

To prove the existence, we will construct the graph of this isomorphism. That is, we will construct a subalgebra $\mathbf{k}$ of $\mathbf{g} \oplus \mathbf{g}'$ whose projections onto the first and onto the second factor are isomorphisms:

Use the x_α and y_α as above, with the corresponding elements $x'_{\alpha'}$ and $y'_{\alpha'}$ in $\mathbf{g}'$. Let

$$\overline{x}_\alpha := x_\alpha \oplus x'_{\alpha'} \in \mathbf{g} \oplus \mathbf{g}'$$

and similarly define

$$\overline{y}_\alpha := y_\alpha \oplus y'_{\alpha'},$$

and

$$\overline{h_\alpha} := h_\alpha \oplus h'_{\alpha'}.$$

Let β be the (unique) maximal root of $\mathbf{g}$, and choose $x \in \mathbf{g}_\beta$. Make a similar choice of $x' \in \mathbf{g}'_{\beta'}$ where β' is the maximal root of $\mathbf{g}'$. Set

$$\overline{x} := x \oplus x'.$$

Let $\mathbf{m} \subset \mathbf{g} \oplus \mathbf{g}'$ be the subspace spanned by all the

$$\operatorname{ad} \overline{y}_{\alpha_{i_1}} \cdots \operatorname{ad} \overline{y}_{\alpha_{i_m}} \overline{x}.$$

The element $\operatorname{ad} \overline{y}_{\alpha_{i_1}} \cdots \operatorname{ad} \overline{y}_{\alpha_{i_m}} \overline{x}$ belongs to $\mathbf{g}_{\beta - \sum \alpha_{i_j}} \oplus \mathbf{g}'_{\beta' - \sum \alpha'_{i_j}}$ so

$$\mathbf{m} \cap (\mathbf{g}_\beta \oplus \mathbf{g}'_{\beta'}) \;\text{ is one dimensional .}$$

In particular $\mathbf{m}$ is a proper subspace of $\mathbf{g} \oplus \mathbf{g}'$.

Let $\mathbf{k}$ denote the subalgebra of $\mathbf{g} \oplus \mathbf{g}'$ generated by the $\overline{x}_\alpha$ the $\overline{y}_\alpha$ and the $\overline{h}_\alpha$. We claim that

$$[\mathbf{k}, \mathbf{m}] \subset \mathbf{m}.$$

Indeed, it is enough to prove that $\mathbf{m}$ is invariant under the adjoint action of the generators of $\mathbf{k}$. For the $\operatorname{ad} \overline{y}_\alpha$ this follows from the definition. For the $\operatorname{ad} \overline{h}_\alpha$ we use the fact that

$$[h, y_\alpha] = -\alpha(h) y_\alpha$$

to move the $\operatorname{ad} \overline{h}_\alpha$ past all the $\operatorname{ad} \overline{y}_\gamma$ at the cost of introducing some scalar multiple, while

$$\operatorname{ad} \overline{h}_\alpha \overline{x} = \langle \beta, \alpha \rangle x_\beta + \langle \beta', \alpha' \rangle x'_{\beta'} = \langle \beta, \alpha \rangle \overline{x}$$

because f is an isomorphism of root systems.

Finally $[x_{\alpha_1}, y_{\alpha_2}] = 0$ if $\alpha_1 \neq \alpha_2 \in \Delta$ since $\alpha_1 - \alpha_2$ is not a root. On the other hand $[x_\alpha, y_\alpha] = h_\alpha$. So we can move the $\operatorname{ad} \overline{x}_\alpha$ past the $\operatorname{ad} \overline{y}_\gamma$ at the expense of introducing an $\operatorname{ad} \overline{h}_\alpha$ every time $\gamma = \alpha$. Now $\alpha + \beta$ is not a root, since β is the maximal root. So $[x_\alpha, x_\beta] = 0$. Thus $\operatorname{ad} \overline{x}_\alpha \overline{x} = 0$, and we have proved that $[\mathbf{k}, \mathbf{m}] \subset \mathbf{m}$. But since $\mathbf{m}$ is a proper subspace of $\mathbf{g} \oplus \mathbf{g}'$, this implies that $\mathbf{k}$ is a proper subalgebra, since otherwise $\mathbf{m}$ would be a proper ideal, and the only proper ideals in $\mathbf{g} \oplus \mathbf{g}'$ are $\mathbf{g}$ and $\mathbf{g}'$.

Now the subalgebra $\mathbf{k}$ can not contain any element of the form $z \oplus 0, z \neq 0$, for it if did, it would have to contain all of the elements of the form $u \oplus 0$ since we could repeatedly apply $\operatorname{ad} x_\alpha$'s until we reached the maximal root space and then get all of $\mathbf{g} \oplus 0$, which would mean that $\mathbf{k}$ would also contain all of $0 \oplus \mathbf{g}'$ and hence all of $\mathbf{g} \oplus \mathbf{g}'$ which we know not to be the case. Similarly $\mathbf{k}$ can not contain any element of the form $0 \oplus z'$. So the projections of $\mathbf{k}$ onto $\mathbf{g}$ and onto $\mathbf{g}'$ are linear isomorphisms. By construction they are Lie algebra homomorphisms. Hence the inverse of the projection of $\mathbf{k}$ onto $\mathbf{g}$ followed by the projection of $\mathbf{k}$ onto $\mathbf{g}'$ is a Lie algebra isomorphism of $\mathbf{g}$ onto $\mathbf{g}'$. By construction it sends x_α to $x'_{\alpha'}$ and h_α to $h_{\alpha'}$ and so is an extension of f. QED