Lie Algebras

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Chapter 7

Cyclic highest weight modules.

In this chapter, $\mathbf{g}$ will denote a semi-simple Lie algebra for which we have chosen a Cartan subalgebra, $\mathbf{h}$ and a base Δ for the roots $\Phi = \Phi^+ \cup \Phi^-$ of $\mathbf{g}$.

We will be interested in describing its finite dimensional irreducible representations. If W is a finite dimensional module for $\mathbf{g}$, then $\mathbf{h}$ has at least one simultaneous eigenvector; that is there is a $\mu \in \mathbf{h}^*$ and a $w \neq 0 \in W$ such that

$$hw = \mu(h)w \quad \forall\, h \in \mathbf{h}. \tag{7.1}$$

The linear function μ is called a **weight** and the vector v is called a **weight vector**. If $x \in \mathbf{g}_\alpha$,

$$hxw = [h,x]w + xhw = (\mu + \alpha)(h)xw.$$

This shows that the space of all vectors w satisfying an equation of the type (7.1) (for varying μ) spans an invariant subspace. If W is irreducible, then the weight vectors (those satisfying an equation of the type (7.1)) must span all of W. Furthermore, since W is finite dimensional, there must be a vector v and a linear function λ such that

$$hv = \lambda(h)v \;\; \forall\, h \in \mathbf{h}, \quad e_\alpha v = 0, \; \forall \alpha \in \Phi^+. \tag{7.2}$$

Using irreducibility again, we conclude that

$$W = U(\mathbf{g})v.$$

The module is **cyclic** generated by v. In fact we can be more precise: Let $h_1, \dots, h_\ell$ be the basis of $\mathbf{h}$ corresponding to the choice of simple roots, let $e_i \in \mathbf{g}_{\alpha_i}$, $f_i \in \mathbf{g}_{-\alpha_i}$ where $\alpha_1, \dots, \alpha_m$ are all the positive roots. (We can choose them so that each e and f generate a little $sl(2)$.) Then

$$\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+,$$

where $e_1, \ldots, e_m$ is a basis of $\mathbf{n}_+$, where $h_1, \ldots, h_\ell$ is a basis of $\mathbf{h}$, and $f_1, \ldots, f_m$ is a basis of $\mathbf{n}_-$. The Poincaré-Birkhoff-Witt theorem says that monomials of the form

$$f_1^{i_1} \cdots f_m^{i_m} h_1^{j_1} \cdots h_\ell^{j_\ell} e_1^{k_1} \cdots e_m^{k_m}$$

form a basis of $U(\mathbf{g})$. Here we have chosen to place all the e's to the extreme right, with the h's in the middle and the f's to the left. It now follows that the elements

$$f_1^{i_1} \cdots f_m^{i_m} v$$

span W. Every such element, if non-zero, is a weight vector with weight

$$\lambda - (i_1\alpha_1 + \cdots + i_m\alpha_m).$$

Recall that

$$\mu \prec \lambda \quad \text{means that } \lambda - \mu = \sum k_i\alpha_i, \quad \alpha_i > 0,$$

where the k_i are non-negative integers. We have shown that every weight μ of W satisfies

$$\mu \prec \lambda.$$

So we make the definition: A cyclic highest weight module for $\mathbf{g}$ is a module (not necessarily finite dimensional) which has a vector v_+ such that

$$x_+v_+ = 0, \quad \forall\, x_+ \in \mathbf{n}_+, \quad hv_+ = \lambda(h)v_+ \ \forall h \in \mathbf{h}$$

and

$$V = U(\mathbf{g})v_+.$$

In any such cyclic highest weight module every submodule is a direct sum of its weight spaces (by van der Monde). The weight spaces V_μ all satisfy

$$\mu \prec \lambda$$

and we have

$$V = \bigoplus V_\mu.$$

Any proper submodule can not contain the highest weight vector, and so the sum of two proper submodules is again a proper submodule. Hence any such V has a unique maximal submodule and hence a unique irreducible quotient. The quotient of any highest weight module by an invariant submodule, if not zero, is again a cyclic highest weight module with the same highest weight.

7.1 Verma modules.

There is a "biggest" cyclic highest weight module, associated with any $\lambda \in \mathbf{h}^*$ called the **Verma module**. It is defined as follows: Let us set

$$\mathbf{b} := \mathbf{h} \oplus \mathbf{n}_+.$$

Given any $\lambda \in \mathbf{h}^*$ let $\mathbf{C}_\lambda$ denote the one dimensional vector space $\mathbf{C}$ with basis z_+ and with the action of $\mathbf{b}$ given by

$$(h + \sum_{\beta \succ 0} x_\beta) z_+ := \lambda(h) z_+.$$

So it is a left $U(\mathbf{b})$ module. By the Poincaré Birkhoff Witt theorem, $U(\mathbf{g})$ is a free right $U(\mathbf{b})$ module with basis $\{f_1^{i_1} \cdots f_\ell^{i_m}\}$, and so we can form the **Verma module**

$$\mathrm{Verm}(\lambda) := U(\mathbf{g}) \otimes_{U(\mathbf{b})} \mathbf{C}_\lambda$$

which is a cyclic module with highest weight vector $v_+ := 1 \otimes z_+$.

Furthermore, any two *irreducible* cyclic highest weight modules with the same highest weight are isomorphic. Indeed, if V and W are two such with highest weight vector v_+, u_+, consider $V \oplus W$ which has (v_+, u_+) as a maximal weight vector with weight λ, and hence $Z := U(\mathbf{g})(v_+, u_+)$ is cyclic and of highest weight λ. Projections onto the first and second factors give non-zero homomorphisms which must be surjective. But Z has a unique irreducible quotient. Hence these must induce isomorphisms on this quotient, V and W are isomorphic.

Hence, up to isomorphism, there is a unique irreducible cyclic highest weight module with highest weight λ. We call it

$$\mathrm{Irr}(\lambda).$$

In short, we have constructed a "largest" highest weight module $\mathrm{Verm}(\lambda)$ and a "smallest" highest weight module $\mathrm{Irr}(\lambda)$.

7.2 When is $\dim \mathrm{Irr}(\lambda) < \infty$?

If $\mathrm{Irr}(\lambda)$ is finite dimensional, then it is finite dimensional as a module over any subalgebra, in particular over any subalgebra isomorphic to $sl(2)$. Applied to the subalgebra $sl(2)_i$ generated by e_i, h_i, f_i we conclude that

$$\lambda(h_i) \in \mathbf{Z}.$$

Such a weight is is called **integral**. Furthermore the representation theory of $sl(2)$ says that the maximal weight for any finite dimensional representation must satisfy

$$\lambda(h_i) = \langle \lambda, \alpha_i \rangle \geq 0$$

so that λ lies in the closure of the fundamental Weyl chamber. Such a weight is called **dominant**. So a necessary condition for $\mathrm{Irr}(\lambda)$ to be finite dimensional is that λ be dominant integral. We now show that conversely, $\mathrm{Irr}(\lambda)$ is finite dimensional whenever λ is dominant integral.

For this we recall that in the universal enveloping algebra $U(\mathbf{g})$ we have

1. $[e_j, f_i^{k+1}] = 0$, if $i \neq j$

2. $[h_j, f_i^{k+1}] = -(k+1)\alpha_i(h_j)f_i^{k+1}$
3. $[e_i, f_i^{k+1}] = -(k+1)f_i^k(k \cdot 1 - h_i)$

where the first two equations are consequences of the fact that ad is a derivation and

$$[e_i, f_j] = 0 \text{ if } i \neq j \text{ since } \alpha_i - \alpha_j \text{ is not a root}$$

and

$$[h_j, f_j] = -\alpha_j(h_i)f_j.$$

The last is a the fact about $sl(2)$ which we have proved in Chapter II. Notice that it follows from 1.) that $e_j(f_i^k)v_+ = 0$ for all k and all $i \neq j$ and from 3.) that

$$e_i f_i^{\lambda(h_i)+1} v_+ = 0$$

so that $f_i^{\lambda(h_i)+1} v_+$ is a maximal weight vector. If it were non-zero, the cyclic module it generates would be a proper submodule of $\mathrm{Irr}(\lambda)$ contradicting the irreducibility. Hence

$$f_i^{\lambda(h_i)+1} v_+ = 0.$$

So for each i the subspace spanned by $v_+, f_i v_+, \cdots, f_i^{\lambda(h_i)} v_+$ is a finite dimensional $sl(2)_i$ module. In particular $\mathrm{Irr}(\lambda)$ contains some finite dimensional $sl(2)_i$ modules. Let V' denote the sum of all such. If W is a finite dimensional $sl(2)_i$ module, then $e_\alpha W$ is again finite dimensional, thus so their sum, which is a finite dimensional $sl(2)_i$ module. Hence V' is $\mathbf{g}$- stable, hence all of $\mathrm{Irr}(\lambda)$.

In particular, the e_i and the f_i act as locally nilpotent operators on $\mathrm{Irr}(\lambda)$. So the operators $\tau_i := (\exp e_i)(\exp -f_i)(\exp e_i)$ are well defined and

$$\tau_i(\mathrm{Irr}(\lambda))_\mu = \mathrm{Irr}(\lambda)_{s_i\mu}$$

so

$$\dim \mathrm{Irr}(\lambda)_{w\mu} = \dim \mathrm{Irr}(\lambda)_\mu \quad \forall w \in \mathcal{W} \tag{7.3}$$

where $\mathcal{W}$ denotes the Weyl group. These are all finite dimensional subspaces: Indeed their dimension is at most the corresponding dimension in the Verma module $\mathrm{Verm}(\lambda)$, since $\mathrm{Irr}(\lambda)_\mu$ is a quotient space of $\mathrm{Verm}(\lambda)_\mu$. But $\mathrm{Verm}(\lambda)_\mu$ has a basis consisting of those $f_1^{k_1} \cdots f_m^{k_m} v_+$. The number of such elements is the number of ways of writing

$$\lambda - \mu = k_1\alpha_1 + \cdots k_m\alpha_m.$$

So $\dim \mathrm{Verm}(\lambda)_\mu$ is the number of m-tuplets of non-negative integers $(k_1, \ldots, k_m)$ such that the above equation holds. This number is clearly finite, and is known as $P_K(\lambda - \mu)$, the Kostant partition function of $\lambda - \mu$, which will play a central role in what follows.

Now every element of E is conjugate under W to an element of the closure of the fundamental Weyl chamber, i.e. to a μ satisfying

$$(\mu, \alpha_i) \geq 0$$

i.e. to a μ that is dominant. We claim that there are only finitely many dominant weights μ which are $\prec \lambda$, which will complete the proof of finite dimensionality. Indeed, the sum of two dominant weights is dominant, so $\lambda + \mu$ is dominant. On the other hand, $\lambda - \mu = \sum k_i \alpha_i$ with the $k_i \geq 0$. So

$$(\lambda, \lambda) - (\mu, u) = (\lambda + \mu, \lambda - \mu) = \sum k_i (\lambda + \mu, \alpha_i) \geq 0.$$

So μ lies in the intersection of the ball of radius $\sqrt{(\lambda, \lambda)}$ with the discrete set of weights $\prec \lambda$ which is finite.

We record a consequence of (7.3) which is useful under very special circumstances. Suppose we are given a finite dimensional representation of $\mathbf{g}$ with the property that each weight space is one dimensional and all weights are conjugate under $\mathcal{W}$. Then this representation must be irreducible. For example, take $\mathbf{g} = sl(n+1)$ and consider the representation of $\mathbf{g}$ on $\wedge^k(\mathbf{C}^{n+1}), 1 \leq k \leq n$. In terms of the standard basis $e_1, \ldots, e_{n+1}$ of $\mathbf{C}^{n+1}$ the elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$ are weight vectors with weights $L_{i_1} + \cdots + L_{i_k}$, Where $\mathbf{h}$ consists of all diagonal traceless matrices and L_i is the linear function which assigns to each diagonal matrix its i-th entry.

These weight spaces are all one dimensional and conjugate under the Weyl group. Hence these representations are irreducible with highest weight

$$\omega_i := L_1 + \cdots + L_k$$

in terms of the usual choice of base, $h_1, \ldots, h_n$ where h_j is the diagonal matrix with 1 in the j-th position, -1 in the $j+1$-st position and zeros elsewhere. Notice that

$$\omega_i(h_j) = \delta_{ij}$$

so that the ω_i form a basis of the "weight lattice" consisting of those $\lambda \in \mathbf{h}^*$ which take integral values on $h_1, \ldots, h_n$.

7.3 The value of the Casimir.

Recall that our basis of $U(\mathbf{g})$ consists of the elements

$$f_1^{i_1} \cdots f_m^{i_m} h_1^{j_1} \cdots h_\ell^{j_\ell} e_1^{k_1} \cdots e_m^{k_m}.$$

The elements of $U(\mathbf{h})$ are then the ones with no e or f component in their expression. So we have a vector space direct sum decomposition

$$U(\mathbf{g}) = U(\mathbf{h}) \oplus (U(\mathbf{g})\mathbf{n}_+ + \mathbf{n}_- U(\mathbf{g})),$$

where $\mathbf{n}_+$ and $\mathbf{n}_-$ are the corresponding nilpotent subalgebras. Let γ denote projection onto the first factor in this decomposition. Now suppose $z \in Z(\mathbf{g})$, the center of the universal enveloping algebra. In particular, $z \in U(\mathbf{g})^{\mathbf{h}}$. The eigenvalues of the monomial above under the action of $h \in \mathbf{h}$ are

$$\sum_{s=1}^{m} (k_s - i_s)\alpha_s(h).$$

So any monomial in the expression for z can not have f factors alone. We have proved that
$$z - \gamma(z) \in U(\mathbf{g})\mathbf{n}_+, \quad \forall\ z \in Z(\mathbf{g}). \tag{7.4}$$

For any $\lambda \in \mathbf{h}^*$, the element $z \in Z(\mathbf{g})$ acts as a scalar, call it $\chi_\lambda(z)$ on the Verma module associated to λ.

In particular, if λ is a dominant integral weight, it acts by this same scalar on the irreducible finite dimensional module associated to λ.

On the other hand, the linear map $\lambda : \mathbf{h} \to \mathbf{C}$ extends to a homomorphism, which we will also denote by λ of $U(\mathbf{h}) = S(\mathbf{h}) \to \mathbf{C}$. Explicitly, if we think of elements of $U(\mathbf{h}) = S(\mathbf{h})$ as polynomials on $\mathbf{h}^*$, then $\lambda(P) = P(\lambda)$ for $P \in S(\mathbf{h})$. Since $\mathbf{n}_+ v = 0$ if v is the maximal weight vector, we conclude from (7.4) that

$$\chi_\lambda(z) = \lambda(\gamma(z)) \quad \forall\ z \in Z(\mathbf{g}). \tag{7.5}$$

We want to apply this formula to the second order Casimir element associated to the Killing form κ. So let $k_1, \ldots, k_\ell \in \mathbf{h}$ be the dual basis to $h_1, \ldots, h_\ell$ relative to κ, i.e.
$$\kappa(h_i, k_j) = \delta_{ij}.$$
Let $x_\alpha \in \mathbf{g}_\alpha$ be a basis (i.e. non-zero) element and $z_\alpha \in \mathbf{g}_{-\alpha}$ be the dual basis element to x_α under the Killing form, so the second order Casimir element is

$$\mathrm{Cas}^\kappa = \sum h_i k_i + \sum_\alpha x_\alpha z_\alpha.$$

where the second sum on the right is over *all* roots. We might choose the $x_\alpha = e_\alpha$ for positive roots, and then the corresponding z_α is some multiple of the f_α. (And, for present purposes we might even choose $f_\alpha = z_\alpha$ for positive α.) The problem is that the z_α for positive α in the above expression for Cas^κ are written to the right, and we must move them to the left. So we write

$$\mathrm{Cas}^\kappa = \sum_i h_i k_i + \sum_{\alpha>0} [x_\alpha, z_\alpha] + \sum_{\alpha>0} z_\alpha x_\alpha + \sum_{\alpha<0} x_\alpha z_\alpha.$$

This expression for Cas^κ has all the $\mathbf{n}^+$ elements moved to the right; in particular, all of the summands in the last two sums annihilate v_λ. Hence

$$\gamma(\mathrm{Cas}^\kappa) = \sum_i h_i k_i + \sum_{\alpha>0} [x_\alpha, z_\alpha]$$

and
$$\chi_\lambda(\mathrm{Cas}^\kappa) = \sum_i \lambda(h_i)\lambda(k_i) + \sum_{\alpha>0} \lambda([x_\alpha, z_\alpha]).$$

For any $h \in \mathbf{h}$ we have

$$\kappa(h, [x_\alpha, z_\alpha]) = \kappa([h, x_\alpha], z_\alpha) = \alpha(h)\kappa(x_\alpha, z_\alpha) = \alpha(h)$$

so
$$[x_\alpha, z_\alpha] = t_\alpha$$

where $t_\alpha \in \mathbf{h}$ is uniquely determined by

$$\kappa(t_\alpha, h) = \alpha(h) \ \ \forall\ h \in \mathbf{h}.$$

Let $(\ ,\ )_\kappa$ denote the bilinear form on $\mathbf{h}^*$ obtained from the identification of $\mathbf{h}$ with $\mathbf{h}^*$ given by κ. Then

$$\sum_{\alpha>0} \lambda([x_\alpha, z_\alpha]) = \sum_{\alpha>0} \lambda(t_\alpha) = \sum_{\alpha>0} (\lambda, \alpha)_\kappa = 2(\lambda, \rho)_\kappa \tag{7.6}$$

where

$$\rho := \frac{1}{2} \sum_{\alpha>0} \alpha.$$

On the other hand, let the constants a_i be defined by

$$\lambda(h) = \sum_i a_i \kappa(h_i, h) \ \ \forall\ h \in \mathbf{h}.$$

In other words λ corresponds to $\sum a_i h_i$ under the isomorphism of $\mathbf{h}$ with $\mathbf{h}^*$ so

$$(\lambda, \lambda)_\kappa = \sum_{i,j} a_i a_j \kappa(h_i, h_j).$$

Since $\kappa(h_i, k_j) = \delta_{ij}$ we have

$$\lambda(k_i) = a_i.$$

Combined with $\lambda(h_i) = \sum_j a_j \kappa(h_j, h_i)$ this gives

$$(\lambda, \lambda)_\kappa = \sum_i \lambda(h_i)\lambda(k_i). \tag{7.7}$$

Combined with (7.6) this yields

$$\chi_\lambda(\mathrm{Cas}^\kappa) = (\lambda + \rho, \lambda + \rho)_\kappa - (\rho, \rho)_\kappa. \tag{7.8}$$

We now use this innocuous looking formula to prove the following: We let $\mathbf{L} = \mathbf{L}_\mathbf{g} \subset \mathbf{h}^*_\mathbf{R}$ denote the lattice of integral linear forms on $\mathbf{h}$, i.e.

$$\mathbf{L} = \{\mu \in \mathbf{h}^* | 2\frac{(\mu, \phi)}{(\phi, \phi)} \in \mathbf{Z}\ \forall \phi \in \Delta\}. \tag{7.9}$$

$\mathbf{L}$ is called the **weight lattice** of $\mathbf{g}$.

For $\mu, \lambda \in \mathbf{L}$ recall that

$$\mu \prec \lambda$$

if $\lambda - \mu$ is a sum of positive roots. Then

Proposition 1 *Any cyclic highest weight module $Z(\lambda), \lambda \in \mathbf{L}$ has a composition series whose quotients are irreducible modules,* $\mathrm{Irr}(\mu)$ *where $\mu \prec \lambda$ satisfies*

$$(\mu+\rho, \mu+\rho)_\kappa = (\lambda+\rho, \lambda+\rho)_\kappa. \tag{7.10}$$

In fact, if

$$d = \sum \dim Z(\lambda)_\mu$$

where the sum is over all μ satisfying (7.10) then there are at most d steps in the composition series.

Remark. There are only finitely many $\mu \in \mathbf{L}$ satisfying (7.10) since the set of all μ satisfying (7.10) is compact and $\mathbf{L}$ is discrete. Each weight is of finite multiplicity. Therefore d is finite.

Proof by induction on d. We first show that if $d = 1$ then $Z(\lambda)$ is irreducible. Indeed, if not, any proper submodule W, being the sum of its weight spaces, must have a highest weight vector with highest weight μ, say. But then

$$\chi_\lambda(\mathrm{Cas}^\kappa) = \chi_\mu(\mathrm{Cas}^\kappa)$$

since W is a submodule of $Z(\lambda)$ and Cas^κ takes on the constant value $\chi_\lambda(\mathrm{Cas}^\kappa)$ on $Z(\lambda)$. Thus μ and λ both satisfy (7.10) contradicting the assumption $d = 1$. In general, suppose that $Z(\lambda)$ is not irreducible, so has a submodule, W and quotient module $Z(\lambda)/W$. Each of these is a cyclic highest weight module, and we have a corresponding composition series on each factor. In particular, $d = d_W + d_{Z(\lambda)/W}$ so that the d's are strictly smaller for the submodule and the quotient module. Hence we can apply induction. QED

For each $\lambda \in \mathbf{L}$ we introduce a formal symbol, $e(\lambda)$ which we want to think of as an "exponential" and so the symbols are multiplied according to the rule

$$e(\mu) \cdot e(\nu) = e(\mu + \nu). \tag{7.11}$$

The *character* of a module N is defined as

$$\mathrm{ch}_N = \sum \dim N_\mu \cdot e(\mu).$$

In all cases we will consider (cyclic highest weight modules and the like) all these dimensions will be finite, so the coefficients are well defined, but (in the case of Verma modules for example) there may be infinitely many terms in the (formal) sum. Logically, such a formal sum is nothing other than a function on $\mathbf{L}$ giving the "coefficient" of each $e(\mu)$.

In the case that N is finite dimensional, the above sum is finite. If

$$f = \sum f_\mu e(\mu) \quad \text{and} \quad g = \sum g_\nu e(\nu)$$

are two finite sums, then their product (using the rule (7.11)) corresponds to convolution:

$$\left(\sum f_\mu e(\mu)\right) \cdot \left(\sum g_\nu e(\nu)\right) = \sum (f \star g)_\lambda e(\lambda)$$

where

$$(f\star g)_\lambda := \sum_{\mu+\nu=\lambda} f_\mu g_\nu.$$

So we let $\mathbf{Z}_{\rm fin}(\mathbf{L})$ denote the set of $\mathbf{Z}$ valued functions on $\mathbf{L}$ which vanish outside a finite set. It is a commutative ring under convolution, and we will oscillate in notation between writing an element of $\mathbf{Z}_{\rm fin}(\mathbf{L})$ as an "exponential sum" thinking of it as a function of finite support.

Since we also want to consider infinite sums such as the characters of Verma modules, we enlarge the space $\mathbf{Z}_{\rm fin}(\mathbf{L})$ by defining $\mathbf{Z}_{\rm gen}(\mathbf{L})$ to consist of $\mathbf{Z}$ valued functions whose supports are contained in finite unions of sets of the form $\lambda - \sum_{\alpha\succ 0} k_\alpha\alpha$. The convolution of two functions belonging to $\mathbf{Z}_{\rm gen}(\mathbf{L})$ is well defined, and belongs to $\mathbf{Z}_{\rm gen}(\mathbf{L})$. So $\mathbf{Z}_{\rm gen}(\mathbf{L})$ is again a ring.

It now follows from Prop.1 that

$$\mathrm{ch}_{Z(\lambda)} = \sum \mathrm{ch}_{\mathrm{Irr}(\mu)}$$

where the sum is over the finitely many terms in the composition series. In particular, we can apply this to $Z(\lambda) = \mathrm{Verm}(\lambda)$, the Verma module. Let us order the $\mu_i \prec \lambda$ satisfying (7.10) in such a way that $\mu_i \prec \mu_j \Rightarrow i \leq j$. Then for each of the finitely many μ_i occurring we get a corresponding formula for $\mathrm{ch}_{\mathrm{Verm}(\mu_i)}$ and so we get collection of equations

$$\mathrm{ch}_{\mathrm{Verm}(\mu_j)} = \sum a_{ij}\mathrm{ch}_{\ \mathrm{Irr}(\mu_i)}$$

where $a_{ii} = 1$ and $i \leq j$ in the sum. We can invert this upper triangular matrix and therefore conclude that there is a formula of the form

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)\mathrm{ch}_{\mathrm{Verm}(\mu)} \tag{7.12}$$

where the sum is over $\mu \prec \lambda$ satisfying (7.10) with coefficients $b(\mu)$ that we shall soon determine. But we do know that $b(\lambda) = 1$.

7.4 The Weyl character formula.

We will now prove

Proposition 2 *The non-zero coefficients in (7.12) occur only when*

$$\mu = w(\lambda+\rho) - \rho$$

where $w \in W$, the Weyl group of $\mathbf{g}$, and then

$$b(\mu) = (-1)^w.$$

Here

$$(-1)^w := \det w.$$

We will prove this by proving some combinatorial facts about multiplication of sums of exponentials.

We recall our notation: For $\lambda \in \mathbf{h}^*$, $\mathrm{Irr}(\lambda)$ denotes the unique irreducible module of highest weight, λ,and $\mathrm{Verm}(\lambda)$ denotes the Verma module of highest weight λ, and more generally, $Z(\lambda)$ denotes an arbitrary cyclic module of highest weight λ. Also

$$\rho := \frac{1}{2}\sum_{\phi\in\Phi^+}\phi$$

is one half the sum of the positive roots. Let $\lambda_i, i = 1,\ldots,\dim\mathbf{h}$ be the basis of the weight lattice, L dual to the base Δ. So

$$\lambda_i(h_{\alpha_j}) = \langle\lambda_i,\alpha_j\rangle = \delta_{ij}.$$

Since $s_i(\alpha_i) = -\alpha_i$ while keeping all the other positive roots positive, we saw that this implied that

$$s_i\rho = \rho - \alpha_i$$

and therefore

$$\langle\rho,\alpha_i\rangle = 1, \quad i = 1,\ldots,\ell := \dim(\mathbf{h}).$$

In other words

$$\rho = \frac{1}{2}\sum_{\phi\in\Phi^+}\phi = \lambda_1 + \cdots + \lambda_\ell. \tag{7.13}$$

The **Kostant partition function**, $P_K(\mu)$ is defined as the number of sets of non-negative integers, k_β such that

$$\mu = \sum_{\beta\in\Phi^+} k_\beta\beta.$$

(The value is zero if μ can not be expressed as a sum of positive roots.)

For any module N and any $\mu \in \mathbf{h}^*$, N_μ denotes the weight space of weight μ. For example, in the Verma module, $\mathrm{Verm}(\lambda)$, the only non-zero weight spaces are the ones where $\mu = \lambda - \sum_{\beta\in\Phi^+} k_\beta\beta$ and the multiplicity of this weight space, i.e. the dimension of $\mathrm{Verm}(\lambda)_\mu$ is the number of ways of expressing in this fashion, i.e.

$$\dim\ \mathrm{Verm}(\lambda)_\mu = P_K(\lambda-\mu). \tag{7.14}$$

In terms of the character notation introduced in the preceding section we can write this as

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = \sum P_K(\lambda-\mu)e(\mu).$$

To be consistent with Humphreys' notation, define the *Kostant function* p by

$$p(\nu) = P_K(-\nu)$$

and then in succinct language

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = p(\cdot - \lambda). \tag{7.15}$$

Observe that if

$$f = \sum f(\mu)e(\mu)$$

then

$$f \cdot e(\lambda) = \sum f(\mu)e(\lambda + \mu) = \sum f(\nu - \lambda)e(\nu).$$

We can express this by saying that

$$f \cdot e(\lambda) = f(\cdot - \lambda).$$

Thus, for example,

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = p(\cdot - \lambda) = p \cdot e(\lambda).$$

Also observe that if

$$f_\alpha = \frac{1}{1 - e(-\alpha)} := 1 + e(-\alpha) + e(-2\alpha) + \cdots$$

then

$$(1 - e(-\alpha))f_\alpha = 1$$

and

$$\prod_{\alpha \in \Phi^+} f_\alpha = p$$

by the definition of the Kostant function.

Define the function q by

$$q := \prod_{\alpha \in \Phi^+} (e(\alpha/2) - e(-\alpha/2)) = e(\rho) \prod (1 - e(-\alpha))$$

since $e(\rho) = \prod_{\alpha \in \Phi^+} e(\alpha/2)$. Notice that

$$wq = (-1)^w q.$$

It is enough to check this on fundamental reflections, but they have the property that they make exactly one positive root negative, hence change the sign of q.

We have

$$qp = e(\rho). \tag{7.16}$$

Indeed,

$$\begin{aligned} qpe(-\rho) &= \left[\prod (1 - e(-\alpha))\right] e(\rho)pe(-\rho) \\ &= \left[\prod (1 - e(-\alpha))\right] p \\ &= \prod (1 - e(-\alpha)) \prod f_\alpha \\ &= 1. \end{aligned}$$

Therefore,

$$q\mathrm{ch}_{\mathrm{Verm}(\lambda)} = qpe(\lambda) = e(\rho)e(\lambda) = e(\lambda + \rho).$$

Let us now multiply both sides of (7.12) by q and use the preceding equation. We obtain

$$q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)e(\mu+\rho)$$

where the sum is over all $\mu \prec \lambda$ satisfying (7.10), and the $b(\mu)$ are coefficients we must determine.

Now ch $_{\mathrm{Irr}(\lambda)}$ is invariant under the Weyl group W, and q transforms by $(-1)^w$. Hence if we apply $w \in W$ to the preceding equation we obtain

$$(-1)^w q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)e(w(\mu+\rho)).$$

This shows that the set of $\mu+\rho$ with non-zero coefficients is stable under W and the coefficients transform by the sign representation for each W orbit. In particular, each element of the form $\mu = w(\lambda+\rho)-\rho$ has $(-1)^w$ as its coefficient. We can thus write

$$q\mathrm{ch}_{V(\lambda)} = \sum_{w\in W}(-1)^w e(w(\lambda+\rho)) + R$$

where R is a sum of terms corresponding to $\mu+\rho$ which are not of the form $w(\lambda+\rho)$. We claim that there are no such terms and hence $R = 0$. Indeed, if there were such a term, the transformation properties under W would demand that there be such a term with $\mu+\rho$ in the closure of the Weyl chamber, i.e.

$$\mu+\rho \in \Lambda := \mathbf{L}\cap D$$

where

$$D = D_{\mathbf{g}} = \{\lambda \in E | (\lambda,\phi) \geq 0 \;\; \forall \phi \in \Delta^+\}$$

and $E = \mathbf{h}^*_{\mathbf{R}}$ denotes the space of real linear combinations of the roots. But we claim that

$$\mu \prec \lambda, \;\; (\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho), \;\; \& \; \mu+\rho\in\Lambda \implies \mu = \lambda.$$

Indeed, write $\mu = \lambda - \pi, \;\; \pi = \sum k_\alpha \alpha, \; k_\alpha \geq 0$ so

$$\begin{aligned} 0 &= (\lambda+\rho,\lambda+\rho) - (\mu+\rho,\mu+\rho) \\ &= (\lambda+\rho,\lambda+\rho) - (\lambda+\rho-\pi,\lambda+\rho-\pi) \\ &= (\lambda+\rho,\pi) + (\pi,\mu+\rho) \\ &\geq (\lambda+\rho,\pi) \;\; \text{since } \mu+\rho\in\Lambda \\ &\geq 0 \end{aligned}$$

since $\lambda+\rho \in \Lambda$ and in fact lies in the interior of D. But the last inequality is strict unless $\pi = 0$. Hence $\pi = 0$. We will have occasion to use this type of argument several times again in the future. In any event we have derived the fundamental formula

$$q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum_{w\in W}(-1)^w e(w(\lambda+\rho)). \tag{7.17}$$

Notice that if we take $\lambda = 0$ and so the trivial representation with character 1 for $V(\lambda)$, (7.17) becomes

$$q = \sum (-1)^w e(w\rho)$$

and this is precisely the denominator in the **Weyl character formula**:

$$\textbf{WCF} \ \mathrm{ch}_{\mathrm{Irr}(\lambda)} = \frac{\sum_{w \in W} (-1)^w e(w(\lambda + \rho))}{\sum_{w \in W} (-1)^w e(w\rho)} \tag{7.18}$$

7.5 The Weyl dimension formula.

For any weight, μ we define

$$A_\mu := \sum_{w \in W} (-1)^w e(w\mu).$$

Then we can write the Weyl character formula as

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \frac{A_{\lambda+\rho}}{A_\rho}.$$

For any weight μ define the homomorphism Ψ_μ from the ring $\mathbf{Z}_{\mathrm{fin}}(\mathbf{L})$ into the ring of formal power series in one variable t by the formula

$$\Psi_\mu(e(\nu)) = e^{(\nu,\mu)_\kappa t}$$

(and extend linearly). The left hand side of the Weyl character formula belongs to $\mathbf{Z}_{\mathrm{fin}}(\mathbf{L})$, and hence so does the right hand side which is a quotient of two elements of $\mathbf{Z}_{\mathrm{fin}}(\mathbf{L})$. Therefore for any μ we have

$$\Psi_\mu(\mathrm{ch}_{\mathrm{Irr}(\lambda)}) = \frac{\Psi_\mu(A_{\rho+\lambda})}{\Psi_\mu(A_\rho)}.$$

$$\Psi_\mu(A_\nu) = \Psi_\nu(A_\mu) \tag{7.19}$$

for any pair of weights. Indeed,

$$\begin{aligned} \Psi_\mu(A_\nu) &= \sum_w (-1)^w e^{(\mu, w\nu)_\kappa t} \\ &= \sum_w (-1)^w e^{(w^{-1}\mu, \nu)_\kappa t} \\ &= \sum (-1)^w e^{(w\mu, \nu)_\kappa t} \\ &= \Psi_\nu(A_\mu). \end{aligned}$$

In particular,

$$\begin{aligned}
\Psi_\rho(A_\lambda) &= \Psi_\lambda(A_\rho) \\
&= \Psi_\lambda(q) \\
&= \Psi_\lambda\left(\prod(e(\alpha/2) - e(-\alpha/2))\right) \\
&= \prod_{\alpha\in\Phi^+}\left(e^{(\lambda,\alpha)_\kappa t/2} - e^{-(\lambda,\alpha)_\kappa t/2)}\right) \\
&= \left(\prod(\lambda,\alpha)_\kappa\right) t^{\#\Phi^+} + \text{ terms of higher degree in } t.
\end{aligned}$$

Hence

$$\Psi_\rho(\mathrm{ch}_{\mathrm{Irr}(\lambda)}) = \frac{\Psi_\rho(A_{\lambda+\rho})}{\Psi_\rho(A_\rho)} = \frac{\prod(\lambda+\rho,\alpha)_\kappa}{\prod(\rho,\alpha)_\kappa} + \quad \text{terms of positive degree in } t.$$

Now consider the composite homomorphism: first apply Ψ_ρ and then set $t = 0$. This has the effect of replacing every $e(\mu)$ by the constant 1. Hence applied to the left hand side of the Weyl character formula this gives the dimension of the representation $\mathrm{Irr}(\lambda)$. The previous equation shows that when this composite homomorphism is applied to the right hand side of the Weyl character formula, we get the right hand side of the **Weyl dimension formula**:

$$\dim \mathrm{Irr}(\lambda) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)_\kappa}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)_\kappa}. \tag{7.20}$$

7.6 The Kostant multiplicity formula.

Let us multiply the fundamental equation (7.17) by $pe(-\rho)$ and use the fact (7.16) that $qpe(-\rho) = 1$ to obtain

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum_{w\in W}(-1)^w pe(-\rho)e(w(\lambda+\rho)).$$

But

$$pe(-\rho)e(w(\lambda+\rho)) = p(\cdot - w(\lambda+\rho)+\rho)$$

or, in more pedestrian terms, the left hand side of this equation has, as its coefficient of $e(\mu)$ the value

$$p(\mu+\rho-w(\lambda+\rho)).$$

On the other hand, by definition,

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum \dim(\mathrm{Irr}(\lambda)_\mu e(\mu).$$

We thus obtain Kostant's formula for the multiplicity of a weight μ in the irreducible module with highest weight λ:

$$\textbf{KMF} \quad \dim\left(\mathrm{Irr}(\lambda)\right)_\mu = \sum_{w\in W}(-1)^w p(\mu+\rho-w(\lambda+\rho)). \tag{7.21}$$

It will be convenient to introduce some notation which simplifies the appearance of the Kostant multiplicity formula: For $w \in W$ and $\mu \in \mathbf{L}$ (or in E for that matter) define

$$w \odot \mu := w(\mu + \rho) - \rho. \tag{7.22}$$

This defines another action of W on E where the "origin of the orthogonal transformations w has been shifted from 0 to $-\rho$". Then we can rewrite the Kostant multiplicity formula as

$$\dim(\mathrm{Irr}(\lambda))_\mu = \sum_{w \in W} (-1)^w P_K(w \odot \lambda - \mu) \tag{7.23}$$

or as

$$\mathrm{ch}(\mathrm{Irr}(\lambda)) = \sum_{w \in W} \sum_{\mu} (-1)^w P_K(w \odot \lambda - \mu) e(\mu), \tag{7.24}$$

where P_K is the original Kostant partition function.

For the purposes of the next section it will be useful to record the following lemma:

Lemma 1 *If ν is a dominant weight and $e \neq w \in W$ then $w \odot \nu$ is not dominant.*

Proof. If ν is dominant, so lies in the closure of the positive Weyl chamber, then $\nu + \rho$ lies in the interior of the positive Weyl chamber. Hence if $w \neq e$, then $w(\nu + \rho)(h_i) < 0$ for some i, and so $w \odot \nu = w(\nu + \rho) - \rho$ is not dominant. QED

7.7 Steinberg's formula.

Suppose that λ' and λ'' are dominant integral weights. Decompose $\mathrm{Irr}(\lambda') \otimes \mathrm{Irr}(\lambda'')$ into irreducibles, and let $n(\lambda) = n(\lambda, \lambda' \otimes \lambda'')$ denote the multiplicity of $\mathrm{Irr}(\lambda)$ in this decomposition into irreducibles (with $n(\lambda) = 0$ if $\mathrm{Irr}(\lambda)$ does not appear as a summand in the decomposition). In particular, $n(\nu) = 0$ if ν is not a dominant weight since $\mathrm{Irr}(\nu)$ is infinite dimensional in this case, so can not appear as a summand in the decomposition. In terms of characters, we have

$$\mathrm{ch}(\mathrm{Irr}(\lambda'))\,\mathrm{ch}(\mathrm{Irr}(\lambda'')) = \sum_{\lambda} n(\lambda)\,\mathrm{ch}(\mathrm{Irr}(\lambda)).$$

Steinberg's formula is a formula for $n(\lambda)$. To derive it, use the Weyl character formula

$$\mathrm{ch}(\mathrm{Irr}(\lambda'')) = \frac{A_{\lambda''+\rho}}{A_\rho}, \quad \mathrm{ch}(\mathrm{Irr}(\lambda)) = \frac{A_{\lambda+\rho}}{A_\rho}$$

in the above formula to obtain

$$\mathrm{ch}(\mathrm{Irr}(\lambda'))A_{\lambda''+\rho} = \sum_{\lambda} n(\lambda) A_{\lambda+\rho}.$$

Use the Kostant multiplicity formula (7.24) for λ':

$$\mathrm{ch}(\mathrm{Irr}(\lambda')) = \sum_{w \in W} \sum_{\mu} (-1)^w P_K(w \odot \lambda' - \mu) e(\mu)$$

and the definition

$$A_{\lambda''+\rho} = \sum_{u \in W} (-1)^u e(u(\lambda'' + \rho))$$

and the similar expression for $A_{\lambda+\rho}$ to get

$$\sum_{\mu} \sum_{u,w \in W} (-1)^{uw} P_K(w \odot \lambda' - \mu)) e(u(\lambda'' + \rho) + \mu) =$$

$$\sum_{\lambda} \sum_{w} n(\lambda)(-1)^w e(w(\lambda + \rho)).$$

Let us make a change of variables on the right hand side, writing

$$\nu = w \odot \lambda$$

so the right hand side becomes

$$\sum_{\nu} \sum_{w} (-1)^w n(w^{-1} \odot \nu) e(\nu + \rho).$$

If ν is a dominant weight, then by Lemma 1 $w^{-1} \odot \nu$ is not dominant if $w^{-1} \neq e$. So $n(w^{-1} \odot \nu) = 0$ if $w \neq 1$ and so the coefficient of $e(\nu + \rho)$ is precisely $n(\nu)$ when ν is dominant.

On the left hand side let

$$\mu = \nu - u \odot \lambda''$$

to obtain

$$\sum_{\nu,u,w} (-1)^{uw} P_K(w \odot \lambda' + u \odot \lambda'' - \nu) e(\nu + \rho).$$

Comparing coefficients for ν dominant gives

$$n(\nu) = \sum_{u,w} (-1)^{uw} P_K(w \odot \lambda' + u \odot \lambda'' - \nu). \tag{7.25}$$

7.8 The Freudenthal - de Vries formula.

We return to the study of a semi-simple Lie algebra **g** and get a refinement of the Weyl dimension formula by looking at the next order term in the expansion we used to derive the Weyl dimension formula from the Weyl character formula.

By definition, the Killing form restricted to the Cartan subalgebra **h** is given by

$$\kappa(h, h') = \sum_{\alpha} \alpha(h)\alpha(h')$$

where the sum is over all roots. If $\mu, \lambda \in \mathbf{h}^*$ with t_μ, t_λ the elements of H corresponding to them under the Killing form, we have

$$(\lambda, \mu)_\kappa = \kappa(t_\lambda, t_\mu) = \sum_\alpha \alpha(t_\lambda)\alpha(t_\mu)$$

so

$$(\lambda, \mu)_\kappa = \sum_\alpha (\lambda, \alpha)_\kappa (\mu, \alpha)_\kappa. \tag{7.26}$$

For each λ in the weight lattice L we have let $e(\lambda)$ denote the "formal exponential" so $\mathbf{Z}_{fin}(L)$ is the space spanned by the $e(\lambda)$ and we have defined the homomorphism

$$\Psi_\rho : \mathbf{Z}_{fin}(\Lambda) \to \mathbf{C}[[t]], \quad e(\lambda) \mapsto e^{(\lambda.\rho)_\kappa t}.$$

Let N and D be the images under Ψ_ρ of the Weyl numerator and denominator. So

$$N = \Psi_\rho(A_{\rho+\lambda}) = \Psi_{\rho+\lambda}(A_\rho)$$

by (7.19) and

$$A_\rho = q = \prod_{\alpha \in \Phi^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) \tag{7.27}$$

and therefore

$$\begin{aligned} N(t) &= \prod_{\alpha > 0} \left(e^{(\lambda+\rho,\alpha)_\kappa t/2} - e^{-(\lambda+\rho,\alpha)_\kappa t/2}\right) \\ &= \prod \left((\lambda+\rho,\alpha)_\kappa t[1 + \frac{1}{24}(\lambda+\rho,\alpha)_\kappa^2 t^2 + \cdots]\right) \end{aligned}$$

with a similar formula for D. Then $N/D \to d(\lambda) =$ the dimension of the representation as $t \to 0$ is the usual proof (that we reproduced above) of the Weyl dimension formula. Sticking this in to N/D gives

$$\frac{N}{D} = d(\lambda)\left(1 + \frac{1}{24}\sum_{\alpha>0}[(\lambda+\rho,\alpha)_\kappa^2 - (\rho,\alpha)_\kappa^2]t^2 + \cdots\right).$$

For any weight μ we have $(\mu, \mu)_\kappa = \sum(\mu, \alpha)_\kappa^2$ by (7.26), where the sum is over *all* roots so

$$\frac{N}{D} = d\left(1 + \frac{1}{48}[(\lambda+\rho, \lambda+\rho)_\kappa - (\rho,\rho)_\kappa]t^2 + \cdots\right),$$

and we recognize the coefficient of $\frac{1}{48}t^2$ in the above expression as $\chi_\lambda(\mathrm{Cas}^\kappa))$, the scalar giving the value of the Casimir associated to the Killing form in the representation with highest weight λ.

On the other hand, the image under Ψ_ρ of the character of the irreducible representation with highest weight λ is

$$\sum_\mu e^{(\mu,\rho)_\kappa t} = \sum_\mu (1 + (\mu,\rho)_\kappa t + \frac{1}{2}(\mu,\rho)_\kappa^2 t^2 + \cdots)$$

where the sum is over all weights in the irreducible representation counted with multiplicity. Comparing coefficients gives

$$\sum_\mu (\mu,\rho)_\kappa^2 = \frac{1}{24} d(\lambda)\chi_\lambda(\mathrm{Cas}^\kappa).$$

Applied to the adjoint representation the left hand side becomes $(\rho,\rho)_\kappa$ by (7.26), while $d(\lambda)$ is the dimension of the Lie algebra. On the other hand, $\chi_\lambda(\mathrm{Cas}^\kappa) = 1$ since $\mathrm{tr}\,\mathrm{ad}(\mathrm{Cas}^\kappa) = \dim(\mathbf{g})$ by the definition of Cas^κ. So we get

$$(\rho,\rho)_\kappa = \frac{1}{24}\dim\mathbf{g} \tag{7.28}$$

for any semisimple Lie algebra $\mathbf{g}$.

An algebra which is the direct sum a commutative Lie and a semi-simple Lie algebra is called reductive. The previous result of Freudenthal and deVries has been generalized by Kostant from a semi-simple Lie algebra to all reductive Lie algebras: Suppose that $\mathbf{g}$ is merely reductive, and that we have chosen a symmtric bilinear form on $\mathbf{g}$ which is invariant under the adjoint representation, and denote the associated Casimir element by $\mathrm{Cas}_{\mathbf{g}}$. We claim that (7.28) generalizes to

$$\frac{1}{24}\,\mathrm{tr}\,\mathrm{ad}(\mathrm{Cas}_{\mathbf{g}}) = (\rho,\rho). \tag{7.29}$$

(Notice that if $\mathbf{g}$ is semisimple and we take our symmetric bilinear form to be the Killing form $(\ ,\ )_\kappa$ (7.29) becomes (7.28).) To prove (7.29) observe that both sides decompose into sums as we decompose $\mathbf{g}$ into as sum of its center and its simple ideals, since this must be an orthogonal decomposition for our invariant scalar product. The contribution of the center is zero on both sides, so we are reduced to proving (7.29) for a simple algebra. Then our symmetric biinear form $(\ ,\ )$ must be a scalar multiple of the Killing form:

$$(\ ,\ ) = c^2(\ ,\ )_\kappa$$

for some non-zero scalar c. If $z_1,\dots,z_N$ is an orthonormal basis of $\mathbf{g}$ for $(\ ,\ )_\kappa$ then $z_1/c,\dots,z_N/c$ is an orthonormal basis for $(\ ,\ )$. Thus

$$\mathrm{Cas}_{\mathbf{g}} = \frac{1}{c^2}\,\mathrm{Cas}^\kappa\,.$$

So

$$\mathrm{tr}\,\mathrm{ad}(\mathrm{Cas}_{\mathbf{g}}) = \frac{1}{c^2}\,\mathrm{tr}\,\mathrm{ad}(\mathrm{Cas}^\kappa) = \frac{1}{c^2}\frac{1}{24}\dim\mathbf{g}.$$

But on $\mathbf{h}^*$ we have the dual relation

$$(\rho,\rho) = \frac{1}{c^2}(\rho,\rho)_\kappa.$$

Combining the last two equations shows that (7.29) becomes (7.28).

Notice that the same proof shows that we can generalize (7.8) as

$$\chi_\lambda(\mathrm{Cas}) = (\lambda+\rho,\lambda+\rho) - (\rho,\rho) \tag{7.30}$$

valid for any reductive Lie algebra equipped with a symmetric bilinear form invariant under the adjoint representation.

7.9 Fundamental representations.

We let ω_i denote the weight which satisfies

$$\omega_i(h_j) = \delta_{ij}$$

so that the ω_i form an integral basis of $\mathbf{L}$ and are dominant. We call these the **basic** weights.If (V, ρ) and (W, σ) are two finite dimensional irreducible representations with highest weights λ and σ, then $V \otimes W, \rho \otimes \sigma$ contains the irreducible representation with highest weight $\lambda + \mu$, and highest weight vector $v_\lambda \otimes w_\mu$, the tensor product of the highest weight vectors in V and W. Taking this "highest" component in the tensor product is known as the **Cartan product** of the two irreducible representations.

Let (V_i, ρ_i) be the irreducible representations corresponding to the basic weight ω_i. Then every finite dimensional irreducible representation of $\mathbf{g}$ can be obtained by Cartan products from these, and for that reason they are called the **fundamental representations**.

For the case of $A_n = sl(n+1)$ we have already verified that the fundamental representations are $\wedge^k(V)$ where $V = \mathbf{C}^{n+1}$ and where the basic weights are

$$\omega_i = L_1 + \cdots + L_i$$

We now sketch the results for the other classical simple algebras, leaving the details as an exercise in the use of the Weyl dimension formula.

For $C_n = sp(2n)$ it is immediate to check that these same expressions give the basic weights. However while $V = \mathbf{C}^{2n} = \wedge^1(V)$ is irreducible, the higher order exterior powers are not: Indeed, the symplectic form $\Omega \in \wedge^2(V^*)$ is preserved, and hence so is the the map

$$\wedge^j(V) \to \wedge^{j-2}(V)$$

given by contraction by Ω. It is easy to check that the image of this map is surjective (for $j = 2, \ldots, n$). the kernel is thus an invariant subspace of dimension

$$\binom{2n}{j} - \binom{2n}{2j-2}$$

and a (not completely trivial) application of the Weyl dimension formula will show that these are indeed the dimensions of the irreducible representations with highest weight ω_j. Thus these kernels are the fundamental representations of C_n. Here are some of the details:

We have

$$\rho = \omega_1 + \cdots + \omega_n = \sum (n - i + 1) L_i.$$

The most general dominant weight is of the form

$$\sum k_i \omega_i = a_1 L_1 + \cdots + a_n L_n$$

where

$$a_1 = k_1 + \cdots + k_n, \quad a_2 = k_2 + \cdots + k_n, \cdots a_n = k_n$$

where the k_i are non-negative integers. So we can equally well use any decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ of integers to parameterize the irreducible representations. We have

$$(\rho, L_i - L_j) = j - i, \quad (\rho, L_i + L_j) = 2n + 2 - i - j.$$

Multiplying these all together gives the denominator in the Weyl dimension formula.

Similarly the numerator becomes

$$\prod_{i<j}(l_i - l_j)\prod_{i\leq j.}(l_i + l_j)$$

where

$$l_i := a_i + n - i + 1.$$

If we set $m_i := n - i + 1$ then we can write the Weyl dimension formula as

$$\dim V(a_1, \ldots, a_n) = \prod_{i<j}\frac{l_i^2 - l_j^2}{m_i^2 - m_j^2}\prod_i \frac{l_i}{m_i},$$

where for the case $i = j$ we have taken out a common factor of 2^n from the numerator and the denominator.

An easy induction shows that

$$\prod_{i<j}(m_i^2 - m_j^2)\prod_i mi = (2n-1)!(2n-3)!\cdots 1!.$$

so if we set

$$r_i = l_i - 1 = a + n - i$$

then

$$\dim V(a_1, \ldots, a_n) = \frac{\prod_{i<j}(r_i - r_j)(r_i + r_j + 2)\prod_i (r_i + 1)}{(2n-1)!(2n-3)!\cdots 1!}.$$

For example, suppose we want to compute the dimension of the fundamental representation corresponding to $\lambda_2 = L_1 + L_2$ so $a_1 = a_2 = 1, a_i = 0, i > 2$. In applying the preceding formula, all of the terms with $2 < i$ are the same as for the trivial representation, as is $r_1 - r_2$. The ratios of the remaining factors to those of the trivial representation are

$$\prod_{j=3}^n \frac{j}{j-1} \cdot \prod_{j=3}^n \frac{j}{j-2} = \prod_{j=3}^n \frac{j}{j-2}$$

coming from the $r_i - r_j$ terms, $i = 1, 2$. Similarly the $r_i + r_j$ terms give a factor

$$\frac{2n+1}{2n-1}\prod_{j=3}^n \frac{2n+2-j}{2n-j}$$

and the terms r_1+1, r_2+1 contribute a factor

$$\frac{n+1}{n-1}.$$

In multiplying all of these terms together there is a huge cancellation and what is left for the dimension of this fundamental representation is

$$\frac{(2n+1)(2n-2)}{2}.$$

Notice that this equals

$$\left(\begin{array}{c} 2n \\ 2 \end{array}\right) - 1 = \dim \ \wedge^2 V - 1.$$

More generally this dimension argument will show that the fundamental representations are the kernels of the contraction maps $i(\Omega) : \wedge^k \to (V) \wedge^{k-2} (V)$ where Ω is the symplectic form.

For B_n it is easy to check that $\omega_i := L_1 + \cdots + L_i \ (i \leq n-1)$, and $\omega_n = \frac{1}{2}(L_1 + \cdots + L_n)$ are the basic weights and the Weyl dimension formula gives the value $\binom{2n+1}{j}$ for $j \leq n-1$ as the dimensions of the irreducibles with these weight, so that they are $\wedge^j(V), \ j = 1, \ldots n-1$ while the dimension of the irreducible corresponding to ω_n is 2^n. This is the spin representation which we will study later.

Finally, for $D_n = o(2n)$ the basic weights are

$$\omega_j = L_1 + \cdots + L_j, \ j \leq n-2,$$

and

$$\omega_{n-1} := \frac{1}{2}(L_1 + \cdots + L_{n-1} + L_n) \text{ and } \omega_n := \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n).$$

The Weyl dimension formula shows that the the first $n-2$ fundamental representations are in fact the representation on $\wedge^j(V), \ j = 1, \ldots, n-2$ while the last two have dimension 2^{n-1}. These are the half spin representations which we will also study later.

7.10 Equal rank subgroups.

In this section we present a generalization of the Weyl character formula due to Ramond-Gross-Kostant-Sternberg. It depends on an interpretation of the Weyl denominator in terms of the spin representation of the orthogonal group $O(\mathbf{g}/\mathbf{h})$, and so on some results which we will prove in Chapter IX. But its logical place is in this chapter. So we will quote the results that we will need. You might prefer to read this section after Chapter IX.

Let $\mathbf{p}$ be an even dimensional space with a symmetric bilinear such that

$$\mathbf{p} = \mathbf{p}_+ \oplus \mathbf{p}_-$$

is a direct sum decomposition of $\mathbf{p}$ into two isotropic subspaces. In other words $\mathbf{p}_+$ and $\mathbf{p}_-$ are each half the dimension of $\mathbf{p}$, and the scalar product of any two vectors in $\mathbf{p}_+$ vanishes, as does the scalar product of any two elements of $\mathbf{p}_-$. For example, we might take $\mathbf{p} = \mathbf{n}_+ \oplus \mathbf{n}_-$ and the symmetric bilinear form to be the Killing form. Then $\mathbf{p}_\pm = \mathbf{n}_\pm$ is such a desired decomposition.

The symmetric bilinear form then puts $\mathbf{p}_\pm$ into duality, i.e. we may identify $\mathbf{p}_-$ with $\mathbf{p}_+^*$ and vice versa. Suppose that we have a commutative Lie algebra $\mathbf{h}$ acting on $\mathbf{p}$ as infinitesimal isometries, so as to preserve each $\mathbf{p}_\pm$, that the e_i^+ are weight vectors corresponding to weights β_i and that the e_i^- form the dual basis, corresponding to the negative of these weights $-\beta_i$. In particular, we have a Lie algebra homomorphism ν from $\mathbf{h}$ to $o(\mathbf{p})$, and the two spin representations of $o(\mathbf{p})$ give two representations of $\mathbf{h}$. By abuse of language, let us denote these two representations by $\mathrm{Spin}_{+\nu}$ and $\mathrm{Spin}_{-\nu}$. We can also consider the characters of these representations of $\mathbf{h}$. According to equation (9.22) (to be proved in Chapter IX) we have

$$\mathrm{ch}_{Spin_+\nu} - \mathrm{ch}_{Spin_-\nu} = \prod_j \left(e(\frac{1}{2}\beta_j) - e(-\frac{1}{2}\beta_j) \right).$$

In the case that $\mathbf{h}$ is the Cartan subalgebra of a semi-simple Lie algebra and and $\mathbf{p}_\pm = \mathbf{n}_\pm$ we recognize this expression as the Weyl denominator.

Now let $\mathbf{g}$ be a semi-simple Lie algebra and $\mathbf{r} \subset \mathbf{g}$ a reductive subalgebra of the same rank. This means that we can choose a Cartan subalgebra of $\mathbf{g}$ which is also a Cartan subalgebra of $\mathbf{r}$. The roots of $\mathbf{r}$ form a subset of the roots of $\mathbf{g}$. The Weyl group $W_{\mathbf{g}}$ acts simply transitively on the Weyl chambers of $\mathbf{g}$ each of which is contained in a Weyl chamber for $\mathbf{r}$. We choose a positive root system for $\mathbf{g}$, which then determines a positive root system for $\mathbf{r}$, and the positive Weyl chamber for $\mathbf{g}$ is contained in the positive Weyl chamber for $\mathbf{r}$.

Let

$$C \subset W_{\mathbf{g}}$$

denote the set of those elements of the Weyl group of $\mathbf{g}$ which map the positive Weyl chamber of $\mathbf{g}$ into the positive Weyl chamber for $\mathbf{r}$. By the simple transitivity of the Weyl group actions on chambers, we know that elements of C form coset representatives for the subgroup $W_{\mathbf{r}} \subset W_{\mathbf{g}}$. In particular, the number of elements of C is the same as the index of $W_{\mathbf{r}}$ in $W_{\mathbf{g}}$.

Let

$$\rho_{\mathbf{g}} \quad \text{and} \quad \rho_{\mathbf{r}}$$

denote half the sum of the positive roots of $\mathbf{g}$ and $\mathbf{r}$ respectively. For any dominant weight λ of $\mathbf{g}$ the weight $\lambda + \rho_{\mathbf{g}}$ lies in the interior of the positive Weyl chamber for $\mathbf{g}$. Hence for each $c \in C$, the element $c(\lambda + \rho_{\mathbf{g}})$ lies in the interior for $\mathbf{r}$ and hence

$$c \bullet \lambda := c(\lambda + \rho_{\mathbf{g}}) - \rho_{\mathbf{r}}$$

is a dominant weight for $\mathbf{r}$, and each of these is distinct.

Let V_λ denote the irreducible representation of $\mathbf{g}$ with highest weight λ. We can consider it as a representation of the subalgebra $\mathbf{r}$. Also the Killing form (or more generally any ad invariant symmetric bilinear form) on $\mathbf{g}$ induces an invariant form on $\mathbf{r}$. Let $\mathbf{p}$ denote the orthogonal complement of $\mathbf{r}$ in $\mathbf{g}$. We thus get a homomorphism of $\mathbf{r}$ into the orthogonal algebra $o(\mathbf{g}/\mathbf{r})$, which is an even dimensional orthogonal algebra, and hence has two spin representations. To specify which of these two spin representations we shall denote by S_+ and which by S_-, we note that there is a one dimensional weight space with weight $\rho_\mathbf{g} - \rho_\mathbf{r}$, and we let S_+ denote the spin representation which contains that one dimensional space. The spaces $S_\pm$ are $o(\mathbf{g}/\mathbf{r})$ modules, and via the homomorphism $\mathbf{r} \to o(\mathbf{g}/\mathbf{r})$ we can consider them as $\mathbf{r}$ modules.

Finally, for any dominant integral weight μ of $\mathbf{r}$ we let U_μ denote the irreducible module of $\mathbf{r}$ with highest weight μ.

With all this notation we can now state

Theorem 1 **[G-K-R-S]** *In the representation ring $R(\mathbf{r})$ we have*

$$V_\lambda \otimes S_+ - V_\lambda \otimes S_- = \sum_{c \in C} (-1)^c U_{c \bullet \lambda}. \tag{7.31}$$

Proof. To say that the above equation holds in the representation ring of $\mathbf{r}$ means that when we take the signed sums of the characters of the representations occurring on both sides we get equality. In the special case that $\mathbf{r} = \mathbf{h}$, we have observed that (7.31) is just the Weyl character formula:

$$\chi(\mathrm{Irr}(\lambda)(\chi(S_{+\mathbf{g}/\mathbf{h}}) - \chi(S_{-\mathbf{g}/\mathbf{h}})) = \sum_{w \in W_\mathbf{g}} (-1)^w e(w(\lambda + \rho_\mathbf{g})).$$

The general case follows from this special case by dividing both sides of this equation by $\chi(S_{+\mathbf{r}/\mathbf{h}}) - \chi(S_{-\mathbf{r}/\mathbf{h}})$. The left hand side becomes the character of the left hand side of (7.31) because the weights that go into this quotient via (9.22) are exactly those roots of $\mathbf{g}$ which are not roots of $\mathbf{r}$. The right hand side becomes the character of the right hand side of (9.22) by reorganizing the sum and using the Weyl character formula for $\mathbf{r}$. QED