Lie Algebras

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Chapter 4

Engel-Lie-Cartan-Weyl

We return to the general theory of Lie algebras. Many of the results in this chapter are valid over arbitrary fields, indeed if we use the axioms to define a Lie algebra over a ring many of the results are valid in this generality. But some of the results depend heavily on the ring being an algebraically closed field of characteristic zero. As a compromise, throughout this chapter we deal with fields, and will assume that all vector spaces and all Lie algebras which appear are finite dimensional. We will indicate the necessary additional assumptions on the ground field as they occur. The treatment here follows Serre pretty closely.

4.1 Engel's theorem

Define a Lie algebra $\mathbf{g}$ to be **nilpotent** if:

$$\exists n|\ [x_1,[x_2,\dots x_{n+1}]\dots]=0 \quad \forall\, x_1,\dots,x_{n+1}\in\mathbf{g}.$$

Example: $\mathbf{n}^+ := \mathbf{n}^+(gl(d)) :=$ all strictly upper triangular matrices. Notice that the product of any $d+1$ such matrices is zero.

The claim is that all nilpotent Lie algebras are essentially like $\mathbf{n}^+$.

We can reformulate the definition of nilpotent as saying that the product of any n operators $\operatorname{ad} x_i$ vanishes. One version of Engel's theorem is

Theorem 1 $\mathbf{g}$ *is nilpotent if and only if* $\operatorname{ad} x$ *is a nilpotent operator for each* $x \in \mathbf{g}$.

This follows (taking $V = \mathbf{g}$ and the adjoint representation) from

Theorem 2 Engel *Let* $\rho : \mathbf{g} \to \operatorname{End}(V)$ *be a representation such that* $\rho(x)$ *is nilpotent for each* $x \in \mathbf{g}$. *Then there exists a basis in terms of which* $\rho(\mathbf{g}) \subset \mathbf{n}^+(gl(d))$, *i.e. becomes strictly upper triangular. Here* d =dim V.

Given a single nilpotent operator, we can always find a non-zero vector, v which it sends into zero. Then on $V/\{v\}$ a non-zero vector which the induced map sends into zero etc. So in terms of such a flag, the corresponding matrix is strictly upper triangular. The theorem asserts that we are can find a single flag which works for all $\rho(x)$. In view of the above proof for a single operator, Engel's theorem follows from the following simpler looking statement:

Theorem 3 *Under the hypotheses of Engel's theorem, if $V \neq 0$, there exists a non-zero vector $v \in V$ such that $\rho(x)v = 0 \ \forall x \in \mathbf{g}$.*

Proof of Theorem 3 in seven easy steps.

- Replace $\mathbf{g}$ by its image, i.e. assume that $\mathbf{g} \subset \mathrm{End}\, V$.
- Then $(\mathrm{ad}\, x)y = L_x y - R_x y$ where L_x is the linear map of $\mathrm{End}\, V$ into itself given by left multiplication by x, and R_x is given by right multiplication by x. Both L_x and R_x are nilpotent as operators since x is nilpotent. Also they commute. Hence by the binomial formula $(\mathrm{ad}\, x)^n = (L_x - R_x)^n$ vanishes for sufficiently large n.
- We may assume (by induction) that for any Lie algebra, $\mathbf{m}$, of smaller dimension than that of $\mathbf{g}$ (and any representation) there exists a $v \in V$ such that $xv = 0 \ \forall x \in \mathbf{m}$.
- Let $\mathbf{k} \subset \mathbf{g}$ be a subalgebra, $\mathbf{k} \neq \mathbf{g}$, and let

  $$N = N(\mathbf{k}) := \{x \in \mathbf{g} | (\mathrm{ad}\, x)\mathbf{k} \subset \mathbf{k}\}$$

  be its normalizer. The claim is that

  1 $N(\mathbf{k})$ is strictly larger than $\mathbf{k}$.

  To see this, observe that each $x \in \mathbf{k}$ acts on $\mathbf{k}$ and on $\mathbf{g}/\mathbf{k}$ by nilpotent maps, and hence there is an $0 \neq \hat{y} \in \mathbf{g}/\mathbf{k}$ killed by all $x \in \mathbf{k}$. But then $y \notin \mathbf{k}$, and $[y, x] = -[x, y] \in \mathbf{k}$ for all $x \in \mathbf{k}$. So $y \in N(\mathbf{k}), \ y \notin \mathbf{k}$.
- If $\mathbf{g} \neq 0$, there is an ideal $\mathbf{i} \subset \mathbf{g}$ such that $\dim \mathbf{g}/\mathbf{i} = 1$. Indeed, let $\mathbf{i}$ be a maximal proper subalgebra of $\mathbf{g}$. Its normalizer is strictly larger, hence all of $\mathbf{g}$, so $\mathbf{i}$ is an ideal. The inverse image in $\mathbf{g}$ of a line in $\mathbf{g}/\mathbf{i}$ is a subalgebra, and is strictly larger than $\mathbf{i}$. Hence it must be all of $\mathbf{g}$.
- Choose such an ideal, $\mathbf{i}$. The subspace

  $$W \subset V, \quad W = \{v | xv = 0, \ \forall x \in \mathbf{i}\}$$

  is invariant under $\mathbf{g}$. Indeed, if $y \in \mathbf{g}, \ w \in W$ then $xyw = yxw + [x, y]w = 0$.
- $W \neq 0$ by induction. Take $y \in \mathbf{g}, \ y \notin \mathbf{i}$. It preserves W and is nilpotent. Hence there is a non-zero $v \in W$ with $yv = 0$. Since y and $\mathbf{i}$ span $\mathbf{g}$, we have $xv = 0 \ \ \forall x \in \mathbf{g}$. QED

No assumptions about the ground field went into this.

4.2 Solvable Lie algebras.

Let $\mathbf{g}$ be a Lie algebra. $D^n\mathbf{g}$ is defined inductively by

$$D^0\mathbf{g} := \mathbf{g},\ D^1(\mathbf{g}) := [\mathbf{g},\mathbf{g}],\ldots,\quad D^{n+1}\mathbf{g} := [D^n\mathbf{g}, D^n\mathbf{g}].$$

If we take $\mathbf{b}$ to consist of all upper triangular $n \times n$ matrices, then $D^1\mathbf{b} = \mathbf{n}^+$ consists of all strictly triangular matrices and then successive brackets eventually lead to zero. We claim that the following conditions are equivalent and any Lie algebra satisfying them is called **solvable**.

1. $\exists n \ \ |D^n\mathbf{g} = 0$.

2. $\exists n$ such that for every family of 2^n elements of $\mathbf{g}$ the successive brackets of brackets vanish; e.g for $n = 4$ this says

   $$[[[[x_1,x_2],[x_3,x_4]],[[x_5,x_6],[x_7,x_8]]],[[[x_9,x_{10}],[x_{11},x_{12}]],[[x_{13}.x_{14}],[x_{15},x_{16}]]]] = 0.$$

3. There exists a sequence of subspaces $\mathbf{g} := \mathbf{i}_1 \supset \mathbf{i}_2 \supset \cdots \supset \mathbf{i}_n = 0$ such each is an ideal in the preceding and such that the quotient $\mathbf{i}_j/\mathbf{i}_{j+1}$ is abelian, i.e. $[\mathbf{i}_j,\mathbf{i}_j] \subset \mathbf{i}_{j+1}$.

Proof of the equivalence of these conditions. $[\mathbf{g},\mathbf{g}]$ is always an ideal in $\mathbf{g}$ so the $D^j\mathbf{g}$ form a sequence of ideals demanded by 3), and hence 1) $\Rightarrow$ 3). We also have the obvious implications 3) $\Rightarrow$ 2) and 2) $\Rightarrow$ 1). So all these definitions are equivalent.

Theorem 4 [Lie.] *Let $\mathbf{g}$ be a solvable Lie algebra over an algebraically closed field k of characteristic zero, and (ρ,V) a finite dimensional representation of $\mathbf{g}$. Then we can find a basis of V so that $\rho(\mathbf{g})$ consists of upper triangular matrices.*

By induction on dim V this reduces to

Theorem 5 [Lie.] *Under the same hypotheses, there exists a (non-zero) common eigenvector v for all the $\rho(y)$, i.e. there is a vector $v \in V$ and a function $\chi : \mathbf{g} \to k$ such that*

$$\rho(y)v = \chi(y)v \quad \forall\ y \in \mathbf{g}. \tag{4.1}$$

Lemma 1 *Suppose that $\mathbf{i}$ is an ideal of $\mathbf{g}$ and (4.1) holds for all $y \in \mathbf{i}$. Then*

$$\chi([x,h]) = 0, \quad \forall\ x \in \mathbf{g} \quad h \in \mathbf{i}.$$

Proof of lemma. For $x \in \mathbf{g}$ let V_i be the subspace spanned by $v, xv, \ldots, x^{i-1}v$ and let $n > 0$ be minimal such that $V_n = V_{n+1}$. So V_n is finite dimensional and $xV_n \subset V_n$. Also $V_n = V_{n+k}\ \forall k$.

Also, for $h \in \mathbf{i}$, (dropping the ρ) we have:

$$\begin{aligned} hv &= \chi(h)v \\ hxv &= xhv - [x,h]v \\ &\equiv \chi(h)xv \mod V_1 \\ hx^2v &= xhxv + [h,x]xv \\ &\equiv \chi(h)x^2v + uxv, \mod V_1 \ u \in I \\ &\equiv \chi(h)x^2v + \chi(u)xv \mod V_1 \\ &= \chi(h)x^2v \mod V_2 \\ &\vdots \\ hx^iv &\equiv \chi(h)x^iv \mod V_i. \end{aligned}$$

Thus V_n is invariant under $\mathbf{i}$ and for each $h \in \mathbf{i}$, $\mathrm{tr}_{|V_n} h = n\chi(h)$. In particular both x and h leave V_n invariant and $\mathrm{tr}_{|V_n}[x,h] = 0$ since the trace of any commutator is zero. This proves the lemma.
Proof of theorem by induction on dim $\mathbf{g}$, which we may assume to be positive. Let $\mathbf{m}$ be any subspace of $\mathbf{g}$ with $\mathbf{g} \supset \mathbf{m} \supset [\mathbf{g},\mathbf{g}]$. Then $[\mathbf{g},\mathbf{m}] \subset [\mathbf{g},\mathbf{g}] \subset \mathbf{m}$ so $\mathbf{m}$ is an ideal in $\mathbf{g}$. In particular, we may choose $\mathbf{m}$ to be a subspace of codimension 1 containing $[\mathbf{g},\mathbf{g}]$. By induction we can find a $v \in V$ and a $\chi : \mathbf{m} \to k$ such that (4.1) holds for all elements of $\mathbf{m}$. Let

$$W := \{w \in V | hw = \chi(h)w \ \forall \ h \in \mathbf{m}\}.$$

If $x \in \mathbf{g}$, then

$$hxw = xhw - [x,h]w = \chi(h)xw - \chi([x,h])w = \chi(h)xw$$

since $\chi([x,h]) = 0$ by the lemma. Thus W is stable under all of $\mathbf{g}$. Pick $x \in \mathbf{g}, \quad x \notin \mathbf{m}$, and let $v \in W$ be an eigenvector of x with eigenvalue λ, say. Then v is a simultaneous eigenvector for all of $\mathbf{g}$ with χ extended as

$$\chi(h + rx) = \chi(h) + r\lambda. \text{ QED}$$

We had to divide by n in the above argument. In fact, the theorem is not true over a field of characteristic 2, with $sl(2)$ as a counterexample.

Applied to the adjoint representation, Lie's theorem says that there is a flag of ideals with commutative quotients, and hence $[\mathbf{g},\mathbf{g}]$ is nilpotent.

4.3 Linear algebra

Let V be a finite dimensional vector space over an algebraically closed field of characteristic zero, and let

$$\det(TI - u) = \prod (T - \lambda_i)^{m_i}$$

be the factorization of its characteristic polynomial where the λ_i are distinct. Let $S(T)$ be any polynomial satisfying

$$S(T) \equiv \lambda_i \bmod (T-\lambda_i)^{m_i}, \quad S(T) \equiv 0 \bmod T,$$

which is possible by the Chinese remainder theorem. For each i let $V_i :=$ the kernel of $(u-\lambda_i)^{m_i}$. Then $V = \bigoplus V_i$ and on V_i, the operator $S(u)$ is just the scalar operator $\lambda_i I$. In particular $s = S(u)$ is semi-simple (its eigenvectors span V) and, since s is a polynomial in u it commutes with u. So

$$u = s + n$$

where

$$n = N(u),\ N(T) = T - S(T)$$

is nilpotent. Also

$$ns = sn.$$

We claim that these two elements are uniquely determined by

$$u = s + n, sn = ns,$$

with s semisimple and n nilpotent. Indeed, since $sn = ns$, $\quad su = us$ so $s(u-\lambda_i)^k = (u-\lambda_i)^k s$ so $sV_i \subset V_i$. Since $s-u$ is nilpotent, s has the same eigenvalues on V_i as u does, i.e. λ_i. So s and hence n is uniquely determined.

If $P(T)$ is any polynomial with vanishing constant term, then if $A \subset B$ are subspaces with $uB \subset A$ then $P(u)B \subset A$. So, in particular, $sB \subset A$ and $nB \subset A$.

Define

$$V_{p,q} := V \otimes V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

with p copies of V and q copies of V^*. Let $u \in \mathrm{End}(V)$ act on V^* by $-u^*$ and on V_{pq} by derivation, so , for example,

$$u_{12} = u \otimes 1 \otimes 1 - 1 \otimes u^* \otimes 1 - 1 \otimes 1 \otimes u^*.$$

Similarly, u_{11} acts on $V_{1,1} = V \otimes V^*$ by

$$u_{11}(x \otimes \ell) = ux \otimes \ell - x \otimes u^*\ell.$$

Under the identification of $V \otimes V^*$ with $\mathrm{End}(V)$, the element $x \otimes \ell$ acts on $y \in V$ by sending it into

$$\ell(y)x.$$

So the element $u_{11}(x \otimes \ell)$ sends y to

$$\ell(y)u(x) - (u^*\ell)(y)x = \ell(y)u(x) - \ell(u(y))x.$$

This is the same as the commutator of the operator u with the operator (corresponding to) $x \otimes \ell$ acting on y. In other words, under the identification of $V \otimes V^*$ with $\mathrm{End}(V)$, the linear transformation u_{11} gets identified with $\operatorname{ad} u$.

Proposition 1 *If $u = s + n$ is the decomposition of u then $u_{pq} = s_{pq} + n_{pq}$ is the decomposition of u_{pq}.*

Proof. $[s_{pq}, n_{pq}] = 0$ and the tensor products of an eigenbasis for s is an eigenbasis for s_{pq}. Also n_{pq} is a sum of commuting nilpotents hence nilpotent. The map $u \mapsto u_{pq}$ is linear hence $u_{pq} = s_{pq} + n_{pq}$. QED

If $\phi : k \to k$ is a map, we define $\phi(s)$ by $\phi(s)_{|V_i} = \phi(\lambda_i)$. If we choose a polynomial such that $P(0) = 0$, $P(\lambda_i) = \phi(\lambda_i)$ then $P(u) = \phi(s)$.

Proposition 2 *Suppose that ϕ is additive. Then*

$$(\phi(s))_{pq} = \phi(s_{pq}).$$

Proof. Decompose V_{pq} into a sum of tensor products of the V_i or V_j^*. On each such space we have

$$\begin{aligned}\phi(s_{p,q}) &= \phi(\lambda_{i_1} + \cdots - \cdots) \\ &= \phi(\lambda_{i_1}) + \phi(..).. \\ &= (\phi(s))_{p,q}\end{aligned}$$

where the middle equation is just the additivity. QED

As an immediate consequence we obtain

Proposition 3 *Notation as above. If $A \subset B \subset V_{p,q}$ with $u_{pq}B \subset A$ then for any additive map, $\phi(s)_{pq}B \subset A$*

Proposition 4 (over **C**) *Let $u = s + n$ as above. If $\operatorname{tr}(u\phi(s)) = 0$ for $\phi(s) = \overline{s}$ then u is nilpotent.*

Proof. $\operatorname{tr} u\phi(s) = \sum m_i \lambda_i \overline{\lambda_i} = \sum m_i |\lambda_i|^2$. So the condition implies that all the $\lambda_i = 0$. QED

4.4 Cartan's criterion.

Let $\mathbf{g} \subset \operatorname{End}(V)$ be a Lie subalgebra where V is finite dimensional vector space over $\mathbf{C}$. Then

$$\mathbf{g} \text{ is solvable} \Leftrightarrow \quad \operatorname{tr}(xy) = 0 \ \forall x \in \mathbf{g}, \quad y \in [\mathbf{g}, \mathbf{g}].$$

Proof. Suppose $\mathbf{g}$ is solvable. Choose a basis for which $\mathbf{g}$ is upper triangular. Then every $y \in [\mathbf{g}, \mathbf{g}]$ has zeros on the diagonal, Hence $\operatorname{tr}(xy) = 0$. For the reverse implication, it is enough to show that $[\mathbf{g}, \mathbf{g}]$ is nilpotent, and, by Engel, that each $u \in [\mathbf{g}, \mathbf{g}]$ is nilpotent. So it is enough to show that $\operatorname{tr} u\overline{s} = 0$, where s is the semisimple part of u, by Proposition 4 above. If it were true that $\overline{s} \in \mathbf{g}$ we would be done, but this need not be so. Write

$$u = \sum [x_i, y_i].$$

Now for $a, b, c \in \mathrm{End}(V)$

$$\begin{aligned} \mathrm{tr}([a,b]c) &= \mathrm{tr}(abc - bac) \\ &= \mathrm{tr}(bca - bac) \\ &= \mathrm{tr}(b[c,a]) \quad \text{so} \\ \mathrm{tr}(u\overline{s}) &= \sum \mathrm{tr}([x_i, y_i]\overline{s}) \\ &= \sum \mathrm{tr}(y_i[\overline{s}, x_i]). \end{aligned}$$

So it is enough to show that $\mathrm{ad}\,\overline{s} : \mathbf{g} \to [\mathbf{g}, \mathbf{g}]$. We know that $\mathrm{ad}\, u : \mathbf{g} \to [\mathbf{g}, \mathbf{g}]$, and we can, by Lagrange interpolation, find a polynomial P such that $P(u) = \overline{s}$. The result now follows from Prop. 3:

Since $\mathrm{End}(V) \sim V_{1,1}$, take $A = [\mathbf{g}, \mathbf{g}]$ and $B = \mathbf{g}$. Then $\mathrm{ad}\, u = u_{1,1}$ so $u_{1,1}\mathbf{g} \subset [\mathbf{g}, \mathbf{g}[$ and hence $\overline{s}_{1,1}\mathbf{g} \subset [\mathbf{g}, \mathbf{g}]$ or $[\overline{s}, x] \in [\mathbf{g}, \mathbf{g}]\ \forall x \in \mathbf{g}$. QED

4.5 Radical.

If $\mathbf{i}$ is an ideal of $\mathbf{g}$ and $\mathbf{g}/\mathbf{i}$ is solvable, then $D^{(n)}(\mathbf{g}/\mathbf{i}) = 0$ implies that $D^{(n)}\mathbf{g} \subset \mathbf{i}$. If $\mathbf{i}$ itself is solvable with $D^{(m)}\mathbf{i} = 0$, then $D^{(m+n)}\mathbf{g} = 0$. So we have proved:

Proposition 5 *If* $\mathbf{i} \subset \mathbf{g}$ *is an ideal, and both* $\mathbf{i}$ *and* $\mathbf{g}/\mathbf{i}$ *are solvable, so is* $\mathbf{g}$.

If $\mathbf{i}$ and $\mathbf{j}$ are solvable ideals, then $(\mathbf{i} + \mathbf{j})/\mathbf{j} \sim \mathbf{i}/(\mathbf{i} \cap \mathbf{j})$ is solvable, being the homomorphic image of a solvable algebra. So, by the previous proposition:

Proposition 6 *If* $\mathbf{i}$ *and* $\mathbf{j}$ *are solvable ideals in* $\mathbf{g}$ *so is* $\mathbf{i}+\mathbf{j}$. *In particular, every Lie algebra* $\mathbf{g}$ *has a largest solvable ideal which contains all other solvable ideals. It is denoted by* $\mathrm{rad}\ \mathbf{g}$ *or simply by* $\mathbf{r}$ *when* $\mathbf{g}$ *is fixed.*

An algebra $\mathbf{g}$ is called **semi-simple** if $\mathrm{rad}\ \mathbf{g} = 0$. Since $D\mathbf{i}$ is an ideal whenever $\mathbf{i}$ is (by Jacobi's identity), if $\mathbf{r} \neq 0$ then the last non-zero $D^{(n)}\mathbf{r}$ is an abelian ideal. So an equivalent definition is: $\mathbf{g}$ is semi-simple if it has no non-zero abelian ideals.

We shall call a Lie algebra **simple** if it is not abelian and if it has no proper ideals. We shall show in the next section that every semi-simple Lie algebra is the direct sum of simple Lie algebras in a unique way.

4.6 The Killing form.

A bilinear form $(\ ,\ ) : \mathbf{g} \times \mathbf{g} \to k$ is called **invariant** if

$$([x, y], z) + (y, [x, z]) = 0 \quad \forall x, y, z \in \mathbf{g}. \tag{4.2}$$

Notice that if $(\ ,\ )$ is an invariant form, and $\mathbf{i}$ is an ideal, then $\mathbf{i}^{\perp}$ is again an ideal.

One way of producing invariant forms is from representations: if(ρ, V) is a representation of $\mathbf{g}$, then

$$(x,y)_\rho := \operatorname{tr}\rho(x)\rho(y)$$

is invariant. Indeed,

$$([x,y],z)_\rho + (y,[x,z])_\rho$$

$$\begin{aligned} &= \operatorname{tr}\{(\rho(x)\rho(y)-\rho(y)\rho(x))\rho(z)) + \rho(y)(\rho(x)\rho(z)-\rho(z)\rho(x))\} \\ &= \operatorname{tr}\{\rho(x)\rho(y)\rho(z)-\rho(y)\rho(z)\rho(x)\} \\ &= 0. \end{aligned}$$

In particular, if we take $\rho = \mathrm{ad}$, $V = \mathbf{g}$ the corresponding bilinear form is called the **Killing form** and will be denoted by $(\ ,\ )_\kappa$. We will also sometimes write $\kappa(x,y)$ instead of $(x,y)_\kappa$.

Theorem 6 $\mathbf{g}$ *is semi-simple if and only if its Killing form is non-degenerate.*

Proof. Suppose $\mathbf{g}$ is not semi-simple and so has a non-zero abelian ideal, $\mathbf{a}$. We will show that $(x,y)_\kappa = 0\ \forall x \in \mathbf{a}, y \in \mathbf{g}$. Indeed, let $\sigma = \mathrm{ad}\,x\,\mathrm{ad}\,y$. Then σ maps $\mathbf{g} \to \mathbf{a}$ and $\mathbf{a} \to 0$. Hence in terms of a basis starting with elements of $\mathbf{a}$ and extending, it (is upper triangular and) has 0 along the diagonal. Hence $\operatorname{tr}\sigma = 0$. Hence if $\mathbf{g}$ is *not* semisimple then its Killing form is degenerate.

Conversely, suppose that $\mathbf{g}$ is semi-simple. We wish to show that the Killing form is non-degenerate. So let $\mathbf{u} := \mathbf{g}^\perp = \{x \mid \operatorname{tr}\mathrm{ad}\,x\,\mathrm{ad}\,y = 0\ \forall y \in \mathbf{g}\}$. If $x \in \mathbf{u}, z \in \mathbf{g}$ then

$$\begin{aligned} \operatorname{tr}\{\mathrm{ad}[x,z]\,\mathrm{ad}\,y\} &= \operatorname{tr}\{\mathrm{ad}\,x\,\mathrm{ad}\,z\,\mathrm{ad}\,y - \mathrm{ad}\,z\,\mathrm{ad}\,x\,\mathrm{ad}\,y)\} \\ &= \operatorname{tr}\{\mathrm{ad}\,x(\mathrm{ad}\,z\,\mathrm{ad}\,y - \mathrm{ad}\,y\,\mathrm{ad}\,z)\} \\ &= \operatorname{tr}\mathrm{ad}\,x\,\mathrm{ad}[z,y] \\ &= 0, \end{aligned}$$

so $\mathbf{u}$ is an ideal. In particular, $\operatorname{tr}_{\mathbf{u}}(\mathrm{ad}\,x_{\mathbf{u}}\,\mathrm{ad}\,y_{\mathbf{u}}) = \operatorname{tr}_{\mathbf{g}}(\mathrm{ad}_{\mathbf{g}}\,x\,\mathrm{ad}_{\mathbf{g}}\,y)$ for $x, y \in \mathbf{u}$, as can be seen from a block decomposition starting with a basis of $\mathbf{u}$ and extending to $\mathbf{g}$.

If we take $y \in D\mathbf{u}$, we see that $\operatorname{tr}\mathrm{ad}\,\mathbf{u}D\,\mathrm{ad}\,\mathbf{u} = 0$, so $\mathrm{ad}\,\mathbf{u}$ is solvable by Cartan's criterion. But the kernel of the map $\mathbf{u} \to \mathrm{ad}\,\mathbf{u}$ is the center of $\mathbf{u}$. So if $\mathrm{ad}\,\mathbf{u}$ is solvable, so is $\mathbf{u}$. QED

Proposition 7 *Let* $\mathbf{g}$ *be a semisimple algebra,* $\mathbf{i}$ *any ideal of* $\mathbf{g}$*, and* $\mathbf{i}^\perp$ *its orthocomplement with respect to its Killing form. Then* $\mathbf{i} \cap \mathbf{i}^\perp = 0$.

Indeed, $\mathbf{i} \cap \mathbf{i}^\perp$ is an ideal on which $tr\,\mathrm{ad}\,x\,\mathrm{ad}\,y \equiv 0$ hence is solvable by Cartan's criterion. Since $\mathbf{g}$ is semi-simple, there are no non-trivial solvable ideals. QED

Therefore

Proposition 8 *Every semi-simple Lie algebra is the direct sum of simple Lie algebras.*

Proposition 9 *$D\mathbf{g} = \mathbf{g}$ for a semi-simple Lie algebra.*

(Since this is true for each simple component.)

Proposition 10 *Let $\phi : \mathbf{g} \to \mathbf{s}$ be a surjective homomorphism of a semi-simple Lie algebra onto a simple Lie algebra. Then if $\mathbf{g} = \bigoplus \mathbf{g}_i$ is a decomposition of $\mathbf{g}$ into simple ideals, the restriction, ϕ_i of ϕ to each summand is zero, except for one summand where it is an isomorphism.*

Proof. Since $\mathbf{s}$ is simple, the image of every ϕ_i is 0 or all of $\mathbf{s}$. If ϕ_i is surjective for some i then it is an isomorphism since $\mathbf{g}_i$ is simple. There is at least one i for which it is surjective since ϕ is surjective. On the other hand, it can not be surjective for for two ideals, $\mathbf{g}_i, \mathbf{g}_j$ $i \neq j$ for then $\phi[\mathbf{g}_i, \mathbf{g}_j] = 0 \neq [\mathbf{s}, \mathbf{s}] = \mathbf{s}$. QED

4.7 Complete reducibility.

The basic theorem is

Theorem 7 **[Weyl.]** *Every finite dimensional representation of a semi-simple Lie algebra is completely reducible.*

Proof.

1. If $\rho : \mathbf{g} \to \operatorname{End} V$ is injective, then the form $(\ ,\ )_\rho$ is non-degenerate. Indeed, the ideal consisting of all x such that $(x, y)_\rho = 0\ \forall y \in \mathbf{g}$ is solvable by Cartan's criterion, hence 0.

2. The **Casimir operator.** Let (e_i) and (f_i) be bases of $\mathbf{g}$ which are dual with respect to some non-degenerate invariant bilinear form, $(,)$. So $(e_i, f_j) = \delta_{ij}$. As the form is non-degenerate and invariant, it defines a map of
$$\mathbf{g} \otimes \mathbf{g} \mapsto \operatorname{End} \mathbf{g}; \quad x \otimes y(w) = (y, w)x.$$
This map is an isomorphism and is a $\mathbf{g}$ morphism. Under this map,
$$\sum e_i \otimes f_i(w) = \sum (w, f_i)e_i = w$$
by the definition of dual bases. Hence under the inverse map
$$\operatorname{End} \mathbf{g} \mapsto \mathbf{g} \otimes \mathbf{g}$$
the identity element, id, corresponds to $\sum e_i \otimes f_i$ (and so this expression is independent of the choice of dual bases). Since id is annihilated by commutator by any element of End($\mathbf{g}$), we conclude that $\sum_i e_i \otimes f_i$ is annihilated by the action of all $(\operatorname{ad} x)_2 = \operatorname{ad} x \otimes 1 + 1 \otimes \operatorname{ad} x$, $x \in \mathbf{g}$. Indeed, for $x, e, f, y \in \mathbf{g}$ we have
$$\begin{aligned} ((\operatorname{ad} x)_2(e \otimes f))\, y &= (\operatorname{ad} xe \otimes f + e \otimes \operatorname{ad} xf)\, y \\ &= (f, y)[x, e] + ([x, f], y)e \\ &= (f, y)[x, e] - (f, [x, y])e \quad \text{by (4.2)} \\ &= ((\operatorname{ad} x)(e \otimes f) - (e \otimes f)(\operatorname{ad} x))\, y. \end{aligned}$$

Set

$$C := \sum_i e_i \cdot f_i \in U(L). \tag{4.3}$$

Thus C is the image of the element $\sum_i e_i \otimes f_i$ under the multiplication map $\mathbf{g} \otimes \mathbf{g} \mapsto U(\mathbf{g})$, and is independent of the choice of dual bases. Furthermore, C is annihilated by $\operatorname{ad} x$ acting on $U(\mathbf{g})$. In other words, it commutes with all elements of $\mathbf{g}$, and hence with all of $U(\mathbf{g})$; it is in the center of $U(\mathbf{g})$.

The C corresponding to the Killing form is called the **Casimir element**, its image in any representation is called the **Casimir operator**.

3. Suppose that $\rho : \mathbf{g} \to \operatorname{End} V$ is injective. The (image of the) central element corresponding to $(\ , \ )_\rho$ defines an element of $\operatorname{End} V$ denoted by C_ρ and

$$\begin{aligned} \operatorname{tr} C_\rho &= \operatorname{tr} \rho(\sum e_i f_i) \\ &= \operatorname{tr} \sum \rho(e_i)\rho(f_i) \\ &= \sum_i (e_i, f_i) \\ &= \dim \mathbf{g} \end{aligned}$$

With these preliminaries, we can state the main proposition:

Proposition 11 *Let $0 \to V \to W \to k \to 0$ be an exact sequence of $\mathbf{g}$ modules, where $\mathbf{g}$ is semi-simple, and the action of $\mathbf{g}$ on k is trivial (as it must be). Then this sequence splits, i.e. there is a line in W supplementary to V on which $\mathbf{g}$ acts trivially.*

The proof of the proposition and of the theorem is almost identical to the proof we gave above for the special case of $sl(2)$. We will need only one or two additional arguments. As in the case of $sl(2)$, the proposition is a special case of the theorem we want to prove. But we shall see that it is sufficient to prove the theorem.

Proof of proposition. It is enough to prove the proposition for the case that V is an irreducible module. Indeed, if V_1 is a submodule, then by induction on $\dim V$ we may assume the theorem is known for $0 \to V/V_1 \to W/V_1 \to k \to 0$ so that there is a one dimensional invariant subspace M in W/V_1 supplementary to V/V_1 on which the action is trivial. Let N be the inverse image of M in W. By another application of the proposition, this time to the sequence

$$0 \to V_1 \to N \to M \to 0$$

we find an invariant line, P, in N complementary to V_1. So $N = V_1 \oplus P$. Since $(W/V_1) = (V/V_1) \oplus M$ we must have $P \cap V = \{0\}$. But since $\dim W = \dim V + 1$, we must have $W = V \oplus P$. In other words P is a one dimensional subspace of W which is complementary to V.

Next we can reduce to proving the proposition for the case that $\mathbf{g}$ acts faithfully on V. Indeed, let $\mathbf{i}=$ the kernel of the action on V. For all $x \in \mathbf{g}$ we have, by hypothesis, $xW \subset V$, and for $x \in \mathbf{i}$ we have $xV = 0$. Hence $D\mathbf{i}$ acts trivially on W. But $\mathbf{i} = D\mathbf{i}$ since $\mathbf{i}$ is semi-simple. Hence $\mathbf{i}$ acts trivially on W and we may pass to $\mathbf{g}/\mathbf{i}$. This quotient is again semi-simple, since $\mathbf{i}$ is a sum of some of the simple ideals of $\mathbf{g}$.

So we are reduced to the case that V is irreducible and the action, ρ, of $\mathbf{g}$ on V is injective. Then we have an invariant element C_ρ whose image in $\operatorname{End} W$ must map $W \to V$ since every element of $\mathbf{g}$ does. (We may assume that $\mathbf{g} \neq 0$.) On the other hand, $C_\rho \neq 0$, indeed its trace is dim $\mathbf{g}$. The restriction of C_ρ to V can not have a non-trivial kernel, since this would be an invariant subspace. Hence the restriction of C_ρ to V is an isomorphism. Hence ker $C_\rho : W \to V$ is an invariant line supplementary to V. We have proved the proposition.

Proof of theorem from proposition. Let $0 \to E' \to E$ be an exact sequence of $\mathbf{g}$ modules, and we may assume that $E' \neq 0$. We want to find an invariant complement to E' in E. Define W to be the subspace of $\mathrm{Hom}_k(E, E')$ whose restriction to E' is a scalar times the identity, and let $V \subset W$ be the subspace consisting of those linear transformations whose restrictions to E' is zero. Each of these is a submodule of $\mathrm{End}(E)$. We get a sequence

$$0 \to V \to W \to k \to 0$$

and hence a complementary line of invariant elements in W. In particular, we can find an element, T which is invariant, maps $E \to E'$, and whose restriction to E' is non-zero. Then ker T is an invariant complementary subspace. QED

As an illustration of construction of the Casimir operator consider $\mathbf{g} = sl(2)$ with

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{tr}(\operatorname{ad} h)^2 &= 8 \\ \operatorname{tr}(\operatorname{ad} e)(\operatorname{ad} f) &= 4 \end{aligned}$$

so the dual basis to the basis h, e, f is $h/8, f/4, e/4$, or, if we divide the metric by 4, the dual basis is $h/2, f, e$ and so the Casimir operator C is

$$\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe.$$

This coincides with the C that we used in Chapter II.