

# Lie Algebras

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# Chapter 11

## The center of $U(\mathfrak{g})$ .

The purpose of this chapter is to study the center of the universal enveloping algebra of a semi-simple Lie algebra  $\mathfrak{g}$ . We have already made use of the (second order) Casimir element.

### 11.1 The Harish-Chandra isomorphism.

Let us return to the situation and notation of Section ???. We have the monomial basis

$$f_1^{i_1} \dots f_m^{i_m} h_1^{j_1} \dots h_\ell^{j_\ell} e_1^{k_1} \dots e_m^{k_m}$$

of  $U(\mathfrak{g})$ , the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})n_+ + n_-U(\mathfrak{g}))$$

and the projection

$$\gamma : U(\mathfrak{g}) \rightarrow U(\mathfrak{h})$$

onto the first factor of this decomposition. This projection restricts to a projection, also denoted by  $\gamma$

$$\gamma : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h}).$$

The projection  $\gamma : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})$  is a bit awkward. However Harish-Chandra showed that by making a slight modification in  $\gamma$  we get an isomorphism of  $Z(\mathfrak{g})$  onto the ring of Weyl group invariants of  $U(\mathfrak{h}) = S(\mathfrak{h})$ . Harish-Chandra's modification is as follows: As usual, let

$$\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha.$$

Recall that for each  $i$ , the reflection  $s_i$  sends  $\alpha_i \mapsto -\alpha_i$  and permutes the remaining positive roots. Hence

$$s_i \rho = \rho - \alpha_i$$

But by definition,

$$s_i \rho = \rho - \langle \rho, \alpha_i \rangle \alpha_i$$

and so

$$\langle \rho, \alpha_i \rangle = 1$$

for all  $i = 1, \dots, m$ . So

$$\rho = \omega_1 + \dots + \omega_m,$$

i.e.  $\rho$  is the sum of the fundamental weights.

### 11.1.1 Statement

Define

$$\sigma : \mathfrak{h} \rightarrow U(\mathfrak{h}), \quad \sigma(h) = h - \rho(h)1. \quad (11.1)$$

This is a linear map from  $\mathfrak{h}$  to the commutative algebra  $U(\mathfrak{h})$  and hence, by the universal property of  $U(\mathfrak{h})$ , extends to an algebra homomorphism of  $U(\mathfrak{h})$  to itself which is clearly an automorphism. We will continue to denote this automorphism by  $\sigma$ . Set

$$\gamma_H := \sigma \circ \gamma.$$

Then Harish-Chandra's theorem asserts that

$$\gamma_H : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})^W$$

and is an isomorphism of algebras.

### 11.1.2 Example of $sl(2)$ .

To see what is going on let's look at this simplest case. The Casimir of degree two is

$$\frac{1}{2}h^2 + ef + fe,$$

as can be seen from the definition. Or it can be checked directly that this element is in the center. It is not written in our standard form which requires that the  $f$  be on the left. But  $ef = fe + [e, f] = fe + h$ . So the way of writing this element in terms of the above basis is

$$\frac{1}{2}h^2 + h + 2fe,$$

and applying  $\gamma$  to it yields

$$\frac{1}{2}h^2 + h.$$

There is only one positive root and its value on  $h$  is 2, so  $\rho(h) = 1$ . Thus  $\sigma$  sends  $\frac{1}{2}h^2 + h$  into

$$\frac{1}{2}(h-1)^2 + h - 1 = \frac{1}{2}h^2 - h + \frac{1}{2} + h - 1 = \frac{1}{2}(h^2 - 1).$$

The Weyl group in this case is just the identity together with the reflection  $h \mapsto -h$ , and the expression on the right is clearly Weyl group invariant.

### 11.1.3 Using Verma modules to prove that $\gamma_H : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})^W$ .

Any  $\mu \in \mathfrak{h}^*$  can be thought of as a linear map of  $\mathfrak{h}$  into the commutative algebra,  $\mathbf{C}$  and hence extends to an algebra homomorphism of  $U(\mathfrak{h})$  to  $\mathbf{C}$ . If we regard an element of  $U(\mathfrak{h}) = S(\mathfrak{h})$  as a polynomial function on  $\mathfrak{h}^*$ , then this homomorphism is just evaluation at  $\mu$ .

From our definitions,

$$(\lambda - \rho)\gamma(z) = \gamma_H(z)(\lambda) \quad \forall z \in Z(\mathfrak{g}).$$

Let us consider the Verma module  $\text{Verm}(\lambda - \rho)$  where we denote its highest weight vector by  $v_+$ . For any  $z \in Z(\mathfrak{g})$ , we have  $hzv_+ = zhv_+ = (\lambda - \rho)(h)zv_+$  and  $e_izv_+ = ze_iv_+ = 0$ . So  $zv_+$  is a highest weight vector with weight  $\lambda - \rho$  and hence must be some multiple of  $v_+$ . Call this multiple  $\varphi_\lambda(z)$ . Then

$$zf_1^{i_1} \cdots f_m^{i_m} v_+ = f_1^{i_1} \cdots f_m^{i_m} \varphi_\lambda(z) v_+,$$

so  $z$  acts as scalar multiplication by  $\varphi_\lambda(z)$  on all of  $\text{Verm}(\lambda - \rho)$ . To see what this scalar is, observe that since  $z - \gamma(z) \in U(\mathfrak{g})n_+$ , we see that  $z$  has the same action on  $v_+$  as does  $\gamma(z)$  which is multiplication by  $(\lambda - \rho)(\gamma(z)) = \lambda(\gamma_H(z))$ . In other words,

$$\varphi_\lambda(z) = \lambda(\gamma_H(z)) = \gamma_H(z)(\lambda).$$

Notice that in this argument we only used the fact that  $\text{Verm}(\lambda - \rho)$  is a cyclic highest weight module: If  $V$  is any cyclic highest weight module with highest weight  $\mu - \rho$  then  $z$  acts as multiplication by  $\varphi_\mu(z) = \mu(\gamma_H(z)) = \gamma_H(z)(\mu)$ . We will use this observation in a moment.

Getting back to the case of  $\text{Verm}(\lambda - \rho)$ , for a simple root  $\alpha = \alpha_i$  let  $m = m_i := \langle \lambda, \alpha_i \rangle$  and suppose that  $m$  is an integer. The element

$$f_i^m v_+ \in \text{Verm}(\lambda - \rho)_\mu$$

where

$$\mu = \lambda - \rho - m\alpha = s_i\lambda - \rho.$$

Now from the point of view of the  $sl(2)$  generated by  $e_i, f_i$ , the vector  $v_+$  is a maximal weight vector with weight  $m - 1$ . Hence  $e_i f_i^m v_+ = 0$ . Since  $[e_j, f_i] = 0, i \neq j$  we have  $e_j f_i^m v_+ = 0$  as well. So  $f_i^m v_+ \neq 0$  is a maximal weight vector with weight  $s_i\lambda - \rho$ . Call the highest weight module it generates,  $M$ . Then from  $M$  we see that

$$\varphi_{s_i\lambda}(z) = \varphi_\lambda(z).$$

Hence we have proved that

$$(\gamma_H(z))(w\lambda) = (\gamma_H(z))(\lambda) \quad \forall w \in W$$

if  $\lambda$  is dominant integral. But two polynomials which agree on all dominant integral weights must agree everywhere. We have shown that the image of  $\gamma$  lies in  $S(\mathfrak{h})^W$ .

Furthermore, we have

$$z_1 z_2 - \gamma(z_1)\gamma(z_2) = z_1(z_2 - \gamma(z_2)) + \gamma(z_2)(z_1 - \gamma(z_2)) \in U(\mathfrak{g})n_+.$$

So

$$\gamma(z_1 z_2) = \gamma(z_1)\gamma(z_2).$$

This says that  $\gamma$  is an algebra homomorphism, and since  $\gamma_H = \sigma \circ \gamma$  where  $\sigma$  is an automorphism, we conclude that  $\gamma_H$  is a homomorphism of algebras.

Equally well, we can argue directly from the fact that  $z \in Z(\mathfrak{g})$  acts as multiplication by  $\varphi_\lambda(z) = \gamma_H(z)(\lambda)$  on  $\text{Verm}(\lambda - \rho)$  that  $\gamma_H$  is an algebra homomorphism.

#### 11.1.4 Outline of proof of bijectivity.

To complete the proof of Harish-Chandra's theorem we must prove that  $\gamma_H$  is a bijection. For this we will introduce some intermediate spaces and homomorphisms. Let  $Y(\mathfrak{g}) := S(\mathfrak{g})^{\mathfrak{g}}$  denote the subspace fixed by the adjoint representation (extended to the symmetric algebra by derivations). This is a subalgebra, and the filtration on  $S(\mathfrak{g})$  induces a filtration on  $Y(\mathfrak{g})$ . We shall produce an isomorphism

$$f : Y(\mathfrak{g}) \rightarrow S(\mathfrak{h})^W.$$

We also have a linear space isomorphism of  $U(\mathfrak{g}) \rightarrow S(\mathfrak{g})$  given by the symmetric embedding of elements of  $S^k(\mathfrak{g})$  into  $U(\mathfrak{g})$ , and let  $s$  be the restriction of this to  $Z(\mathfrak{g})$ . We shall see that  $s : Z(\mathfrak{g}) \rightarrow Y(\mathfrak{g})$  is an isomorphism. Finally, define

$$S_k(\mathfrak{g}) = S^0(\mathfrak{g}) \oplus S^1(\mathfrak{g}) \oplus \cdots \oplus S^k(\mathfrak{g})$$

so as to get a filtration on  $S(\mathfrak{g})$ . This induces a filtration on  $S(\mathfrak{h}) \subset S(\mathfrak{g})$ . We shall show that for any  $z \in U_k(\mathfrak{g}) \cap Z(\mathfrak{g})$  we have

$$(f \circ s)(z) \equiv \gamma_H(z) \pmod{S_{k-1}(\mathfrak{g})}.$$

This proves inductively that  $\gamma_H$  is an isomorphism since  $s$  and  $f$  are. Also, since  $\sigma$  does not change the highest order component of an element in  $S(\mathfrak{h})$ , it will be enough to prove that for  $z \in U_k(\mathfrak{g}) \cap Z(\mathfrak{g})$  we have

$$(f \circ s)(z) \equiv \gamma(z) \pmod{S_{k-1}(\mathfrak{g})}. \tag{11.2}$$

We now proceed to the details.

### 11.1.5 Restriction from $S(\mathfrak{g}^*)^{\mathfrak{g}}$ to $S(\mathfrak{h}^*)^W$ .

We first discuss polynomials on  $\mathfrak{g}$  — that is elements of  $S(\mathfrak{g}^*)$ . Let  $\tau$  be a finite dimensional representation of  $\mathfrak{g}$ , and consider the symmetric function  $F$  of degree  $k$  on  $\mathfrak{g}$  given by

$$(X_1, \dots, X_k) \mapsto \sum \operatorname{tr} (\tau(X_{\pi_1}) \cdots \tau(X_{\pi_k}))$$

where the sum is over all permutations. For any  $Y \in \mathfrak{g}$ , by definition,

$$YF(X_1, \dots, X_k) = F([Y, X_1], X_2, \dots, X_k) + \cdots + F(X_1, \dots, X_{k-1}, [Y, X_k]).$$

Applied to the above

$$F(X_1, \dots, X_k) = \operatorname{tr} \tau(X_1) \cdots \tau(X_k)$$

we get

$$\begin{aligned} & \operatorname{tr} \tau(Y) \tau(X_1) \cdots \tau(X_k) - \operatorname{tr} \tau(X_1) \tau(Y) \cdots \tau(X_k) \\ & \quad + \operatorname{tr} \tau(X_1) \tau(X) \tau(X_2) \cdots \tau(X_k) - \cdots \\ & = \operatorname{tr} \tau(Y) \tau(X_1) \cdots \tau(X_k) - \operatorname{tr} \tau(X_1) \cdots \tau(X_k) \tau(Y) = 0. \end{aligned}$$

In other words, the function

$$X \mapsto \operatorname{tr} \tau(X)^n$$

belongs to  $S(\mathfrak{g}^*)^{\mathfrak{g}}$ . Now since  $\mathfrak{h}$  is a subspace of  $\mathfrak{g}$ , the restriction map induces a homomorphism,

$$r : S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow S(\mathfrak{h}^*)^W.$$

If  $F \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ , then, as a function on  $\mathfrak{g}$  it is invariant under the automorphism  $\tau_i := (\exp \operatorname{ad} e_i)(\exp \operatorname{ad} -f_i)(\exp \operatorname{ad} e_i)$  and hence

$$r : S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow S(\mathfrak{h}^*)^W.$$

If  $F \in S(\mathfrak{g}^*)^{\mathfrak{g}}$  is such that its restriction to  $\mathfrak{h}$  vanishes, then its value at any element which is conjugate to an element of  $\mathfrak{h}$  (under  $\mathcal{E}(\mathfrak{g})$  the subgroup of automorphisms of  $\mathfrak{g}$  generated by the  $\tau_i$ ) must also vanish. But these include a dense set in  $\mathfrak{g}$ , so  $F$ , being continuous, must vanish everywhere. So the restriction of  $r$  to  $S(\mathfrak{g}^*)^{\mathfrak{g}}$  is injective.

To prove that it is surjective, it is enough to prove that  $S(\mathfrak{h}^*)^W$  is spanned by all functions of the form  $X \mapsto \operatorname{tr} \tau(X)^k$  as  $\tau$  ranges over all finite dimensional representations and  $k$  ranges over all non-negative integers. Now the powers of any set of spanning elements of  $\mathfrak{h}^*$  span  $S(\mathfrak{h}^*)$ . So we can write any element of  $S(\mathfrak{h}^*)^W$  as a linear combination of the  $\mathcal{A}\lambda^k$  where  $\mathcal{A}$  denotes averaging over  $W$ . So it is enough to show that for any dominant weight  $\lambda$ , we can express  $\lambda^k$  in terms of  $\operatorname{tr} \tau^k$ . Let  $E_\lambda$  denote the (finite) set of all dominant weights  $\prec \lambda$ . Let  $\tau$  denote the finite dimensional representation with highest weight  $\lambda$ . Then  $\operatorname{tr} \tau(X)^k - \mathcal{A}\lambda^k(X)$  is a combination of  $\mathcal{A}\mu(X)^k$  where  $\mu \in E_\lambda$ . Hence by induction on the finite set  $E_\lambda$  we get the desired result. In short, we have proved that

$$r : S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow S(\mathfrak{h}^*)^W$$

is bijective.

### 11.1.6 From $S(\mathfrak{g})^{\mathfrak{g}}$ to $S(\mathfrak{h})^W$ .

Now we transfer all this information from  $S(\mathfrak{g}^*)$  to  $S(\mathfrak{g})$ : Use the Killing form to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  and hence get an isomorphism

$$\alpha : S(\mathfrak{g}) \rightarrow S(\mathfrak{g}^*).$$

Similarly, let

$$\beta : S(\mathfrak{h}) \rightarrow S(\mathfrak{h}^*)$$

be the isomorphism induced by the restriction of the Killing form to  $\mathfrak{h}$ , which we know to be non-degenerate. Notice that  $\beta$  commutes with the action of the Weyl group. We can write

$$S(\mathfrak{g}) = S(\mathfrak{h}) + J$$

where  $J$  is the ideal in  $S(\mathfrak{g})$  generated by  $n_+$  and  $n_-$ . Let

$$j : S(\mathfrak{g}) \rightarrow S(\mathfrak{h})$$

denote the homomorphism obtained by quotienting out by this ideal. We claim that the diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\alpha} & S(\mathfrak{g}^*) \\ j \downarrow & & \downarrow r \\ S(\mathfrak{h})' & \xrightarrow{\beta} & S(\mathfrak{h}^*) \end{array}$$

commutes. Indeed, since all maps are algebra homomorphisms, it is enough to check this on generators, that is on elements of  $\mathfrak{g}$ . If  $X \in \mathfrak{g}$ , then

$$\langle h, r\alpha(X) \rangle = \langle h, \alpha(X) \rangle = \langle h, X \rangle$$

where the scalar product on the right is the Killing form. But since  $h$  is orthogonal under the Killing form to  $n_+ + n_-$ , we have

$$(h, X) = (h, jX) = \langle h, \beta(jX) \rangle. \quad QED$$

Upon restriction to the  $\mathfrak{g}$  and  $W$ -invariants, we have proved that the right hand column is a bijection, and hence so is the left hand column, since  $\beta$  is a  $W$ -module morphism. Recalling that we have defined  $Y(\mathfrak{g}) := S(\mathfrak{g})^{\mathfrak{g}}$ , we have shown that the restriction of  $j$  to  $Y(\mathfrak{g})$  is an isomorphism, call it  $f$ :

$$f : Y(\mathfrak{g}) \rightarrow S(\mathfrak{h})^W.$$

### 11.1.7 Completion of the proof.

Now we have a canonical linear bijection of  $S(\mathfrak{g})$  with  $U(\mathfrak{g})$  which maps

$$S(\mathfrak{g}) \ni X_1 \cdots X_r \mapsto \frac{1}{r!} \sum_{\pi \in \Sigma_r} X_{\pi_1} \cdots X_{\pi_r},$$

where the multiplication on the left is in  $S(\mathfrak{g})$  and the multiplication on the right is in  $U(\mathfrak{g})$  and where  $\Sigma_r$  denotes the permutation group on  $r$  letters. This map is a  $\mathfrak{g}$  module morphism. In particular this map induces a bijection

$$s : Z(\mathfrak{g}) \rightarrow Y(\mathfrak{g}).$$

Our proof will be complete once we prove (11.2). This is a calculation: write

$$u_{AJB} := f^A h^J e^B$$

for our usual monomial basis, where the multiplication on the right is in the universal enveloping algebra. Let us also write

$$p_{AJB} := f^A h^J e^B = f_\alpha^{i_\alpha} \cdots h_1^{j_1} \cdots h_\ell^{j_\ell} \cdots e_\gamma^{k_\gamma} \in S(\mathfrak{g})$$

where now the powers and multiplication are in  $S(\mathfrak{g})$ . The image of  $u_{AJB}$  under the canonical isomorphism with of  $U(\mathfrak{g})$  with  $S(\mathfrak{g})$  will not be  $p_{AJB}$  in general, but will differ from  $p_{AJB}$  by a term of lower filtration degree. Now the projection  $\gamma : U(\mathfrak{g}) \rightarrow U(\mathfrak{h})$  coming from the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (n_- U(\mathfrak{g}) + U(\mathfrak{g}) n_+)$$

sends  $u_{AJB} \mapsto 0$  unless  $A = 0 = B$  and is the identity on  $u_{0J0}$ . Similarly,

$$j(p_{AJB}) = 0 \quad \text{unless } A = 0 = B$$

and

$$j(p_{0J0}) = p_{0J0} = h^J.$$

These two facts complete the proof of (11.2). *QED*