Math 128, Problem set # 9

Signature formulas.

April 22, 2004 due April 29

In combinatorics, one frequently counts a point lying in the interior of a polytope with weight one, a point which lies on the interior of an (n-1)-dimensional face with weight 1/2, a point which lies on the interior of an (n-2) dimensional face with weight 1/4 etc.

Let $2m$ denote the number of roots of a Lie algebra. Multiplying by a power of two to clear denominators, this suggest the following modification of the Kostant partition function: For a given choice Φ^+ of positive roots, and for any weight μ let P_μ denote the polytope in $\mathbb{R}^m$ given by

$$P_\mu := \left\{ (r_\alpha) \in \mathbb{R}^m_{\geq 0} | \sum r_\alpha \alpha = \mu \right\}.$$

Let Q_μ denote the set of integer points in P_μ. So

$$Q_\mu := \left\{ (k_\alpha) \in \mathbb{Z}^m_{\geq 0} | \sum k_\alpha \alpha = \mu \right\}.$$

We count the points of Q_μ weighting an interior point (i.e. a point where all the $k_\alpha > 0$) by 2^m, a point where exactly one $k_\alpha = 0$ with all others strictly positive by 2^{m-1} etc. In other words, we define

$$K_2(\mu) := \sum_{(k_\alpha) \in Q_\mu} 2^{\#\{k_\alpha > 0\}}.$$

This problem set presents some as yet unpublished results of Guillemin and Rassart. The exposition follows the treatment in Etienne Rassart's thesis (unpublished) MIT 2004.

1. The Taylor expansion of $\frac{1+(q-1)z}{1-z}$ is $1 + qz + qz^2 + qz^3 \cdots$. (This follows from multiplying the geometric series by the polynomial $1 + (q-1)z$.) Take $q = 2$ and conclude from this that

$$\prod_{\alpha \in \Phi^+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} = \sum_\mu K_2(\mu) e^{-\mu}.$$

Recall that for any weight μ we defined

$$A_\mu := \sum_{w \in W} (-1)^w e^{w\mu}$$

where W is the Weyl group and $(-1)^w$ is short for $(-1)^{\ell(w)}$ where $\ell(w)$ is the length of w in terms of (a choice of) simple reflections. In other words,

$$(-1)^w = \det w.$$

Let

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

2. Show that

$$A_\rho = \prod_{\alpha \in \Phi^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) = e^\rho \prod_{\alpha \in \Phi^+} \left(1 - e^{-\alpha}\right)$$

(we actually did this in class) and that

$$A_{2\rho} = \prod_{\alpha \in \Phi^+} \left(e^{\alpha} - e^{-\alpha}\right) = e^{2\rho} \prod_{\alpha \in \Phi^+} \left(1 - e^{-2\rho}\right).$$

[Hint: use the fact that $\ell(w) = n(w)$, the number of positive roots made negative by w.]

For any dominant weight λ let $\chi(\lambda)$ denote the character of $\mathrm{Irr}(\lambda)$, the irreducible representation associated to λ.

3. Show that

$$\chi(\rho) = e^\rho \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha}).$$

A weight is called **strictly dominant** if it lies in the interior of the Weyl chamber. Equivalently, this condition says that λ is strictly dominant if and only if $\lambda - \rho$ is dominant. For any strictly dominant weight λ let $W(\lambda)$ denote the representation

$$W(\lambda) := \mathrm{Irr}(\lambda - \rho) \otimes \mathrm{Irr}(\rho).$$

Let $\tilde{\chi}(\lambda)$ denote the character of $W(\lambda)$ so that

$$\tilde{\chi}(\lambda) = \chi(\lambda - \rho)\chi(\rho).$$

4. Show that

$$\tilde{\chi}(\lambda) = \sum_{w \in W} (-1)^w e^{w\lambda} \prod_{\alpha \in \Phi^+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} = \sum_{\mu} \sum_{w \in W} (-1)^w K_2(\mu) e^{w\lambda - \mu}.$$

5. Let $\tilde{m}_\lambda(\nu) := \dim W(\lambda)_\nu$. In other words, $\tilde{m}_\lambda(\nu)$ is the multiplicity with which the weight ν occurs in $W(\lambda)$. Show that

$$\tilde{m}_\lambda(\nu) = \sum_{w \in W} K_2(w\lambda - \nu).$$

[This result was obtained by geometric means in the paper "Signature Quantization" by Guillemin-Sternberg-Weitsman (to appear (2004)) in the Journal of Differential Geometry.]

6. Let λ and μ be strictly dominant weights. Show that $W(\lambda) \otimes W(\mu)$ is a direct sum (with multiplicities) of representations of the form $W(\nu)$. In other words show that there exist integers $\tilde{N}^\nu_{\lambda\mu}$ such that

$$W(\lambda) \otimes W(\mu) = \bigoplus_\nu \tilde{N}^\nu_{\lambda\mu} W(\nu).$$

[Hint:Write

$$W(\lambda) \otimes W(\mu) = (\mathrm{Irr}(\lambda - \rho) \otimes \mathrm{Irr}(\rho) \otimes \mathrm{Irr}(\mu - \rho)) \otimes \mathrm{Irr}(\rho)$$

decompose $\mathrm{Irr}(\lambda - \rho) \otimes \mathrm{Irr}(\rho) \otimes \mathrm{Irr}(\mu - \rho)$ into irreducibles and write each component as $\mathrm{Irr}(\nu - \rho)$ where ν is strictly dominant.]

The purpose of the next three problems is to determine the integers $\tilde{N}^\nu_{\lambda\mu}$.

7. Show that

$$\sum_{w \in W} (-1)^w e^{w\lambda} \prod_{\alpha \in \Phi^+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \tilde{\chi}(\mu) = \sum_\nu \tilde{N}^\nu_{\lambda\mu} \sum_{\tau \in W} (-1)^\tau e^{\tau\nu} \prod_{\alpha \in \Phi^+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}}.$$

8. Conclude that

$$\sum_\beta \sum_{w \in W} \sum_{u \in W} (-1)^{uw} K_2(u\mu - \beta) e^{w\lambda + \beta} = \sum_\nu \sum_{\tau \in W} (-1)^\tau \tilde{N}^\nu_{\lambda\mu} e^{\tau(\nu)}$$

where the outer sum on the right is over all strictly dominant roots.

Substituting $\gamma = w\lambda + \beta$ in the sum on the left hand side of this equation and $\gamma = \tau(\nu)$ in the sum on the right hand side of this equation gives

$$\sum_\gamma \sum_{w \in W} \sum_{u \in W} (-1)^{uw} K_2(u(\mu) + w(\lambda) - \gamma) e^\gamma = \sum_\gamma (-1)^\tau \tilde{N}^{\tau_\gamma^{-1}(\gamma}_{\lambda\mu} e^\gamma$$

where the sum on the right is over all γ conjugate to a strictly dominant root, i.e. over all γ in the interior of some Weyl chamber, and the τ_γ is the unique element of the Weyl group which moves γ into the positive Weyl chamber.

9. Conclude that

$$\tilde{N}_{\lambda\mu}^{\nu} = \sum_{w \in W} \sum_{u \in W} (-1)^{uw} K_2(w\lambda + u\mu - \nu).$$

Notice that we have obtained variants of the Kostant and Steinberg formulas. We have replaced $\mathrm{Irr}(\lambda)$ by $W(\lambda)$ and the Kostant partition function by K_2. The results - Problems **5** and **9** - do not involve any "ρ shiflts" as occur in the usual Kostant and Steinberg formulas.