Math 128 - Problem set # 7

Clifford algebras.

April 8, 2004 due April 22

Because of disruptions due the the Jewish holidays and surgery on my knee there will be no class next week (April 13 or 15). (Doctor's orders.) So I probably will not cover the basics of representation theory by the end of this week. One of the basic representations of the orthogonal group is the spin representation. This, like all representations we will be studying, is a representation on a complex finite dimensional vector space and involves the complex Clifford algebras. However in this problem set I want to study the real Clifford algebras.

When Dirac wrote his famous paper in 1928 on relativistic wave equations (in which he gave a stunning explanation for the magnetic moment of the electron and which led to the prediction of the existence of anti-particles) he created the whole subject - in particular the famous Dirac equation - completely on his own. However some of the algebraic constructs which we now view as underlying this equation had been developed by William Clifford (1845 - 1879) and are today known as Clifford algebras. They are essential for understanding the generalization of the Dirac equation to spaces other than our four dimensional space-time. The purpose of this exercise set is to explain the ideas involved.

As usual, there will be some choices that I make as to conventions. I will follow the conventions of the algebraists rather than the geometers in the fundamental definition.

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1 Review: The tensor algebra.

Let V be a vectors space. (In this section the ground field can be arbitrary. But you might as well think of V as a vector space over the real numbers as we will eventually specialize to this case.)

Consider the the following "universal problem": Find an associative algebra U and a linear map $i : V \to U$ such that for any associative algebra A and any linear map $f : V \to A$ there exists a unique *algebra homomorphism* $\phi : U \to A$ such that

- $$\phi(\mathbf{1}_U) = \mathbf{1}_A$$
  where $\mathbf{1}_U$ denotes the identity element of U and $\mathbf{1}_A$ denotes the identity element of A, and

- $$\phi(i(v)) = f(v)$$
  for all $v \in V$.

As is always true in such problems if a solution exists then it is unique up to a unique homomorphism. Indeed, if (i_1, U_1) and (i_1, U_2) were two such solutions, then the conditions above with (i_2, U_2) playing the role of (f, A) say that there exists a unique homomorphism $\phi : U_1 \to U_2$ such that $i_2 = \phi \circ i_1$. Interchanging the roles implies that there is a unique homomorphism $\psi : U_2 \to U_1$ such that $i_1 = \psi \circ i_2$. Then $\psi \circ \phi : U_1 \to U_1$ is a homomorphism satisfying

$$(\psi \circ \phi) \circ i_1 = i_1.$$

But the identity map id:$U_1 \to U_1$ satisfies id$\circ i_1 = i_1$. Hence the uniqueness implies that

$$\psi \circ \phi = \text{id}.$$

So ϕ is an isomorphism with inverse ψ and similarly ψ is an isomorphism with inverse ϕ.

So the problem is to construct one such solution to our universal problem. Consider the direct sum

$$T(V) := \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

So an element of $T(V)$ is a finite sum of elements of

$$T^r(V) := V \otimes V \otimes \cdots \otimes V \qquad r \text{ factors}$$

and every element of $T^r(V)$ is a finite sum of elements of the form

$$v_1 \otimes \cdots \otimes v_r, \qquad v_i \in V.$$

The space $T^0(V)$ is just $\mathbb{R}$ (or more generally the ground field over which V is defined). There is a unique bilinear map

$$m = m_{k,\ell} : T^p(V) \otimes T^q(V) \to T^{k+\ell}(V)$$

determined by

$$(v_1 \otimes \cdots \otimes v_k) \otimes (w_1 \otimes \cdots \otimes w_\ell) \mapsto v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$$

(just remove the parentheses).

So we know how to multiply $a_k \in T^K(V)$ with $b_\ell \in T^\ell(V)$. Then extend this multiplication (by the distributive law) so we know how to multiply $a_0 + \cdots + a_r$ with $b_0 + \cdots + b_s$. This makes $T(V)$ into an associative algebra with the identity element being $1 \in \mathbb{R}$.

Define the map $i : V \to T(V)$ to be the map which identifies V as the $r = 1$ piece in $T(V)$. The image of i, that is the $r = 1$ piece $T^1(V)$ sitting inside $T(V)$ generates $T(V)$ as an algebra in the sense that every element of $T(V)$ can be written as a sum of an element of $\mathbb{R}$ and of products of elements of $T^1(V) = V$.

This has the following consequence: Suppose that we are given a linear map $f : V \to A$ where A is an associative algebra with unit.Then $f(v_1)f(v_2)$ is bilinear in v_1 and v_2 and so defines a map

$$\phi_2 : V \otimes V \to A$$

determined by

$$\phi_2(v_1 \otimes v_2) = f(v_1)f(v_2).$$

Similarly, there is a well defined map

$$\phi_r : T^r(V) \to A$$

determined by

$$\phi_r(v_1 \otimes \cdots v_r) = f(v_1) \cdots f(v_r)$$

the multiplication on the right taking place in A. (For $r = 0$ we take $\phi(a_0) = a_0 \mathbf{1}_A$.)

Then define

$$\phi(a_0 + \cdots + a_r) = \phi_0(a_0) + \phi_1(a_1) + \phi_2(a_2) + \cdots + \phi_r(a_r)$$

(where, of course, $\phi_1(a_1) = f(a_1)$). It is easy to check that ϕ is an algebra homomorphism and that it is uniquely determined by $\phi_1 = f$ since $V = T^1(V)$ generates $T(V)$ as an algebra.

The kernel of ϕ, that is, the set of elements of $T(V)$ satisfying $\phi(t) = 0$, form a two sided ideal in the sense that

$$a \in T(V), \quad b \in \ker(\phi) \Rightarrow ab \in \ker(\phi) \quad \text{and} \quad ba \in \ker(\phi).$$

Conversely, if I is any two sided ideal in $T(V)$ meaning that I is a subspace and

$$a \in T(V),\ b \in I\ \Rightarrow ab \in I \quad \text{and} \quad ba \in I$$

then we can make the quotient space $T(V)/I$ into an algebra by defining the multiplication

$$[a/I] \cdot [a'/I] := [aa'/I]$$

where $[a/I]$ denotes the equivalence class of a modulo I.

2 Definition and basic properties of Clifford algebras

2.1 Definition.

Let $\mathbf{V}$ be a vector space with a symmetric bilinear form $(\ ,\ )$. The **Clifford algebra** associated to this data is the algebra

$$C(\mathbf{V}) := T(\mathbf{V})/I$$

where $T(\mathbf{V})$ denotes the tensor algebra

$$T(\mathbf{V}) = k \oplus \mathbf{V} \oplus (\mathbf{V} \otimes \mathbf{V}) \oplus \cdots$$

and where I denotes the ideal in $T(\mathbf{V})$ generated by all elements of the form

$$y_1y_2 + y_2y_1 - 2(y_1, y_2)\mathbf{1}, \quad y_1,\ y_2\ \in \mathbf{V}$$

and $\mathbf{1}$ is the unit element of the tensor algebra. The space $\mathbf{V}$ injects as a subspace of $C(\mathbf{V})$ and generates $C(\mathbf{V})$ as an algebra.

A linear map f of $\mathbf{V}$ to an associative algebra A with unit 1_A is called a **Clifford map** if

$$f(y_1)f(y_2) + f(y_2)f(y_1) = 2(y_1, y_2)1_A, \quad \forall y_1,\ y_2\ \in \mathbf{V}$$

or what amounts to the same thing (by polarization since we are not over a field of characteristic 2) if

$$f(y)^2 = (y,y)1_A \quad \forall\, y \in \mathbf{V}.$$

Any Clifford map gives rise to a unique algebra homomorphism of $C(\mathbf{V})$ to A whose restriction to $\mathbf{V}$ is f. The Clifford algebra is "universal" with respect to this property.

If the bilinear form is identically zero, then $C(\mathbf{V}) = \wedge\mathbf{V}$, the exterior algebra. But we will be interested in the opposite extreme, the case where the bilinear form is non-degenerate, i.e. gives what we may call a (not necessarily positive definite) scalar product when over the reals.

If we are over the real numbers, any scalar product has a signature (p,q) where p denotes the number of plus signs and q the number of minus signs in an "orthonormal" basis for the metric. For example, if we choose the metric on Minkowski space to be $t^2 - x^2 - y^2 - z^2$ the signature is $(1,3)$. The Clifford algebra for a non-degenerate metric of signature (p,q) will be denoted by $C(p,q)$.

2.2 Examples.

2.2.1 $C(1,0)$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ as an algebra.

The algebra $\mathbb{R}\oplus\mathbb{R}$ denotes the space of all pairs of real numbers with componentwise addition and multiplication

Here $\mathbf{V}$ is a one dimensional vector space so has an orthonormal basis consisting of a single vector e. We have $\mathbf{1}, e$ as a basis for the Clifford algebra where $\mathbf{1}$ is the identity element and $e^2 = \mathbf{1}$. The map

$$x\mathbf{1} + ye \mapsto (x-y, x+y)$$

gives the isomorphism. Indeed

$$(x\mathbf{1} + ye)(x'\mathbf{1} + y'e) = (xx' + yy')\mathbf{1} + (xy' + yx')e$$

in the Clifford algebra. But

$$xx' + yy' - xy' - yx' = (x-y)(x'-y')$$

and

$$xx' + yy' + xy' + yx' = (x+y)(x'+y')$$

showing that the map is an algebra homomorphism. The inverse map given by

$$(r,s) \mapsto \frac{1}{2}(r+s)\mathbf{1} + \frac{1}{2}(s-r)e$$

is equally easily checked to be an algebra homomorphism, so we do get the desired isomorphism. I emphasize that this isomorphism is an isomorphism of algebras, **not** of superalgebras when we introduce the gradation below.

2.2.2 $C(0,1)$ is isomorphic to the complex numbers.

Here $e^2 = -1$ so the isomorphism is

$$x\mathbf{1} + ye \mapsto x + iy.$$

Again this is only an isomorphism of algebras; the the superalgebra structure (to be discussed later on) is lost.

2.2.3 $C(2,0)$ is isomorphic to the algebra of all two by two matrices over the real numbers.

Consider the map f of $\mathbb{R}^2$ to real two by two matrices given by

$$f: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x & y \\ y & -x \end{pmatrix}.$$

Squaring the right hand side gives

$$\begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix} = (x^2+y^2)\mathbf{1}$$

where $\mathbf{1}$ is the identity matrix. So f is a Clifford map and hence determines a homomorphism ϕ from $C(2,0)$ to the algebra of all real two by two matrices.

Let

$$e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be the standard orthonormal basis of $\mathbb{R}^2$. Then as a vector space our Clifford algebra $C(2,0)$ has basis

$$\mathbf{1}, e_1, e_2, e_1 \cdot e_2.$$

These elements are clearly linearly independent, and any other element can be reduced to a linear combination of these four using the Clifford identities. For example,

$$e_1 \cdot e_2 \cdot e_1 = -e_1 e_1 e_2$$

since $e_1e_2 + e_2e_1 = 0$. But $e_1^2 = \mathbf{1}$ so $e_1 \cdot e_2 \cdot e_1 = -e_2$. Similarly any product of e_1's and e_2's in any order can be rearranged (at the possible cost of a minus sign) to a product where all the e_1's come first and then the e_2' s. The end result will be one of the four elements $\mathbf{1}, e_1, e_2, e_1 \cdot e_2$.

The image of $\mathbf{1}$ under our homomorphism is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so

$$e_1 \cdot e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

form a basis for the space of all two by two matrices we see that ϕ is an isomorphism.

We will denote the algebra of $k \times k$ matrices over the real numbers by $\mathbb{R}(k)$. So we have shown that $C(2,0)$ is isomorphic to $\mathbb{R}(2)$.

We will denote the algebra of $k \times k$ matrices over the complex numbers by $\mathbb{C}(k)$ and will denote the algebra of $k \times k$ matrices over the quaternians by $\mathbb{H}(k)$.

2.2.4 $C(1,1)$ is isomorphic to the algebra of creation and annihilation operators for a fermion with one state which is isomorphic to $\mathbb{R}(2)$.

The algebra of creation and annihilation operators for a fermion with one state looks as follows: There is the identity operator $\mathbf{1}$, the creation operator ϵ, the annihilation operator ι. These last two are subject to the "anti-commutation relation"

$$\epsilon\iota + \iota\epsilon = \mathbf{1}$$

together with $\epsilon^2 = 0$ and $\iota^2 = 0$. So the map

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto u\epsilon + v\iota$$

is a Clifford map provided that we endow $\mathbb{R}^2$ with the quadratic form

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = uv$$

which clearly has signature $(1,1)$.

The "Pauli exclusion principle" for fermions says that (in our situation where there is only one fermion state) the space on which our creation and annihilation operators act is two dimensional, spanned by the vacuum and the one fermion state. If we take these as our basis then

$$\epsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \iota = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

and it is easy to check that this determines an isomorphism of $C(1,1)$ with $\mathbb{R}(2)$.

2.2.5 $C(0,2)$ is isomorphic to the quaternions.

Here we let i be the element of $\begin{pmatrix}1\\0\end{pmatrix}$ in the Clifford algebra so $i^2 = -1$ and let $j = \begin{pmatrix}0\\1\end{pmatrix}$ in the Clifford algebra so $j^1 = -1$ and

$$ij + ji = 0.$$

Let $k := ij$. Then

$$k^2 = ijij = -iijj = -1$$

and

$$ik + ki = -j - i^2 j = 0$$

and similarly $jk + kj = 0$. We get the formulas that Hamilton scratched onto the bridge walking into town from the observatory.

2.3 Chirality and the element γ.

Suppose we choose an orientation on our vector space of type (p, q) and we choose an oriented "orthonormal" basis

$$e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}$$

and define γ to be their product in the given order

$$\gamma := e_1 \cdot e_2 \cdots e_n \qquad (n = p + q).$$

(In Minkowski space this is what the physicists call γ^5 up to a factor of i.) This depends on the orientation - choosing the opposite orientation replaces γ by $-\gamma$. We have

$$\gamma^2 = (-1)^{\frac{n(n-1)}{2}+q}. \tag{1}$$

The $n(n-1)/2$ comes from moving e_1 past $e_2 \cdots e_n$ then e_2 past $e_3 \cdots e_n$ etc. The q comes from the fact that $e_{p+1}^2 = \cdots = e_{p+q}^2 = -1$. Also

$$\gamma e_i = (-1)^{n-i} e_1 \cdots e_i e_i \cdots e_n$$

while

$$e_i \gamma = (-1)^{i-1} e_1 \cdots e_i e_i \cdots e_n.$$

So $e_i \gamma = (-1)^{n-1} \gamma e_i$. Since the e_i form a basis of the vector space, we see that

$$v\gamma = (-1)^{n-1} \gamma v \tag{2}$$

for any vector in the vector space.

2.4 Bott periodicity.

This says that we have the following isomorphism as algebras, where the tensor product in the formulas below is taken in the sense of algebras, not the "correct" sense of superalgebras:

$$\begin{aligned} C(p,q)\otimes C(2,0) &= C(q+2,p)\\ C(p,q)\otimes C(1,1) &= C(p+1,q+1)\\ C(p,q)\otimes C(0,2) &= C(q,p+2). \end{aligned}$$

I will prove the first of these, the proof of the other two will be left as a problem for you. Let $\mathbf{V}$ be a vector space with an inner product of signature (p,q) and let $\mathbf{W}$ be a two dimensional vector space with a positive definite scalar product and orthonormal basis e_1, e_2. Let $\gamma = e_1 e_2$ be the "gamma" of $\mathbf{W}$. Consider the map

$$\psi : \mathbf{V}\oplus\mathbf{W} \to C(\mathbf{V})\otimes C(\mathbf{W}$$

given by

$$\psi(v) := v\otimes\gamma, \qquad v\in\mathbf{V}$$

and

$$\psi(w) := \mathbf{1}\otimes w, \qquad w\in\mathbf{W}.$$

We have

$$(v\otimes\gamma)(\mathbf{1}\otimes w) + (\mathbf{1}\otimes w)(v\otimes\gamma) = v\otimes(\gamma w + w\gamma) = 0.$$

so

$$\psi(v)\psi(w) + \psi(w)\psi(v) = 0.$$

Also

$$\psi(w)^2 = \mathbf{1}\otimes w^2 = \|w\|^2\mathbf{1}\otimes\mathbf{1} = \|w\|^2\mathbf{1}$$

where the last $\mathbf{1}$ denotes the identity element in the tensor product. and

$$\psi(v)^2 = (v\otimes\gamma)(v\otimes\gamma) = v^2\otimes\gamma^2 = v^2\otimes(-\mathbf{1}) = -Q(v)\mathbf{1}\otimes\mathbf{1},$$

where Q denotes the quadratic form on V. So if we change Q to $-Q$ on V and keep the $\|\cdot\|^2$ on W we get a Clifford map which must be an isomorphism on dimensional grounds.

1. Prove the remaining two assertions.

Here are some examples:

2.4.1 $C(3,0)$ is isomorphic to the algebra of all two by two complex matrices.

Take $p = 0$ and $q = 1$. We know that $C(2,0)$ is isomorphic to the algebra of all two by two matrices over the real numbers and that $C(0,1)$ is isomorphic to the algebra of complex numbers. Applying the first of the isomorphisms above says that $C(3,0)$ is isomorphic to the tensor product of the algebra of all two by two matrices over the real numbers with the complex numbers which is just the algebra of all two by two matrices over the complex numbers.

2.4.2 $C(3,1)$ is isomorphic to the algebra of all four by four matrices over the real numbers.

Take $p = q = 1$ and apply the first of the above isomorphisms or take $p = 2, q = 0$ and apply the second of the above isomorphisms.

We see that the algebra $C(3,1)$ has a four dimensional representation. Later we will identify this representation with the space of Majorana spinors.

2.4.3 $C(1,3)$ is isomorphic to the algebra of all two by two matrices over the quaternions.

Take $p = 0$ and $q = 2$ and use the second of the above isomorphisms or take $p = q = 1$ and use the third of the above isomorphisms. As a consequence:

2.4.4 The algebras $C(3,1)$ and $C(1,3)$ are not isomorphic.

In fact, the algebra $C(3,1)$ does not have a four dimensional real representation. The minimal representation is on $\mathbb{H} \oplus \mathbb{H}$ which is eight dimensional over the real numbers!

2.5 Determining all Clifford algebras.

2 Show that $\mathbb{H} \otimes \mathbb{H}$ is isomorphic to $\mathbb{R}(4)$ as algebras over the real numbers.

3. Determine all Clifford algebras $C(p,q)$ with $0 \leq p \leq 8, 0 \leq q \leq 8$. For Eddie' s sake arrange the answer as a table in matrix notation so that $C(p.q)$ occurs in the p-th row and q-th column.

4. What is the relation between $C(p+8,q)$ and $C(p,q)$?