Math 128 Problem set #2.

Simplicity of the orthogonal algebras.

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Contents

This problem set is a continuation of Problem set #1. In that problem set we proved that the algebra $sl(n+1)$ is simple by showing that it is isomorphic to the algebra of infinitesimal collineations in n dimensions. Basically, this isomorphism arises from the fact that that the group $Sl(n+1)$ acts on the space of lines through the origin in $n+1$ dimensional space, and this space is just the n dimensional projective space. Even though this action is not linear, or perhaps because it is not linear, we were able to use this action to prove that $sl(n+1)$ is simple. In the proof, it did not matter whether we were over the real or the complex numbers.

In this problem set we will use a non-linear action of the orthogonal group of $n+2$ dimensional space on a space of n dimensions to prove that the Lie algebras $o(n)$ are simple for all $n \geq 5$ (and also for $n=3$). The algebra $o(4)$ is not simple. Here the use of the complex numbers is crucial. The reason is that over the complex numbers all non-degenerate quadratic forms in n dimensions are the same. Over the real numbers a non-degenerate quadratic form has a signature - how many plus and minus signs appear when we write the quadratic form as $\pm x_1^2 \pm x_2^2 + \cdots \pm x_n^2$ and this determines the structure of the Lie algebra.

Despite the above comments, we will begin our discussion over the real or complex numbers.

1 The algebra $o(n+1,1)$.

1.1 Description of the algebra.

Let Q be the $(n+2)\times(n+2)$ symmetric matrix

$$Q := \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

where I is the $n-$dimensional identity matrix. Define $\mathfrak{k} = o(n+1,1)$ to consist of all real $(n+2)\times(n+2)$ matrices M which satisfy

$$MQ + QM^t = 0. \qquad (1)$$

Here M^t denotes the transpose of M. (In the case of the complex numbers, I want to emphasize that the is the transpose, not the adjoint, no complex conjugation is involved.)

1. Show that $\mathfrak{k}$ is a Lie subalgebra of the algebra of all $(n+2)\times(n+2)$ matrices.

2. Write the matrix M in block form

$$M = \left(\begin{array}{c|c|c} a & x & b \\ \hline u & X & y \\ \hline c & v & d \end{array}\right)$$

where a, b, c, d are scalars, x and v are row vectors of length n, u and y are column vectors of length n, and X is an $n\times n$ matrix. What does condition (1) say about these entries?

1.2 Gradation of the algebra.

3. Show that the algebra $\mathfrak{k} := o(n+1,1)$ decomposes as

$$\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1 \qquad (2)$$

with

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j}$$

(so for example $[\mathfrak{k}_1, \mathfrak{k}_1] = 0$) where

$$\mathfrak{k}_0 = o(n) \oplus \mathbb{K}E$$

where $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$, where left bracket by E is the degree derivation and where $o(n)$ acts as the orthogonal algebra on the n-dimensional spaces $\mathfrak{k}_{-1}$ and $\mathfrak{k}_1$.

4. Show that every ideal $\mathbb{I}$ in $\mathfrak{k}$ is graded, i.e. satisfies

$$\mathbb{I} = \mathbb{I} \cap \mathfrak{k}_{-1} \oplus \mathbb{I} \cap \mathfrak{k}_0 \oplus \mathbb{I} \cap \mathfrak{k}_1.$$

5. Show that if $y \in \mathfrak{k}_0$ or in $\mathfrak{k}_1$ satisifes $[x,y] = 0$ for all $x \in \mathfrak{k}_{-1}$ then $y = 0$. Conclude that if $\mathfrak{k}_0$ acts irreducibly on $\mathfrak{k}_{-1}$ then $o(n+1,1)$ is simple.

The algebra $o(2)$ acts irreducibly on the *real* plane $\mathbf{R}^2$ since no line through the origin is fixed by any rotation through an angle other than a multiple of π. But it does *not* act irreducibly on the *complex* plane $\mathbf{C}^2$ since in the complex plane there are null eigenvectors :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

for example. So acting irreducibly over the complex numbers is more restrictive.

I want to repeat that over the complex numbers there is only one orthogonal algebra in any dimension. There is no signature, so $o(n+1,1)$ is the same as $o(n+2)$ over the complex numbers, Over the real numbers there is a big difference between $o(3,1)$ and $o(4)$ for example.

1.3 The Lie algebra $o(m)$ over the complex numbers is simple if $m \geq 5$.

Remember that all non-degenerate bilinear forms on a vector space of dimension n over the complex numbers are the same, up to isomorphism. So we can talk of the algebra $o(n)$ consising of all $n \times n$ matrices satisfying

$$(Ax, y) + (x, Ay) = 0$$

where $(\, , \,)$ is a non-degenerate bilinear form (*not* a scalar product: $(x,y) = (y,x)$ with no complex conjugation). Given a pair of vectors x and y we can define the transformation $A_{x,y}$ by

$$A_{x,y}u = (y,u)x - (x,u)y$$

and check that

$$(A_{x,y}u, v) + (u, A_{x,y}v) = 0.$$

6. Show that if z is a vector in $\mathbb{C}^n$ such that $(z,z) \neq 0$ then $U(o(n))z = \mathbb{C}^n$.

The trouble that can arise comes from the existence of a vector e with $(e,e) = 0$. In two dimensions we saw that there exists such a vector e with $U(o(2))e = \mathbb{C} \cdot e$.

7. Show that if $(e, e) = 0$ there is an f with $(f, f) = 0$ and $(f, e) = 1$. If $n \geq 3$ show that the restriction of $(\ ,\ )$ to the space orthogonal to the plane spanned by e and f is non-degenerate. Conclude that there is a z in this space with $(z, z) \neq 0$ and so $A_{z.f} e = z$. Conclude that $o(n)$ acts irreducibly on $\mathbb{C}^n$ when $n \geq 3$.

2 An isomorphism of $o(n+1, 1)$ with a subalgebra of the polynomial vector fields in n variables.

I will use the notation

$$\|x\|^2 := x_1^2 + x_2^2 + \cdots + x_n^2.$$

Consider the following graded subspace $\mathfrak{h}$ of $\mathbb{V}$ where $\mathbb{V}$ is the Lie algebra of polynomial vector fields in n-variables:

$$\mathfrak{h}_{-1} := \mathbb{V}_{-1}.$$

The space $\mathfrak{h}_{-1}$ consists of all vector fields of the form

$$\sum_{ij} A_{ij} x_i \partial_j + z \sum_i x_i \partial_i, \quad \text{where } A_{ij} = -A_{ji}.$$

In other words, an element of $\mathfrak{h}_1$ is the sum of an anti-symmetric linear vector field and a multiple of the Euler vector field E.

The space $\mathfrak{h}_1$ consists of all quadratic polynomial vector fields of the form

$$\frac{1}{2}\|x\|^2 u \cdot \partial - (u \cdot x)E$$

where $u = (u_1, \ldots, , u_n)$ is a vectcor of constants and we use the shorthand notation

$$u \cdot \partial := u_1\partial_1 + \cdots + u_n\partial_n \quad \text{and} \quad u \cdot x := u_1 x_1 + \cdots + u_n x_n,$$

and $\mathfrak{h}_k = \{0\}$ for $k > 1$.

8. Show that $\mathfrak{h}$ is a Lie subalgebra of $\mathbb{V}$ and that $\mathfrak{h}$ is isomorphic to $o(n + 1, 1)$.

3 The conformal action of $o(n + 1, 1)$ on $\mathbf{R}^n$.

In this section I want to explain the geometric meaning of the isomorphism of Problem **8**. We will specialize to the real variable case. A differentiable map of some open subset of $\mathbb{R}^n$ to another is called **conformal** if its differential preserves angles at every point. In more detail: If f is differentiable map and if x is a point in the domain of definition of f and if u and v are two tangent vectors at x then the requirement is that

$$(df_x u, df_x v) = c(x)(u, v).$$

For example, translations, orthogonal (linear transformations) and the linear transformation consisting of multiplication by an overall scale factor are globally defined conformal transformations.

Consider the "inversion with respect to the unit sphere" given by

$$\mathcal{J}: \quad x \mapsto \frac{1}{\|x\|^2} x.$$

This map is defined everywhere except at the origin. In 19th century language we would say that this inversion maps the origin to the point at infinity and the point at infinity to the origin. Notice that

$$\mathcal{J}^2 = \text{identity},$$

9. Show that $\mathcal{J}$ is conformal.

Let u be a constant vector. Let $\phi_{t,u}$ be the one parameter group consisting of translations by tu so

$$\phi_{t,u}(x) := x + tu.$$

Now consider the one parameter family of transformations

$$\psi_{t,u} := \mathcal{J} \circ \phi_t \circ \mathcal{J} = \mathcal{J} \circ \phi_t \circ \mathcal{J}^{-1}.$$

For any $x \neq 0$ this is defined for sufficiently small $|t|$. From Problem **9** we know that $\psi_{t,u}$ is conformal where defined. (For $x = 0$ we will define $\psi_{t,u}(0) := 0$.)

10. What is the vector field generating the one parameter group $\psi_{t,u}$?

We have now identified the Lie algebra $\mathfrak{h}$ as an algebra of conformal vector fields. Once again we will find that there is a crucial difference between the cases $n = 2$ and $n > 2$. For $n = 2$ the algebra of conformal vector fields is infinite dimensional. This follows from the key fact discovered by Riemann that the condition that a map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

defined on a region of the plane be orientation preserving and conformal is the same as the condition that $w = u + iv$ be a holomorphic function of $z = x + iy$.

For $n > 2$ it is a theorem of Liouville that the Lie algebra of conformal vector fields is precisely our algebra $\mathfrak{h} \sim o(n+1, 1)$. We will present a poor man's version of Liouville's theorem in the next section.

4 Liouville's theorem.

Let's go back to the graded Lie algebra $\mathbb{V}$ of polynomial vector fields (in n variables). If $X \in \mathbb{V}_1$ (and so is a vector field with homogeneous quadratic polynomial coefficients) and if u and v are constant vectors, then

$$[u \cdot \partial, [v \cdot \partial, X]] = [v \cdot \partial, [u \cdot \partial, X]]$$

by Jacobi's identity since $[u \cdot \partial, v \cdot \partial] = 0$. The common value is a constant vector field, call it $w \cdot \partial$. Let me denote the linear map sending u to w for a given X and v by X_v. So the notation is

$$X_v u = w \quad \text{where } [u \cdot \partial, [v \cdot \partial, X]] = w \cdot \partial$$

and we know that

$$X_u v = X_v u.$$

We have identified the algebra $o(n)$ with the subspace of $\mathbb{V}_0$ consisting of $\sum A_{ij} x_i \partial_j$ where $A_{ij} = -A_{ji}$. So if we write $A(v)$ instead of $[v\partial, \sum A_{ij} x_i \partial_j]$ we have

$$(A(v), w) = -(v, A(w)).$$

11. Show that there is no $X \in \mathbb{V}_1$ other than $X = 0$ which satisfies

$$X_v \in o(n) \quad \text{for all vectors } v.$$

[Hint: consider $(X_v(u), w)$. This is symmetric in u and v and anti-symmetric in u and w. Show that this implies that $(X_v(u), w) = 0$ for all u, v, w and hence that $X_v(u) = 0$ for all u and hence that $X_v = 0$ for all v implying that $X = 0$.]

This implies that the only elements of $\mathbb{V}$ which are infinitesimal isometries are the vector fields of the form $v \cdot \partial + \sum A_{ij} x_i \partial_j, \quad A_{ij} = -A_{ji}$. This is a poor man's version of a theorem of Euclid which says that the only isometries of Euclidean space are the Euclidean motions.

Now suppose that $X \in \mathbb{V}_1$ satisfies

$$[v \cdot \partial, X] \in \mathfrak{h}_0.$$

This is the same as saying that for any u, v, w

$$(X_v u, w) + (u, X_v w) = \lambda_v^X (u, w).$$

Here λ_v^X depends linearly on v for fixed X and the map

$$X \mapsto \lambda_\bullet^X$$

is a linear map from $\mathbb{V}_1$ to the space of linear function of v. Problem **11** implies that this map is injective, i.e. if $\lambda_v^X = 0$ for all v then $X = 0$. This proves that the subspace $\mathfrak{h}_1 \subset \mathbb{V}_1$ consists precisely of all $X \in \mathbb{V}_1$ which satisfy

$$[v \cdot \partial, X] \in \mathfrak{h}_0 \quad \forall\ v.$$

We now want to study the subspace of $\mathbb{V}_2$ consisting of all Y such that

$$[u \cdot \partial, [v \cdot \partial, Y] \in \mathfrak{h}_0 \quad \forall\ u, v.$$

This expression depends symmetrically on u and v. Let us call it $Y_{u,v}$. This is an element of $\mathbb{V}_0$, and we can consider the linear transformation

$$w \mapsto [w \cdot \partial, Y_{u,v}]$$

which we shall denote by

$$Y_{u,v}(w).$$

Notice that $Y_{u,v}(w)$ depends symmetrically on all three vectors u, v, w. Since $Y_{u,v} \in \mathfrak{h}_o$ we have

$$(Y_{u,v}w, z) + (w.Y_{u,v}z) = \lambda^Y_{u,v}(w, z).$$

As before, if $\lambda^Y_{u,v} = 0$ for all u and v then $Y = 0$. Since $Y_{u,v} = Y_{v,u}$ and we have $\lambda^Y_{u,v} = \lambda^Y_{v,u}$. Also $\lambda^Y_{u,v}$ depends bilinearly on u and v for fixed Y. So

$$\lambda^Y_{u,u} = 0 \text{ for all } u \text{ implies that } Y = 0.$$

12. Show that if u and v are orthogonal vectors that

$$\lambda^Y_{u,u}(v, v) = -\lambda^Y_{v,v}(u, u)$$

and hence if u and v are orthogonal unit vectors that

$$\lambda^Y_{u,u} = -\lambda^Y_{v,v}.$$

Conclude that if $n \geq 3$ then $Y = 0$.

This implies that for $n \geq 3$ there is no $Z \in \mathbb{V}_k, k \geq 2$ (other than $Z = 0$) for which

$$[v_1 \cdot \partial, [v_2 \cdots \partial, [\cdots, [v_k, Z] \cdots] \in \mathfrak{h}_0 \quad \forall\ v_1, \ldots, v_k.$$

In other words, the algebra of infinitesimal conformal polynomial vector fields is precisely $\mathfrak{h}$ when $n \geq 3$. This is a version of Liouville's theorem.