Math 128 Problem set #1.

Derivations, graded Lie algebras, and simplicity.

February 5, 2002, due Feb. 12.

Contents

1 Derivations.

Let $\mathbb{A}$ be an algebra. For the moment, all that we mean by the word "algebra" is that $\mathbb{A}$ is a vector space together with a bilinear map

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A}$$

called "multiplication". We make no further demands such as associativity, commutativity, etc. We denote the product of two elements $A, B \in \mathbb{A}$ by AB.

1.1 The definition of a derivation.

A linear map

$$D : \mathbb{A} \to \mathbb{A}$$

is called a **derivation** if it satisfies **Leibnitz's rule**

$$D(AB) = (DA)B + A(DB) \qquad \forall A, B \ \in \mathbb{A}.$$

1. Show that if D_1 and D_2 are derivations of $\mathbb{A}$ then so is their commutator $[D_1, D_2] = D_1 D_2 - D_2 D_1$. This shows that the set of all derivations of A form a Lie algebra which we will denote by $\mathrm{Der}(\mathbb{A})$.

2. Show that if $\mathbb{A}$ has a unit element $\mathbf{1}$ (so that $\mathbf{1}A = A\mathbf{1} = A$ for all $A \in \mathbb{A}$) and if D is a derivation then

$$D\mathbf{1} = 0.$$

Let $A_1, \ldots, A_n$ be a set of generators of $\mathbb{A}$ meaning that every element of $\mathbb{A}$ can be written as a sum of products of the A_i. It follows from Leibnitz's rule that if D is a derivation then D is completely determined by the values DA_i.

1.2 Derivations and automorphisms.

The intuitive idea behind the concept of a derivation is the following: A linear map $\phi : \mathbb{A} \to \mathbb{A}$ is called an **automorphism** of $\mathbb{A}$ if

$$\phi(AB) = \phi(A)\phi(B) \quad \forall A, B \in \mathbb{A}.$$

Suppose (in this subsection) that our ground field is the field of real numbers, so that $\mathbb{A}$ is a vector space over the real numbers, and suppose that $t \mapsto \phi_t$ is a differentiable curve of automorphisms with $\phi_0 = \mathrm{id}$. Let

$$D := \left. \frac{d}{dt}\phi_t \right|_{t=0}.$$

Then differentiating the equation

$$\phi_t(AB) = (\phi_t A)(\phi_t B)$$

with respect to t, using Leibnitz's rule for the derivative of a product from elementary calculus and setting $t = 0$ shows that D is a derivation.

For any linear transformation X on $\mathbb{A}$ the curve $t \mapsto \exp tX$ where

$$\exp tX := I + tX + \frac{1}{2}t^2X^2 + \frac{1}{3!}t^3X^3 + \cdots$$

is a well defined curve of linear transformations and satisfies

$$\frac{d}{dt}\exp tX = X(\exp tX) = (\exp tX)X.$$

This implies that if $A \in \mathbb{A}$ and we set

$$A(t) := (\exp tX)A$$

then $t \mapsto A(t)$ is a smooth curve of elements of $\mathbb{A}$ which satisfies

$$A'(t) = XA(t), \qquad A(0) = A.$$

3. Let D be a derivation of $\mathbb{A}$. Show that $\exp tD$ is an automorphism of $\mathbb{A}$ forall t. [Hint: Use the uniqueness theorem for ordinary differential equations with initial conditions.]

This shows that we can think of a derivation as an "infinitesimal automorphism" if we are over the real numbers.

1.3 The degree derivation of a graded algebra.

The algebra $\mathbb{A}$ is said to be **graded** (over the integers) if for each integer n there is a subspace $\mathbb{A}_n$ (possibly zero) of $\mathbb{A}$ such that

$$\mathbb{A} = \bigoplus \mathbb{A}_n$$

and

$$\mathbb{A}_m \cdot \mathbb{A}_n \subset \mathbb{A}_{m+n}.$$

The first condition means that every element A of $\mathbb{A}$ can be written in a unique way as a *finite* sum

$$A = \sum A_i, \qquad A_i \in \mathbb{A}_i,$$

where the sum is over distinct i.

The second condition means that if $A_m \in \mathbb{A}_m$ and $A_n \in \mathbb{A}_n$ then $A_m A_n \in \mathbb{A}_{m+n}$. If $\mathbb{A}$ is a graded algebra, the operator E determined by

$$EA_n = nA_n \quad \text{if} \qquad A_n \in \mathbb{A}_n$$

is a derivation since if $A_m \in \mathbb{A}_m$ and $A_n \in \mathbb{A}_n$ then $E(A_m A_n) = mA_m A_n + nA_m A_n = (m+n)(A_m A_n)$. If, for example

$$A = A_2 + A_5, \qquad A_2 \in \mathbb{A}_2 \quad \text{and} \quad A_5 \in \mathbb{A}_5$$

then

$$EA = 2A_2 + 5A_5.$$

The derivation E is called the **degree derivation**. I use the letter E in honor of Euler (see below).

4. Let $\mathbb{B}$ be a linear subspace of $\mathbb{A}$ with property that $E\mathbb{B} \subset \mathbb{B}$. (This means that if $B \in \mathbb{B}$ then $EB \in \mathbb{B}$.) Show that

$$\mathbb{B} = \bigoplus_n \mathbb{B} \cap \mathbb{A}_n.$$

In other words show that if $B \in \mathbb{B}$ and

$$B = B_{i_1} + \cdots + B_{i_k}, \qquad B_{i_j} \in \mathbb{A}_{i_j}$$

is the decomposition of B into its (distinct) components, then each of the B_{i_j} belongs to $\mathbb{B}$. [Hint: Use induction on the number of components or use the non-vanishing of the Vandermonde determinant.]

1.4 Derivations and multiplications.

If $P \in \mathbb{A}$ then P defines the operator on $\mathbb{A}$ consisting of (left) multiplication by P:

$$A \mapsto PA, \quad \forall A \in \mathbb{A}.$$

Suppose that the algebra $\mathbb{A}$ is associative and commutative. Let P be an element of $\mathbb{A}$ and D a derivation of $\mathbb{A}$. Then the operator PD consisting of first applying D and then multiplying by P is again a derivation. Indeed,

$$\begin{aligned}(PD)(AB) &= P(D(AB)) \\ &= P((DA)B + A(DB)) \\ &= (PDA)B + A(PDB).\end{aligned}$$

If P and Q are elements of $\mathbb{A}$ and D_1 and D_2 are derivations, and if $A \in \mathbb{A}$ then

$$(PD_1)((QD_2A) = P(D_1Q)D_2A + PQD_1D_2A.$$

In other words,

$$(PD_1) \circ (QD_2) = P(D_1Q)D_2 + PQD_1D_2.$$

Doing things in reverse order and subtracting gives

$$[PD_1, QD_2] = (PD_1Q)D_2 - (QD_2P)D_1 + PQ[D_1, D_2].$$

2 The Lie algebra of polynomial vector fields.

Take $\mathbb{A}$ to be the algebra of polynomials in n variables $x_1, \ldots, x_n$. Then $\mathbb{A}$ is a commutative associative algebra with unit and is graded:

$$\mathbb{A} = \bigoplus_{k=0}^{\infty} \mathbb{A}_k$$

where $\mathbb{A}_k$ consists of the polynomials which are homogeneous of (total) degree k. Furthermore, $\mathbb{A}$ is generated as an algebra by $x_1, \ldots, x_n$. So a derivation D of this algebra is completely determined by the values $Dx_1, \ldots Dx_n$.

For example, consider the derivation ∂_i determined by

$$\partial_i x_i = 1, \quad \partial_i x_j = 0, \quad j \neq i.$$

Then ∂_i is the derivation which consists of taking the partial derivative of any polynomial with respect to the i-th variable.

Let D be any derivation of $\mathbb{A}$ and let

$$X_i := Dx_i$$

so that $X_i \in \mathbb{A}$. Then

$$D = X_1\partial_1 + X_2\partial_2 + \cdots + X_n\partial_n$$

since the right hand side of this equation is a derivation, and takes the same values on the generators $x_1, \dots, x_n$ as does D. We have $[\partial_i, \partial_j] = 0$ (since this is true on the generators) and so the last formula of the preceding section becomes

$$[X_1\partial_1 + \cdots + X_n\partial_n, Y_1\partial_1 + \cdots Y_n\partial_n] = \sum_k \sum_i (X_i\partial_i Y_k - Y_i\partial_i X_k)\partial_k.$$

For those who have taken a course in multivariable calculus, this is just the familiar formula for the Lie bracket of two vector fields, except that we are restricting ourselves to vector fields whose coefficients are polynomials.

We will call $\mathrm{Der}(\mathbb{A})$ the Lie algebra of polynomial vector fields and denote it by $\mathbb{V}$. The Lie algebra $\mathbb{V}$ is itself graded (starting from -1). Indeed

$$\mathbb{V} = \bigoplus_{k=-1}^{\infty} \mathbb{V}_k$$

where $\mathbb{V}_k$ consists of those

$$X = X_1\partial_1 + \cdots + X_n\partial_n$$

where all the X_i are homogeneous of degree $k+1$. Indeed, these are precisely the derivations with the property that

$$X: \quad \mathbb{A}_j \to \mathbb{A}_{j+k}$$

for all j. (Taking the partial derivative lowers the degree by one, when we follow this by multiplication by a polynomial of degree $k+1$ the total effect is to raise the degree by k.)

Thus $\mathbb{V}_{-1}$ consists of the constant vector fields - vector fields of the form $v_1\partial_1 + \cdots + v_n\partial_n$ where the v's are constants. We can thus identify $\mathbb{V}_{-1}$ with the vector space of row vectors of size n over the ground field.

Here is an obvious but important fact: Suppose that $X \in \mathbb{V}_k, k \geq 0$ has the property that $[v, X] = 0$ for all $v \in \mathbb{V}_{-1}$. Then (if our ground field is of characteristic zero) $X = 0$. Indeed the condition says that $\partial_i X_j = 0$ for all i and each j implying that all the $X_j = 0$.

2.1 The action of $\mathbb{V}_0$ on $\mathbb{V}_{-1}$.

The fact that $[\mathbb{V}_0, \mathbb{V}_i] \subset \mathbb{V}_i$ implies that $\mathbb{V}_0$ which is the set of linear vector fields is a subalgebra, and that it has a representation on on each $\mathbb{V}_j$. A typical element of $\mathbb{V}_0$ is of the form

$$L = \sum L_{ij} x_i \partial_j$$

where the L_{ij} are constants. If $v = v_1\partial_1 + \cdots v_n\partial_n \in \mathbb{V}_{-1}$ then our general formula gives

$$[L, v] = -[v, L] = -(v_1L_{11} + v_2L_{21} + \cdots + v_nL_{n1})\partial_1 - \cdots - (v_1L_{1n} + \cdots v_nL_{nn})\partial_n$$

If we consider v as a row vector this is precisely the effect of multiplying v on the right by the matrix $-(L_{ij})$ i.e.

$$(v_1, \ldots, v_n) \mapsto -(v_1, \ldots, v_n) \begin{pmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ldots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{pmatrix}.$$

This shows that there is no subspace of $\mathbb{V}_1$ other than the zero subspace or the whole space $\mathbb{V}_{-1}$ which is invariant under the action of $\mathbb{V}_0$. We say that the action of $\mathbb{V}_0$ on $\mathbb{V}_{-1}$ is **irreducible**.

2.2 The degree derivation aka the Euler vector field.

The vector field

$$E = x_1\partial_1 + \cdots + x_n\partial_n$$

is the degree derivation of $\mathbb{A}$. Indeed it is a derivation and coincides with the degree derivation on the generators.

So

$$EP = kP$$

if P is a homogeneous polynomial of degree k. This is a famous observation of Euler.

Notice that left bracket by E is also the degree derivation on $\mathbb{V}$, that is, if $X \in \mathbb{V}_k$ then

$$[E, X] = kX.$$

You can see this either by direct computation or from the defining property: if $P \in \mathbb{A}_j$ then

$$[E, X]P = EXP - XEP = (j + k)XP - jXP = kXP.$$

2.3 The algebra of polynomial vector fields is simple.

Let $\mathbb{L}$ be a Lie algebra. A subspace $\mathbb{I} \subset \mathbb{L}$ is called an **ideal** if

$$[\mathbb{L}, \mathbb{I}] \subset \mathbb{I}.$$

In other words, $\mathbb{I}$ is an ideal if $Y \in \mathbb{I}$ implies that $[X, Y] \in \mathbb{I}$ for any $X \in \mathbb{L}$.

A Lie algebra $\mathbb{L}$ is said to be **simple** if its only ideals are $\{0\}$ or $\mathbb{L}$ (we say that $\mathbb{L}$ has no non-trivial ideals) and if, furthermore, $\mathbb{L}$ is not commutative. (This last condition is meant to exclude the case of the one dimensional trivial Lie algebra.)

5. Show that the Lie algebra $\mathbb{V}$ of all polynomial vector fields is simple. [Hint: First show that any ideal $\mathbb{I}$ in $\mathbb{V}$ must be graded, that is show that $\mathbb{I} = \bigoplus \mathbb{I}_j$ where $\mathbb{I}_j = \mathbb{I} \cap \mathbb{V}_j$. Next show that either $\mathbb{I}_{-1} = \{0\}$ or $\mathbb{I}_{-1} = \mathbb{V}_{-1}$. Then show that $\mathbb{I}_{-1} = \{0\}$ implies that $\mathbb{I} = \{0\}$ while $\mathbb{I}_{-1} = \mathbb{V}_{-1}$ implies that $\mathbb{I} = \mathbb{V}$.]

3 The Lie algebra of infinitesimal collineations.

Consider the following linear subspace $\mathfrak{g}$ of $\mathbb{V}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where

$$\mathfrak{g}_{-1} = \mathbb{V}_{-1}, \quad \mathfrak{g}_0 = \mathbb{V}_0$$

and where $\mathfrak{g}_1 \subset \mathbb{V}_1$ consists of those elements of the form

$$\ell E$$

where ℓ is a linear function. Recall that $\mathbb{V}_1$ consists of vector fields whose coefficients are homogeneous quadratic functions. The subspace $\mathfrak{g}_1$ are those vector fields of the form

$$x_1 \ell \partial_1 + x_2 \ell \partial_2 + \cdots + x_n \ell \partial_n$$

where ℓ is a linear function. So $\mathfrak{g}$ is a finite dimensional subspace, in fact of dimension

$$n + n^2 + n = (n+1)^2 - 1.$$

6. Show that $\mathfrak{g}$ is a Lie subalgebra of $\mathbb{V}$. This amounts to showing that

$$[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$$

and that

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \{0\}.$$

7. Show that the Lie algebra $\mathfrak{g}$ is simple.

4 The Lie algebra of infinitesimal collineations in n dimensions is isomorphic to $sl(n+1)$.

By definition, the algebra $sl(n+1)$ consists of all $(n+1) \times (n+1)$ matrices of trace zero. We can write the most general such matrix in "block form" as

$$\left(\begin{array}{c|ccc} -\operatorname{tr} A & v_1 & \cdots & v_n \\ \hline \ell_1 & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ \ell_n & A_{n1} & \cdots & A_{nn} \end{array}\right)$$

where $A = (A_{ij})$ is an arbitrary $n \times n$ matrix. We may write the above matrix more succinctly as

$$\left(\begin{array}{c|c} -\operatorname{tr} A & v \\ \hline \ell & A \end{array}\right).$$

8. In $\mathfrak{g}$ compute the commutator $[v\partial, \ell E]$ where $v\partial$ is short for $v_1\partial_1 + \cdots + v_n\partial_n$ and where $\ell = \ell_1 x_1 + \cdots + \ell_n x_n$ (and where the v_i and ℓ_j are constants).

9. In $sl(n+1)$ compute the commutator

$$\left[\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \ell & 0 \end{pmatrix}\right].$$

10. Show that the map

$$v\partial \mapsto \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, \quad \ell E \mapsto \begin{pmatrix} 0 & 0 \\ -\ell & 0 \end{pmatrix}$$

$$\sum_{ij} L_{ij} x_i \partial_j \mapsto \begin{pmatrix} -\frac{1}{n+1}\operatorname{tr} L & 0 \\ 0 & L - \frac{1}{n+1}\operatorname{tr} L \cdot I \end{pmatrix}$$

where $L = (L_{ij})$ and I is the $n \times n$ identity matrix, is a Lie algebra isomorphism of $\mathfrak{g}$ with $sl(n+1)$.

If we combine Problem **7** with Problem **10** we have a proof of the fact that the algebra $sl(n+1)$ is simple.