MATH 126
SECTION LECTURE NOTES

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1. General Representation Theory

Representation theory is no longer the study of representations of finite groups. Rather, it is now the study of modules over algebras.

Recall that a module is like a vector space, only over a ring, and that an algebra is a special kind of ring. In general, these algebras are not as well-behaved as the group algebra $K[G]$, but we can still study them. If you've taken 123, you know that we can completely classify modules over a PID, and that this has lots of wonderful applications: classification of commutative groups, Jordan normal form, enumeration of endomorphisms and automorphisms of these modules, etc. So we want to do the same for modules over an algebra. (*This is to justify the abstraction that follows.*)

2. Restricting and Inducing Modules, with tensors

Ken covered the module/tensor interpretation of induced representations really quickly, and I think that it is a very instructive way of looking at this. However, it is rather abstract, so please stop me if things are ever not completely clear.

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A professor here, Dick Gross, says that: "Tensors are the hardest thing in math". I hope you don't think so after this.

For this section, let's fix rings $S \subset R$ and modules M, N.

So if you have an R*-module* M*, how do you get an* S*-module?*

Well, you just forget the R-module structure. For example, take $S = \mathbf{Z} \subset \mathbf{R} = R$. Now any $\mathbf{R}$ vector space is a commutative group, i. e., a $\mathbf{Z}$-module. Let's denote this by $\mathrm{Res}^R_S(M)$.

Indeed, as Ken mentioned, this is functorial. That is, given any map *of* R*-modules*, it is a fortiori a map *of* S*-modules*. So given any $M \xrightarrow{\varphi} N$ of R-modules, we get $\mathrm{Res}^R_S(M) \xrightarrow{\mathrm{Res}^R_S(\varphi)} \mathrm{Res}^R_S(N)$. Note that *as sets*, $M = \mathrm{Res}^R_S(M)$, $N = \mathrm{Res}^R_S(N)$, $\varphi = \mathrm{Res}^R_S(\varphi)$, but they live in different worlds. This can be confusing.

So restricting is easy. Conversely, given an S-module M, how do you get an R-module?

This is just as easy, so long as we use the right notation. Think of M as

$$M = \{s_1 m_1 + s_2 m_2 + \cdots + s_k m_k | s_i \in S, m_i \in M\} \tag{0.1}$$

that is, as linear combinations of elements of M with coefficients in S. If you want, you can pick a generating set or (if it exists) a basis, etc. Now the obvious way to make M into an R-module is...

Replace the S by an R, so define

$$\mathrm{Ind}^R_S(M) = R \otimes_S M = \{r_1 m_1 + r_2 m_2 + \cdots + r_k m_k | r_i \in R, m_i \in M\} \tag{0.2}$$

We often write such elements as $r_1 \otimes m_1 + r_2 \otimes m_2 + \cdots + r_k \otimes m_k$. This agrees with the definition given in class; the tensor just means that we remember the relations in M. More formally, both of these "equalities" are not quite right—I should say that we mod out by the relations in M.

At this point we desperately need examples, so here's some: as before, take $S = \mathbf{Z} \subset \mathbf{R} = R$. Now, assume M is a *free* $\mathbf{Z}$-module, that is, it has no torsion. Then if S is generated as a $\mathbf{Z}$-module by $\{e_1, \ldots, e_n\}$, then *what is* Ind^R_S*?*

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It's the $\mathbf{R}$-module (vector space) with basis $\{e_1, \dots, e_n\}$. Note that it is not the $\mathbf{R}$-module with basis $e_1, e_1 + e_2, e_{49} - 13e_7, \dots$ *because* we tensored over $\mathbf{Z}$, so it remembers those relations.

By the way, what happens to torsion if we tensor with $\mathbf{R}$? *That is, given an element* $m \in M(\exists r \in Z(rm = 0))$*, what happens to it in the natural injection* $M \to R \otimes_S M$*, where* $m \mapsto 1 \otimes m$*?*

It goes to zero:

$$1 \otimes m = \frac{1}{r} r \otimes m = \frac{1}{r} \otimes rm = \frac{1}{r} \otimes 0 = 0 \tag{0.3}$$

In fact, this happens if we tensor with any divisible ring such as $\mathbf{Q}, \mathbf{R}, \mathbf{C}$. I don't know if this term is standard—I'm using it with analogy with divisible group, and it is *not* the same as division ring)

Also, why is this functorial? *That is, given* $M \xrightarrow{\varphi} N$*, how do we get* $R \otimes M \xrightarrow{R \otimes \varphi} R \otimes N$*?*

Well, just map $\varphi(r_i m_i) = r_i \varphi(n_i)$, and this works. Formally, what we're doing is:

$$R \otimes M \xrightarrow{\mathrm{id} \otimes \varphi} R \otimes N \tag{0.4}$$

that is, setting our map to act as the identity on R.

3. Adjoints

So now we understand Res, Ind. I claim that these are in some sense inverse to each other. That is, they are adjoints. More correctly, I claim we have

$$\frac{\mathrm{Ind}_S^R M \to N}{M \to \mathrm{Res}_S^R N} \tag{0.5}$$

which means that there is a natural correspondence between maps on the top and maps on the bottom. Note that on top everything is an R-module, and on the bottom everything is an S-module.

To make this correspondence clear, let's take $S = \varnothing \subset \mathbf{R} = R$, so objects on the bottom are modules over the empty set, i. e., sets, and objects on the top are $\mathbf{R}$ vector spaces.

So given a set $M = \{m_1, \dots, m_k\}$ *and a vector space* $N = \mathbf{R}^l$*, what are* $\mathrm{Ind}_S^R M, \mathrm{Res}_S^R N$*?*

$\mathrm{Ind}_S^R M$ is the vector space with basis $\{m_1, \dots, m_k\}$, and $\mathrm{Res}_S^R N$ is the set $\mathbf{R}^l$.

So here's the correspondence: Any map between vector spaces is determined by where it sends a basis, and the basis can be sent anywhere.

Could someone explain why Ind, Res *are not inverses of each other for this choice of* R, S*?*

Well, $\mathrm{Ind}(\mathrm{Res}\,\mathbf{R})$, for example, will be an $\mathbf{R}$-vector space with infinite basis.

In fact, for this we really didn't need our particular choices of S, R, they work for any $S \subset R$, and in particular for $S = K[H] \subset K[G] = R$. So that's what's going on with restricting and inducing.