# MATH 126
# SECTION LECTURE NOTES

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## 1. Notation

Throughout, we will be considering $\rho\colon G \to GL(V)$, where $\rho$ is a representation, $G$ is a group, and $V$ is a vector space over some field (unspecified, but often $\mathbf{C}$). Note that we use subscripts to distinguish representations, (so $\rho_1, \rho_2\colon G \to GL(V)$ are different maps), while some authors (such as Serre) use subscripts to denote the image of an element (so $\rho_g = \rho(g) \in GL(V)$. This is so that we don't get abominations like $\rho(g)(v)$, for the image of $v$ under $\rho(g)$ (which these authors write $\rho_g(v)$). We instead will generally fix a representation and write $gv$ for $\rho(g)(v)$; this notation is preferred when considering representations as $K[G]$–modules, as we will.

## 2. Examples of Representations

Given any group $G$, we have some canonical representations:

**2.1. Trivial (Unit) Representation.** Send everything to the identity map. This is not facile: it is a legitimate representation, and is useful in classifying representations.

*What $K[G]$ module corresponds to this representation?*

Remember that every $K[G]$ module is simply a $K$ vector space on which there is a multiplication. Thus, the trivial $K[G]$ module is simply the 1 dimensional vector space spanned by one element $e_1$, with multiplication defined by $ge_1 = e_1$.

Note that $n$ dimensional representations correspond to $K[G]$ modules that have an underlying $n$ dimensional vector space.

Note that the multiplication is often easier to visualize if you have a basis.

**2.2. Regular Representation.** Consider the vector space $V$ generated by $e_g$, and where $\rho_g(e_{g'}) = e_{gg'}$ (which we will henceforth write $ge'_g = e_{gg'}$).

As you can see, this defines a representation, which is noteworthy in that *every simple representation is a subrepresentation of the regular representation.*

*What $K[G]$ corresponds to this representation?*

The vector spaces spanned by $e_g$, with multiplication defined by $ge'_g = e_{gg'}$. See the complete equivalence of the two notions?

**2.3. Permutation Representation.** In what we did in the previous example was view $G$ as acting on itself by left multiplication. In fact, given *any* group action $G \times S \to S$ we have an associated representation, given by... $ge_s = e_g s$.

*What are some standard group action that we know?*

- $S_n \times [n]$, the permutations on $n$ letters
- $D_n \times [n]$, the isometries of a regular $n$–gon
- $C_n \times [n]$, the orientation-preserving isometries of a regular $n$–gon

Received by the editors September 28, 1998.

2.4. **More on the Regular Representation, and Some References.** As noted above, every simple representation is a subrepresentation of the regular representation. Indeed, every $n$ dimensional simple representation occurs $n$ times as a subrepresentation! More formally, given any decomposition into simple ones, each $n$ dimensional simple representation occurs $n$ times as a direct summand. This can be done directly; if you want details, see Serre. In fact, this book is a good reference, and the first part (or two) are quite accessible. You see, the first part was written for quantum chemists, so it's completely trivial and very easy. The second part was written for second year graduate students at one of France's elite universities, so you should find it accessible, with work. The third part was written for Grothendieck, an insane Russian-German-French mathematician— his parents were Russian and German (hence the non-too French name), and he was born a bastard in France. He sometimes disappears for years at a time, and when he returns cannot remember where he was. Once, he was found herding sheep. There's a picture of him downstairs on the Institute des Hautes Etudes Scientific poster. I do not recommend that you try reading this part; it may be covered in 250a, and if you'd like, I can tell you when it is covered if you'd like to attend lecture or get notes.

The above result, on multiplicity of representations, as noted by Ken today, follows directly from the technical work with semisimple rings and modules that we've done in class over the past two days. If you'd like a reference for *that*, see either Lang's "Algebra" or Bourbaki's "Algebra", which are both in the math library.