MATH 126 PROBLEM SET 9: INTEGRALITY II

This problem set is due Wednesday December 13. All groups are assumed to be finite and all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0. This is a long example sheet. If you don't complete it don't worry too much. It is meant as an example of how the techniques we have learnt, together with a certain amount of ingenuity and patience, allow one to compute a reasonably complicated character table.

1) Show that the integral elements of $\mathbb{Q}(\sqrt{-11})$ are the elements $(x+y\sqrt{-11})/2$ where $x, y \in \mathbb{Z}$ have the same parity.

2) [OPTIONAL GALOIS THEORY QUESTION - BUT NOTE THE RESULT FOR THE NEXT QUESTION] Let $\zeta = e^{2\pi i/11}$. Show that $\mathrm{Gal}\,(\mathbb{Q}(\zeta)/\mathbb{Q}) \hookrightarrow (\mathbb{Z}/11\mathbb{Z})^\times$ ($\sigma \mapsto r$ where $\sigma(\zeta) = \zeta^r$) and deduce that $\mathbb{Q}(\zeta)$ contains at most one quadratic extension of $\mathbb{Q}$. Show that $\zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9$ is a root of $X^2 + X + 3$ and deduce that $\mathbb{Q}(\sqrt{-11})$ is the unique quadratic extension of $\mathbb{Q}$ in $\mathbb{Q}(\zeta)$.

Also show that $\mathbb{Q}(e^{\pi i/4}) = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$ has 3 quadratic subfields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$ and hence that $\mathbb{Q}(\sqrt{-11}) \cap \mathbb{Q}(e^{\pi i/4}) = \mathbb{Q}$.

3) Let G be a simple group of order $7920 = 16.9.5.11$ with 10 conjugacy classes $[g_1], ..., [g_{10}]$ with cardinalities $1, 1584, 720, 720, 990, 440, 165, 1320, 990, 990$ respectively. Let $g_1, ..., g_{10}$ have orders $1, 5, 11, 11, 4, 3, 2, 6, 8, 8$. Let χ_0 denote the trivial character of G.

Let P denote a Sylow 11 subgroup of G and let N denote its normaliser (i.e. $N = \{g \in G : gPg^{-1} = P\}$). Show that N is a non-abelian group of order 15 and that $g_3^{-1} \sim g_4$. Let ψ be a non-trivial one dimensional representation of N/P. Calculate the character of $\mathrm{Ind}_N^G \psi$ and show that it equals $\chi_1 + \chi_2 + \chi_3$ for distinct irreducible characters χ_1, χ_2, χ_3 of G. Show that for $i = 1, 2, 3$ we have

$$\chi_i(g_1) + 11\chi_i(g_2) + 5(\chi_i(g_3) + \chi_i(g_4)) = 55.$$

Show that the $[g_2]$ column in the character table of G either consists of $1, \pm 2$ and 8 zeros, or of $1, \pm 1, \pm 1, \pm 1, \pm 1$ and 5 zeros. After perhaps reordering χ_1, χ_2, χ_3 show that either $\chi_1(g_2) = \chi_2(g_2) = 0$ and $\chi_3(g_2) = -1$, or $\chi_1(g_2) = \chi_2(g_2) = -1$ and $\chi_3(g_2) = 1$. Show moreover that $\chi_3(g_3) = \chi_3(g_4) \in \mathbb{Z}$.

Show that either $\chi_i(g_3) = \chi_i(g_4) \in \mathbb{Z}$ for $i = 1, 2$, or that there are $x, y \in \mathbb{Z}$ of the same parity and with $y \neq 0$ such that

$$\chi_1(g_3) = \chi_2(g_4) = (x + y\sqrt{-11})/2$$

and

$$\chi_1(g_4) = \chi_2(g_3) = (x - y\sqrt{-11})/2.$$

We will refer to these two cases as the first and second cases.

In the first case show that for any $i = 1, 2, 3$ we have $|\chi_i(g_3)| \leq 3$, $|\chi_i(g_4)| \leq 3$ and

$$\chi_i(g_3) = \chi_i(g_4) \equiv \chi_i(g_1) \bmod 11.$$

Show that if $\chi_i(g_2) = 0$ then $\chi_i(g_1) \in \{10, 30, 45, 55, 80\}$; that if $\chi_i(g_2) = -1$ then $\chi_i(g_1) \in \{9, 24, 44\}$; and that if $\chi_i(g_2) = 1$ then $\chi_i(g_1) \in \{36, 66\}$. Deduce that the only possibilities for the first four entries in the χ_i row of the character table are

45	0	1	1
55	0	0	0
24	-1	3	3
44	-1	0	0
36	1	3	3
66	1	0	0

and hence that in the first case the following must be the top left hand corner of the character table of G

	$[g_1]$	$[g_2]$	$[g_3]$	$[g_4]$
χ_0	1	1	1	1
χ_1	55	0	0	0
χ_2	45	0	1	1
χ_3	44	-1	0	0.

In the second case show that $y = \pm 1$, $|x| \leq 3$, $|\chi_3(g_3)| = |\chi_3(g_4)| \leq 3$,

$$\chi_1(g_1) = \chi_2(g_1) \equiv 6x \bmod 11$$

and

$$\chi_3(g_3) = \chi_3(g_4) \equiv \chi_3(g_1) \bmod 11.$$

Show that the only possibilities for the first four entries in the χ_3 row of the character table are

24	-1	3	3
44	-1	0	0
66	1	0	0.

Also show that the only possibilities for the triple $(\chi_1(g_1), \chi_1(g_2), x)$ are

40	0	2
45	0	1
55	0	0
60	0	-1
66	-1	0.

Conclude that the second case can not in fact arise.

Complete the $[g_1]$ and $[g_2]$ columns of the character table of G.

Compute the real part of the $[g_3]$ and $[g_4]$ columns of the character table, for instance by using the equality $(\chi_i, \operatorname{Ind}_N^G \psi) = 0$ for $i > 3$. Then use the column orthogonality relations to compute these columns completely. Next show that there are at most two possibilities for the $[g_5]$ column.

Show that the entries in the row of the 16 dimensional representations which remain unfilled must lie in $\mathbb{Z}$. Then use row orthogonality relations to compute the rows corresponding to the 16 dimensional representations. Next use the row orthogonality relations to compute the $[g_6]$ column.

Compute the $[g_4]$ and $[g_7]$ columns. Then compute the row of the 44 dimensional representation and use this to compute the $[g_8]$ column. Finally complete the character table of G.

[There is a (unique) such group G, the Mathieu group M_{11}. (11 because it is a subgroup of S_{11}.) I suspect that when finite group theorists do this sort of computation it is unusual to know all the conjugacy class information ahead of time, rather it has to be figured out at the same time as one figures out the character table.]