MATH 126 PROBLEM SET 8: INTEGRALITY

This problem set is due Wednesday December 6. All groups are assumed to be finite and all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0. The first two questions really belong to the section on induction.

1) If $\phi : G \to \mathbb{Q}$ is a class function such that $\phi|_H \in R(H)_{\mathbb{Q}}$ for all cyclic subgroups $H \subset G$, show that $\phi \in R(G)_{\mathbb{Q}}$.

2) If $\phi : G \to \mathbb{Q}$ is a class function such that $\phi(g^m) = \phi(g)$ for all m coprime to $\#G$, show that $\phi \in R(G)_{\mathbb{Q}}$.

3) Which of the following numbers are integral?

$$\sqrt{5}, \sqrt{5}/2, (1+\sqrt{5})/2, (1+\sqrt{3})/2, (\sqrt{3}+\sqrt{7})/2.$$

4) Let p be a prime and G be a non-abelian group of order p^3. Show that G has p^2 representations of degree 1 and $p-1$ irreducible representations of degree p. Deduce that G has p^2+p-1 conjugacy classes.

5) Let V be an irreducible representation of G. Let $H_m \subset Z(G)^m$ denote the subset of $(z_1, ..., z_m) \in Z(G)^m$ such that $z_1 z_2 ... z_m = 1$. Show that H_m is a normal subgroup of G^m and that $V \otimes V ... \otimes V$ (m factors) is an irreducible representation of G^m/H_m. Deduce that

$$\#Z(G)(\#G/(\#Z(G)\dim V))^m$$

is integral for all $m \in \mathbb{Z}_{>0}$ and hence that

$$\dim V | \#(G/Z(G)).$$

6) Suppose that A is an abelian normal subgroup of G and that V is an irreducible representation of G. Show that $\dim V | \#(G/A)$. [HINT: Use induction on $\#G$. Reduce to the case V is a faithful representation of G and distinguish the cases $A \subset Z(G)$ and $A \not\subset Z(G)$.]

7) Suppose that G is a non-trivial perfect group (i.e. $G = [G,G]$, where $[G,G]$ denotes the smallest normal subgroup containing all elements of the form $ghg^{-1}h^{-1}$ with $g,h \in G$) and that G has a faithful two dimensional representation ρ. Show that ρ is irreducible, $\det \rho$ is trivial and that G has an element of order 2. [Hint: Recall the Sylow theorems.] Deduce that $2 | \#Z(G)$ and that $GL_2(\mathbb{C})$ contains no non-abelian finite simple group.

8) Show that the character of any irreducible representation of dimension greater than 1 assumes the value 0 on some conjugacy class. [HINT: Let χ denote the character. It takes values in some finite Galois extension $\mathbb{Q}(\zeta)/\mathbb{Q}$, where ζ is a root of unity. If $\sigma \in \mathrm{Gal}\,(\mathbb{Q}(\zeta)/\mathbb{Q})$ show that $\sigma \circ \chi$ is also an irreducible character of G. Let $\chi_1 = \chi, ..., \chi_r$ denote the distinct characters obtained in this way. If $\chi(g) \neq 0$ show that

$$(\sum_i |\chi_i(g)|^2)/r \geq |\prod_i \chi_i(g)|^{2/r} \geq 1.$$

and that if we have equality then $|\chi_i(g)| = 1$ for all $i = 1, ..., r$.] (I was told by one of the authors of the "Atlas of Finite Simple Groups" that this is a very useful result to know when computing complicated character tables, though I can't say that I have ever used it myself.)