

Math 126, midterm exam

March 27, 2000

Problem 1. Suppose G is a group of order 68, generated by an element x of order 17 and an element y of order 4, satisfying the relation $y^{-1}xy = x^4$. Find the degrees of the irreducible representations. For each irreducible character χ , give the values of $\chi(1)$, $\chi(x)$ and $\chi(y)$.

Problem 2. Let G be a finite group, let

$$G = \Omega_1 \cup \dots \cup \Omega_r$$

be its decomposition into conjugacy classes, and let χ_1, \dots, χ_r be its irreducible characters.

Let $\sigma : G \rightarrow G$ be an automorphism of G (a bijective homomorphism from G to itself). Show that, for each i , the subset $\sigma(\Omega_i)$ is a conjugacy class of G . Show that, for each i , the function

$$\chi_i^\sigma(g) \stackrel{\text{def}}{=} \chi_i(\sigma(g))$$

is an irreducible character of G .

Show that the number of irreducible characters χ_i with $\chi_i^\sigma = \chi_i$ is equal to the number of conjugacy classes Ω_j with $\sigma(\Omega_j) = \Omega_j$.

For the next question, recall the basic properties of a Frobenius group N . Such a group is a transitive permutation group with the property that no element of the group other than the identity fixed more than 1 point. The elements fixing no points, together with the identity, form a normal subgroup K , the Frobenius kernel. The stabilizer of any chosen point is a subgroup H with the property that $KH = N$. The conjugates of H intersect only at the identity, and the number of conjugates of H is $|K|$. In particular, this tells us that no element of K can commute with any element of $N \setminus K$.

Problem 3. Let N be a Frobenius group, with $G = KH$ as above. Let $K = K_1 \cup \dots \cup K_s$ be the decomposition of K into the conjugacy classes of K , with K_1 being the conjugacy class of the identity. Let ψ_1, \dots, ψ_s be the irreducible characters of K , with ψ_1 the trivial character.

Fix an element $h \in N \setminus K$. Show that, for $\gamma \in K_i$ with $i \geq 2$, the element $h\gamma h^{-1}$ is not in K_i . Use the previous question to show that, for $i \geq 2$, the function

$$\psi_i^h(z) \stackrel{\text{def}}{=} \psi_i(hzh^{-1}) \quad (z \in K)$$

is an irreducible character of K and is not equal to ψ_i .

Show that the induced character ψ_i^N is zero on $N \setminus K$, and that, for $\gamma \in K$,

$$\psi_i^N(\gamma) = \sum_{h \in H} \psi_i^h(\gamma).$$

Deduce that the induced character ψ_i^N is an irreducible character of N for $i \geq 2$. (We did a similar thing in class, in rather less generality.)

Problem 4. Let G be a finite simple group. Let χ be a non-trivial irreducible character, and Ω a conjugacy class other than the conjugacy class of the identity. Show that if $|\Omega|$ is prime to the degree of χ , then $\chi(\gamma) = 0$ for $\gamma \in \Omega$. (We essentially proved this in class, on the way to proving Burnside's theorem. Extract the proof from your notes, from Ledermann, or from elsewhere.)

Let G be a finite simple group having a faithful, irreducible representation of dimension p (a prime) and suppose p^2 does not divide $|G|$. Show that G cannot have an abelian subgroup of order pq , for any integer $q > 1$.

Exhibit two groups G satisfying the hypotheses of the last part.