Solution Set 7

1. Consider the space W of solutions of the differential equation. If f satisfies a differential equation at every point x, then it certainly satisfies the differential equation at the point $x-t$. That is, $\tau_t f$ will also be a solution. Therefore W is G-invariant. From ODE, we know that W is finite dimensional.

Let H be a finite dimensional G-invariant subspace of smooth functions in V. Consider the basis $f_1, \ldots, f_n$. By assumption, for all t, there exists a matrix $A(t)$ such that

$$\begin{pmatrix} \tau_t f_1 \\ \vdots \\ \tau_t f_n \end{pmatrix} = A(t) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

This holds for all x. So for any $x_1, \ldots, x_n$,

$$\begin{pmatrix} f_1(x_1-t) & \cdots & f_1(x_n-t) \\ \vdots & & \vdots \\ f_n(x_1-t) & \cdots & f_n(x_n-t) \end{pmatrix} = A(t) \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{pmatrix}$$

By linear independence of the f_i's, we can choose $x_1, \ldots, x_n$ such that the matrix on the right is invertible. It follows that $A(t)$ is a smooth function of t. Thus

$$\begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix} = \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \tau_t f_1 \\ \vdots \\ \tau_t f_n \end{pmatrix} = A'(t) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Therefore $f_1', \ldots, f_n' \in H$. The rest of the proof is not hard.

2. One direction of implication is easy. The hard part is the forward direction. Here's a sketch of what I thought of, which may or may not be the intended solution. Let H be a finite dimensional G-invariant subspace with $f \in H$. Using a similar argument as in problem 1, we can show that $\frac{\partial f}{\partial x} \in H$ and that all elements in H satisfy a certain differential equation in x. We do a similar thing with y and z. It follows that f must be a linear combination of things that look like $p(x,y,z)e^{ax+by+cz}$ where p is a polynomial. The hard part, which is intuitively clear, is that if the exponential parts don't all vanish, then we can act by rotations to get infinitely many linearly independent elements of H.

3. One way to do this is to define $(Tf)(x) = \mu^{\lfloor x \rfloor} f(x)$ and $(Sf)(x) = f(x+1)$. This representation has a finite dimensional subrepresentation spanned by the functions $\mu^{k\lfloor x \rfloor}$ for $k = 0, \ldots, n-1$.

Now let V be any finite dimensional representation of G with $Uv = \mu v$. Let u be an eigenvector of T with eigenvalue λ. Then $\mu u = Uu = STS^{-1}T^{-1}u = \frac{1}{\lambda}STS^{-1}u$. So $T(S^{-1}u) = \lambda\mu(S^{-1}u)$. Therefore $\mu\lambda$ is an eigenvalue of T. By induction, $\mu^k\lambda$ is an eigenvalue of T for any integer $k \geq 0$. Since V is finite dimensional, T has finitely many eigenvalues, therefore μ is a root of unity. This argument also shows that if $\mu = e^{2\pi i/n}$, then the dimension must of V be at least n.