Solution Set 5

1.
• Conjugacy classes of S_n are determined by cycle structure. So the the conjugacy class of an n-cycle g has $(n-1)!$ elements, and therefore the centralizer of g has $|S_n|/(n-1)! = n$ elements. Surely the n powers of g commute with g, and therefore these are the only elements commuting with g.
• Let C be the subgroup generated by such a g, and let $N = N_G(C)$. Let n be prime. We claim that N is Frobenius. Assume the contrary. Then some non-trivial $h \in N$ fixes distinct numbers k_1, k_2. For some i, $g^i(k_1) = k_2$. Then

$$hg^i h(k_1) = hg^i(k_1) = h(k_2) = k_2.$$

It follows that $hg^i h = g^i$ and since n is prime, $hgh = g$. So by the previous bullet, h is a power of g and therefore cannot fix anything.
• Further assume that $(n-1)/2$ is prime, and let $G \subset A_n$ be a non-Abelian simple group containing C. Note that there are $(n-1)!$ elements of order n in A_n. Since n is prime, Sylow n-subgroups of A_n contain n elements—the identity together with $n-1$ elements of order n. Therefore there are $(n-1)!/(n-1) = (n-2)!$ Sylow n-subgroups of A_n. By Sylow's Theorem, we observe that $|N_{A_n}(C)| = |A_n|/(n-2)! = \frac{1}{2}n(n-1)$. Then since $C \subset N_G(C) \subset N_{A_n}(C)$ and $(n-1)/2$ is prime, it follows that $N_G(C)$ must be either C or $N_{A_n}(C)$. I'll get back to you on why it cannot be C.
• By Sylow's Theorem, G has $|G|/|N_G(C)|$ Sylow n-subgroups. Multiplying this by $n-1$, we see that G has $2|G|/n$ elements of order n. The size of a conjugacy class of an element $g \in G$ of order n is $|G|/|C_G(g)| = |G|/n$ by the first bullet. So there must be only 2 conjugacy classes of elements of order n in G.

2. (a) We know that $N_G(C) \subset N_{S_n}(C)$ so by the previous problem, $N = N_G(C)$ is Frobenius. Furthermore, it has order $\frac{1}{2}11 \cdot 10 = 55$. Since C is a Sylow 11-subgroup of G, it follows that $n_{11} = |G|/N_G(C)| = 2^4 \cdot 3^2$.
(b)
• Sylow 5-subgroups of G each contain 5 elements—the identity and 4 elements of order 5—and are conjugate to each other. It follows that all elements of order 5 in G must have the same cycle structure; they're either all 5-cycles or all have cycle structure 5+5+1. Since $N \subset G$ and contains elements of order 5 and is also Frobenius, it follows that the structure must be 5+5+1.
• Now let us establish that $n_5 = 2^2 \cdot 3^2 \cdot 11$.
Claim: $11|n_5$.
If not, then since $|G| = |N_G(Q_5)| \cdot n_5$ we have 11 divides $|N_G(Q_5)|$. That is, there exists an 11-cycle x normalizing Q_5. This gives us an action of x on Q_5. This action is necessarily trivial, that is, x commutes with the elements of Q_5. But this contradicts the hint. (The hint itself is not too hard to prove.)
Claim: $9|n_5$.
Assume otherwise. As before, we have 3 divides $|N_G(Q_5)|$. Therefore some element x of order 3 normalizes Q_5. It is obvious that this action of x on Q_5 must fix one of the non-trivial elements of $Q_5 \subset G$. This contradicts the hint.

Now use Sylow's Theorem and observe that the only possibility for n_5 divisible by 11 and 9 is $n_5 = 2^2 \cdot 3^2 \cdot 11$.
• Given non-trivial $y \in Q_5$, we know that $|C_G(y)|$ divides $|N_G(Q_5)| = |G|/n_5 = 20$. We claim that $|C_G(y)| = 5$. If not, then there is an element $z \in C_G(y)$ of order 2, contradicting the hint.
(c) We know N is non-Abelian and has order $5 \cdot 11$, so by a prior problem set, we know that N has 5 characters of degree 1, call them $\psi_1, \ldots, \psi_5$ with ψ_1 trivial. Here is the table for these characters:

class size	1	11	11	11	11	5	5
	1	g_5	g_5^2	g^3	g^4	g_{11}	g'_{11}
ψ_1	1	1	1	1	1	1	1
ψ_2	1	a	a^2	a^3	a^4	1	1
ψ_3	1	a^3	a	a^4	a^2	1	1
ψ_4	1	a^2	a^4	a	a^3	1	1
ψ_5	1	a^4	a^3	a^2	a	1	1

where $a = e^{\frac{2\pi i}{5}}$. Now we apply our formula for induced characters (omitting the details) to find the following table, for $i \neq 1$:

class size	1	$2^4 \cdot 3^2 \cdot 11$	$2^4 \cdot 3^2 \cdot 5$	$2^4 \cdot 3^2 \cdot 5$	NA
	1	g_5	g_{11}	g'_{11}	other junk
ψ_1^G	$2^4 \cdot 3^2$	4	1	1	0
ψ_i^G	$2^4 \cdot 3^2$	-1	1	1	0

where we have used (b) and the last bullet of problem 1. For $i \neq 1$, we readily compute $\langle \psi_i^G, \psi_i^G \rangle = 3$. This can only happen if ψ_i^G is a sum of 3 distinct 1-dimensional representations of G.

3. No one had trouble with this one.