Math 126, sheet 5

March 15, 2000

Problem 1. Show that an n-cycle in S_n commutes only with its powers. Let C be a subgroup of S_n generated by an n-cycle. Let N be the normalizer of C in S_n. If n is prime, show that N is a Frobenius group, as we defined it in class (i.e. no group element stabilizes more than one point in $\{1, \ldots, n\}$).

Continue to suppose that n is prime, and suppose also that $(n-1)/2$ is prime. Show that if $G \subset A_n$ is a nonabelian simple group containing the cyclic group C, then $N_G(C) = N_{A_n}(C)$. Show that G contains exactly two conjugacy classes of elements of order n.

Problem 2. Let G be a non-abelian simple subgroup of A_{11} of order $11 \cdot 10 \cdot 9 \cdot 8 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. (There is such a group, the Matthieu group M_{11}.) Let n_p denote the number of Sylow-p subgroups.

(a) Show that G contains a Frobenius subgroup N of order 55 and that $n_{11} = 2^4 \cdot 3^2$.

(b) Show that $n_5 = 2^2 \cdot 3^2 \cdot 11$. Show that elements of order 5 commute only with their powers and comprise one conjugacy class (Hint: use the fact that $(a\ b\ c\ d\ e)(f\ g\ h\ i\ j)$ commutes only with elements of order 5 and 1 in A_{11}.)

(c) Show that N has 5 characters of degree 1 (including the trivial one). Let these by $\psi_1, \ldots, \psi_5$, with ψ_1 the trivial character. Compute the induced characters ψ_i^G of G. By computing $\langle \psi_i^G, \psi_i^G \rangle$, show that, for $i \neq 1$, the character ψ_i^G is a sum of three distinct irreducibles. (Remember that $2\cos(2\pi/5) = (-1+\sqrt{5})/2$.)

Problem 3. Let g be an element of a finite group G. Suppose that the order of the element g is a prime p, and that g is conjugate to each of its powers, $g, g^2, \dots, g^{p-1}$. Show that for all characters χ of G, the value $\chi(g)$ is an integer, and $\chi(g) \equiv \chi(1) \bmod p$.

Problem 4. *Optional.* Try and narrow down the possibilities for the degrees of the three irreducible representations of G that arose at the end of Problem 2.

To do this, study the characters of the group N of order 55. To get going, recall that we now have

$$\psi_i^G = \phi_1 + \phi_2 + \phi_3, \tag{1}$$

where the ϕ_n are distinct irreducible characters of G. (The restriction of ϕ_n to N is a character of N having an additional property: if h and g are elements of N which are conjugate in G.)

Let g_5 be a representative of the single conjugacy class of elements of order 5 in G, and let g_{11} and g'_{11} be representatives of the two conjugacy classes of elements of order 11. For $n = 1, 2, 3$, let

$$d_n = \phi_n(1), \qquad \alpha_n = \phi_n(g_5), \qquad \beta_n = \phi_n(g_{11}) + \phi_n(g'_{11}).$$

Use the method of the previous question to show that α_n and β_n are integers. Use the orthogonality relations on the columns to get a bound on the size of the α_n, and try and bound the β_n similarly. This, together with the relation

$$\langle \psi_i^G, \phi_n \rangle = 1,$$

eventually leads to a rather short list of possibilities. Try and get to the unique answer: the degrees are 44, 45 and 55.

Peter Kronheimer