

## Math 126, sheet 4

March 3, 2000

All representations here are over the field  $\mathbf{C}$ . All groups called  $G$  are finite.

**Problem 1.** Here is something we might have done earlier, to help with the question about groups of order  $p^3$ . The *commutator subgroup*  $G' \leq G$  is the subgroup generated by the commutators  $ghg^{-1}h^{-1} \in G$ . Verify the following standard facts about the commutator subgroup.

(a) The commutator subgroup is a normal subgroup. (b) The quotient  $G/G'$  is abelian. (c) If  $\psi : G \rightarrow H$  is a homomorphism, then the image of  $\psi$  is abelian if and only if  $G'$  is contained in the kernel.

Deduce that the number of 1-dimensional representations of  $G$  is equal to the order of  $G/G'$ .

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**Problem 2.** Suppose  $G$  is a non-abelian group of order  $pq$ , where  $p$  and  $q$  are prime and  $p > q$ .

- (a) What are the degrees of the irreducible representations of  $G$ ?
  - (b) Show (using the answer to the previous part) that  $q$  divides  $p - 1$ .
  - (c) What is the order of  $G'$ ?
  - (d) How many conjugacy classes does  $G$  have?
  - (e) Show that no element of order  $p$  can commute with any element of order  $q$  and deduce that the center  $Z(G)$  is trivial.
  - (f) What are the sizes of the conjugacy classes?
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**Problem 3.** Let  $G$  be a non-abelian group of order 21. Show that  $G$  has two conjugacy classes of size 3, which together with the identity element make up the group  $G'$  of order 7. Show that  $G$  has three 1-dimensional characters. Work out what the corresponding rows of the character table look like.

Show that  $G$  has two 3-dimensional characters. Use the orthogonality relations etc. to find what the corresponding rows of the table are.

Are the 3-dimensional characters of  $G$  obtainable as induced characters  $\xi^G$ , from some characters  $\xi$  of  $G'$ ?

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**Problem 4.** Let  $G$  be  $S_4$  and  $H$  the subgroup generated by  $x = (1\ 2\ 3\ 4)$ . Let  $B$  be the 1-dimensional representation of  $H$  with  $B(x) = \sqrt{-1}$ . Let  $\xi$  be its character.

Write out the character table of  $S_4$  (we did it in class), and for each irreducible character  $\chi$  of  $S_4$ , calculate  $\langle \chi_H, \xi \rangle_H$ . Use Frobenius reciprocity to deduce how the induced character  $\xi^G$  decomposes into irreducibles. (There are, of course, other ways to go about it.)

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**Problem 5.** Show that  $(1 + \sqrt{13})/2$  is an algebraic integer and that  $(1 + \sqrt{11})/2$  is not.

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