Problem Set 2

1. $\langle \chi, \chi_{triv} \rangle = \chi(1)/|G|$ equals the multiplicity of the trivial representation in χ, which must be an integer. Thus $|G|$ divides $\chi(1)$.

2. If all the irreducible representations are 1-dimensional, then since $|G|$ equals the sum of the squares of the dimensions of the irreducibles, there must be $|G|$ irreducibles and therefore $|G|$ conjugacy classes. This means that every element in G is its own conjugacy class, which means that every element in G is central in G.

3. Let χ_r be labelled as in page 62 of Ledermann. Simple calculation reveals that $\chi_1\chi_i = \chi_i$ and $\chi_5\chi_i = \chi_5$ for all i, $\chi_2^2 = \chi_3^2 = \chi_4^2 = \chi_1$, $\chi_2\chi_3 = \chi_4$, $\chi_2\chi_4 = \chi_3$, $\chi_3\chi_4 = \chi_2$, and $\chi_5^2 = \chi_1 + \chi_2 + \chi_3 + \chi_4$. The last equation can be obtained either by inspection or by calculating $\langle \chi^2, \chi_i \rangle$ for each i.

4. Using labelling from page 79 of Ledermann, $\chi = \chi_2$. Plug and chug to get $\langle \chi^2, \chi_1 \rangle = \langle \chi^2, \chi_2 \rangle = \langle \chi^2, \chi_3 \rangle = \langle \chi^2, \chi_4 \rangle = \langle \chi^2, \chi_5 \rangle = 1$. Therefore χ^2 has 5 summands.

5. Let A, B be representations on G and H, respectively, with characters ξ and η. Then it is simple to check that $C(g,h) = A(g)B(h)$ is a representation of $G \times H$ with character $\chi(g,h) = \xi(g)\eta(h)$. Now pick characters ξ' of G and η' of H. Let $\chi(g,h) = \xi(g)\eta(h)$ and $\chi'(g,h) = \xi'(g)\eta'(h)$. Then

$$\begin{aligned}
\langle \chi, \chi' \rangle &= \sum_{(g,h)\in G\times H} \xi(g)\eta(h)\overline{\xi'(g)\eta'(h)} \\
&= \sum_{g\in G} \xi(g)\overline{\xi'(g)} \sum_{h\in H} \eta(h)\overline{\eta'(h)} \\
&= \langle \xi, \xi' \rangle \langle \eta, \eta' \rangle
\end{aligned}$$

From this it follows that the if ξ and η are irreducible, then so is χ. Furthermore, it shows that distinct pairs of irreducibles (ξ_i, η_j) gives rise to distinct χ_{ij}'s. If we sum the squares of the dimensions of these χ_{ij}'s, we get $\sum_{i,j} \chi_{ij}(1)^2 = \sum_{i,j} \xi_i(1)^2\eta_j(1)^2 = \sum_i \xi_i(1)^2 \sum_j \eta_j(1)^2 = |G||H| = |G \times H|$. Therefore this list of irreducibles must be exhaustive.

6. Let χ_A be the character of a degree 2 representation A. Note that $1 = \langle \chi_A, \chi_A \rangle = \langle \chi_A\overline{\chi_A}, \chi_{triv} \rangle$. This shows that $A \otimes \overline{A}$ =(trivial representation)$\oplus B$ where B is degree 3 and does not contain the trivial representation. Then since there are no other degree 1 representations, it follows that B must be irreducible.

Since A is a degree 2 irreducible, G has even order and therefore there exists an element $g_0 \in G$ of order 2. So $A(g_0)$ must have eigenvalues ± 1. Now observe that $g \mapsto \det(A(g))$ is a degree 1 representation of G and is therefore trivial. Thus $\det(A(g_0)) = 1$, so $A(g_0) = \pm I$. It follows that $\chi_B(g_0) = 3$. So $B(g_0) = I$ and thus B is not faithful.

Now suppose that there is a non-Abelian finite simple group G contained in GL(2,C). Then there is a faithful degree 2 representation of G. Suppose D is a degree 1 representation of G. Since G is simple, ker D must be 1 or G. If ker D=1, then G injects into the complex numbers which is impossible since G is non-Abelian. So ker $D = G$ and thus D is the trivial representation. By the previous argument, there exists a non-faithful degree 3 irreducible representation B of G. Thus ker $B = G$ since G is simple. But this is impossible since such a representation is reducible.

7. First, the dimensions of the irreducibles divide $|G| = p^3$, so they may only be $1, p, p^2, p^3$. Second, $|G| = p^3$ is the sum of the squares of these dimensions. Thus we have $p^3 = a_0 + a_1 p^2$ where a_i is the number of degree p^i irreducibles. Clearly, $p^2 | a_0$ and $a_0 > 0$. The number of conjugacy classes equals the number of irreducibles which is $a_0 + a_1 = a_0 + p - \frac{a_0}{p^2} = kp^2 + p - k$ for some $1 \leq k$.

The size of a conjugacy class divides p^3, so the possible sizes are $1, p, p^2$. The number of conjugacy classes of size 1 is equal to the order of the center of G, which properly divides $|G|$ since G is non-Abelian. Thus there are at most p^2 conjugacy classes of size 1. Clearly, there are at most p^2 conjugacy classes of size greater than 1. So the total number of conjugacy classes is at most $2p^2$. Combining this with the previous paragraph shows that $k = 1$, so that the number of conjugacy classes is $p^2 + p - 1$.