Math 126, sheet 2

February 18, 2000

All representations here are over the field **C**. *We shall prove later a result we stated in class: the degree of an irreducible representation divdides the order of the group. You may assume this result when answering the last two questions.*

Problem 1. Let V be a representation of a finite group G, and suppose its character χ has $\chi(g) = 0$ for $g \neq 1$. Show that $\chi(1)$ is an integer multiple of $|G|$.

Problem 2. Suppose that all the irreducible representations of a finite group G are 1-dimensional. Show that G is abelian.

Problem 3. Let Q be the quaternion group of order 8. For each pair V, V' of irreducible representations of Q (including the cases $V = V'$), calculate the decomposition of $V \otimes V'$ into irreducibles by using the characters.

Problem 4. Let V be the 5-dimensional permutation representation of the group A_5, and write $V = U \oplus W_4$ as usual, where U is the 1-dimensional trivial summand. Let χ be the character of W_4. Consider the representation $W_4 \otimes W_4$ with character $\chi\chi = \chi^2$. How many summands are there in the decomposition of $W_4 \otimes W_4$ into irreducibles?

Problem 5. Let G and H be finite groups. If ξ and η are characters of G and H respectively, show that the function $\chi(g, h) = \xi(g)\eta(h)$ is a character of $G \times H$. If the characters ξ and η correspond to irreducible representations, show that the representation corresponding to ξ is irreducible also. Show that all irreducible representations of $G \times H$ arise in this way.

Problem 6. Suppose the group G has no non-trivial 1-dimensional representations and has an irreducible 2-dimensional matrix representation A.

Show that G must have an irreducible 3-dimensional representation B.

Show that the representation B is not faithful. (Consider elements of G of order 2, perhaps.)

Deduce that there is no non-abelian finite simple group contained in $GL(2, \mathbf{C})$.

Problem 7. Let G be a non-abelian group of order p^3, where p is prime. Show, using only the character theory, that G has p^2 representations of degree 1 and $(p-1)$ irreducible representations of degree p. Deduce that G has exactly $p^2 + p - 1$ conjugacy classes.

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