

Problem Set 1

1. Let U be the subspace spanned by the elements gv where $g \in G$. Since there are $|G|$ such elements, $\dim U \leq |G|$. And since each generator of gv of U is sent to another generator of U under the action of any $h \in G$, it follows that U is a G -module.

2. Let V be an irreducible G -module. Choose a nonzero $v \in V$ and construct U as in the previous problem. As before, $\dim U \leq |G|$. Suppose $\dim U = |G|$. Then the elements gv for $g \in G$ form a basis for V . So $\sum_{g \in G} gv \neq 0$, and by the same arguments made in class, this element is invariant under the action of any $h \in G$. So if $|G| > 1$, this element spans a proper G -submodule of V , which is impossible by assumption. Thus $\dim U < |G|$.

3. Let $A : G \rightarrow \text{GL}(V)$ be an irreducible complex representation over an Abelian group G . For any $g \in G$, $A(g) : V \rightarrow V$ and $A(g)A(h) = A(h)A(g)$ for all $h \in G$. Thus $A(g)$ is a G -module homomorphism, and so $A(g) = \lambda_g I$ for some λ_g by the Corollary to Schur's Lemma. Since this is true for all $g \in G$, it is clear that any 1-dimensional subspace of V is a G -module. This is a contradiction unless V was 1-dimensional to begin with.

Now let A be a matrix with $A^k = I$. Then if x is a generator of the cyclic group C_k , then $B(x^i) = A^i$ gives a representation of C_k , which is Abelian. Therefore this representation is a direct sum of 1-dimensional representations. This gives the desired eigenspace decomposition of A .

4. In general, if G is a group presented by generators x_1, \dots, x_n and relations, then every representation A can be described its values on the generators (that is, the values of $A(x_1), \dots, A(x_n)$). Furthermore, given arbitrary values of $A(x_1), \dots, A(x_n)$, these values will give rise to a representation of G iff they satisfy the same relations as x_1, \dots, x_n .

Let A be a 1-dimensional representation of D_{2k} . Then $A(x)^k = 1$, $A(y)^2 = 1$, and $A(x^{-1}) = A(y)A(x)A(y) = A(x)$. So $A(y) = \pm 1$; and $A(x) = 1$ if k is odd, or $A(x) = \pm 1$ if k is even. All of these possibilities give rise to representations by the discussion above. So we have 2 degree 1 representations when k is odd, and we have 4 when k is even.

Now let A be an irreducible degree 2 representation of D_{2k} . As before, $A(x)^k = I$, $A(y)^2 = I$, and $A(x^{-1}) = A(y)A(x)A(y)$. By Problem 3, we can work in a basis which diagonalizes $A(x)$. Furthermore, the eigenvalues of $A(x)$ must be k -th roots of unity. Then $A(x^{-1}) = \overline{A(x)}$. By the third relation, $A(x)$ is similar to $\overline{A(x)}$, so they have the same eigenvalues. Thus

$$A(x) = \begin{pmatrix} e^{2\pi im/k} & 0 \\ 0 & e^{-2\pi im/k} \end{pmatrix}$$

for some m . By the third relation again, $A(y)$ interchanges the two eigenspaces of $A(x)$. And since $A(y)^2 = I$, we can make a change of basis (one that doesn't

affect $A(x)$ in order to get

$$A(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now if $e^{2\pi im/k} = \pm 1$, then this representation is reducible. Also, choosing m or $k - m$ gives equivalent representations. So we are left with $0 < m < \frac{k}{2}$. By our construction, this list certainly *includes* all irreducible degree 2 representations. To see that these representations are irreducible, observe that $A(x)$ and $A(y)$ have different invariant subspace decompositions. To see that these representations are inequivalent, just look at the trace of $A(x)$. Hence there are exactly $\lfloor \frac{k-1}{2} \rfloor$ degree 2 irreducible representations.

To finish our proof that these irreducible representations are all of the irreducible representations, we just need to show that there are no degree 3 or higher irreducible representations.

Let A be any representation of D_{2k} with $\dim V \geq 3$. Let $v \in V$ be any eigenvector of $A(x)$ with eigenvalue λ . We claim that the span of v and $A(y)v$ forms a D_{2k} -invariant subspace of V . It's clearly $A(y)$ -invariant. We just need to show that it's $A(x)$ -invariant. $A(x)A(y)v = A(y)\overline{A(x)v} = \overline{\lambda}A(y)v$ by the third relation plus the arguments we made in the degree 2 case. This completes the proof.

Verifying the theorem is simple except for determining the conjugacy classes of D_{2k} . For k odd, the conjugacy classes are

$$\{1\}, \{y, yx, \dots, yx^{k-1}\}, \{x, x^{k-1}\}, \{x^2, x^{k-2}\}, \dots, \{x^{\frac{k-1}{2}}, x^{\frac{k+1}{2}}\}$$

For k even, the conjugacy classes are

$$\{1\}, \{y, yx^2, \dots, yx^{k-2}\}, \{yx, yx^3, \dots, yx^{k-1}\}, \{x, x^{k-1}\}, \{x^2, x^{k-2}\}, \dots, \{x^{\frac{k}{2}-1}, x^{\frac{k}{2}+1}\}, \{x^{\frac{k}{2}}\}$$

5. By the same argument used in Problem 3, any element of Z , the center of G , must act as multiplication by a scalar. Since the subgroup Z is finite, this scalar must be a $|Z|$ -th root of unity. But since the representation is faithful, different elements of Z must act as a different $|Z|$ -th roots of unity. Therefore the representation restricted to Z gives an isomorphism between Z and the $|Z|$ -th roots of unity.