IRREDUCIBLE REPRESENTATIONS OF $GL_2(\mathbb{F}_q)$

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1. Preliminaries

The group $G = GL_2(\mathbb{F}_q)$ is not only the slightly overweight cousin of $GL_2(\mathbb{C})$ (it is a "Flabby" S_2) but also a tremendously interesting finite group. It is both the automorphism group of the vector space of dimension 2 over the field $\mathbb{F}_q$ and a finite group of order $q(q-1)(q^2-1)$, so naturally, representation theorists would like very much to understand its representations. We follow the exposition in [1].

Because the representations of $GL_2(\mathbb{F}_q)$ of dimension higher than 1 are difficult to describe, we need to better describe the structure of G. Probably the most important subgroup of G is B, the Borel subgroup. B, together with the non-trivial permutation matrix w allow G to be decomposed via the Bruhat decomposition. In order to find the representations of G, we will induce representations from B. B contains the group P of "shears," matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$, in addition to the group D of diagonal matrices and $Z = Z(G)$, the scalar multiples of the identity. Finally, B contains a copy of $\mathbb{F}_q^+$, the group U of matrices of the form $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$.

We are now in a position to begin describing the representation of G, but how many are there? This question is equivalent to the question of the number of conjugacy classes of G, so we begin by computing this. Linear algebra tells us that two matrices in GL_2 are conjugate iff they satisfy the same minimal polynomial. Thus, we can determine the conjugacy classes by looking at the minimal polynomial of an arbitrary matrix. Since G is the group of 2×2 matrices, the characteristic polynomial is the minimal polynomial whenever the matrix is not a scalar multiple of the identity, our job is actually quite easy. We classify the characteristic polynomial of a matrix A based on its roots:

If the roots α_i are equal (and the minimal polynomial is not the characteristic polynomial), then A is $\alpha_i \cdot I$

If the roots α_i are equal and the minimal polynomial is the characteristic polynomial, then A is conjugate to $\left[\begin{smallmatrix} \alpha_1 & 1 \\ 0 & \alpha_1 \end{smallmatrix}\right]$

If the roots α_i are unequal, then A is conjugate to $\left[\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right]$

If the roots are not in the field $\mathbb{F}_q$, then they are in the unique quadratic extension $\mathbb{F}_{q^2}$. If $\bar{\cdot}$ denotes the conjugate of an element of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, then $\mathrm{N}(\alpha) = \alpha\bar{\alpha} \in \mathbb{F}_q$, and $\mathrm{Tr}(\alpha) = \alpha + \bar{\alpha} \in \mathbb{F}_q$, and clearly, A is conjugate to $\left[\begin{smallmatrix} 0 & -\mathrm{N}(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{smallmatrix}\right]$, $\alpha \in \mathbb{F}_{q^2} - \mathbb{F}_q$.

So, how many conjugacy classes are there? Clearly, any given matrix in G falls in one of the above conjugacy classes, so we need only find the number of conjugacy classes of each type. There are $q-1$ conjugacy classes of the first two types (1 for each possible $\alpha \in \mathbb{F}_q^\times$), $\frac{1}{2}(q-1)(q-2)$ conjugacy classes of the third type, and $\frac{1}{2}(q^2-q)$ of the fourth type. Thus, there are exactly q^2-1 conjugacy classes.

Now that we know the number of representations, we can begin finding them. The easiest place to start is with the representations of B. Since $B = Z \times P$, we can find the irreducible representations of B by finding those of Z and P. Z is isomorphic to $\mathbb{F}_q^\times$, the cyclic group of order $q-1$, and so it has $q-1$ irreducible representations $\rho_i\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$. P clearly also has $q-1$ 1-dimensional irreducible representations $\rho^j = \rho_j \circ \det$. Thus, B has $(q-1)^2$ 1-dimensional characters $\rho_i^j = \rho_i \otimes \rho^j$.

P has one more irreducible representation. Fix a non-trivial representation $\psi\colon \mathbb{F}_q^+ \to \mathbb{C}^\times$, and for each $a \in \mathbb{F}_q^\times$, define a representation of U by

$$\psi_a(u) = \psi(\alpha u \alpha^{-1}), \quad \alpha = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

The representation ψ induces a representation on P which satisfies the interesting property that

$$\operatorname{Res}_U^P \operatorname{Ind}_U^P(\psi) = \bigoplus_{a \in \mathbb{F}_q^\times} \psi_a$$

Frobenius reciprocity shows then that $\operatorname{Ind}_U^P(\psi)$ is irreducible, and the direct sum decomposition of the restriction shows that it is of dimension $q-1$. Thus,

$$\phi = \operatorname{Ind}_U^P(\psi)$$

is the final irreducible representation of P (the sums of the squares of the dimensions of the irreducible representations is $|P|$).

We lift this representation to B to find the final $q-1$ representations of B:

$$\phi_i = \rho_i \otimes \phi$$

2. Inducing Representations of G from B

First, we observe that $[G : B] = q+1$, so

$$\dim(\operatorname{Ind}_B^G(\rho_i^j)) = q+1$$

We begin by associating to each 1-dimensional representation σ of B an ordered pair of 1-dimensional representations of $\mathbb{F}_q^\times$ (σ_1, σ_2), such that

$$\sigma\left(\begin{bmatrix} a & * \\ 0 & b \end{bmatrix}\right) = \sigma_1(a)\sigma_2(b)$$

That this can be done is immediate from the fact that $B/U = D$ and $D \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$.

For each $\sigma = (\sigma_1, \sigma_2)$, we also define σ_w by

$$\sigma_w = (\sigma_2, \sigma_1)$$

Thus,

$$\sigma_w(d) = \sigma(wdw^{-1}) \quad \forall d \in D$$

Using this characterization of a 1-dimensional representation, we analyze the induced representations. Since it provides a very nice classification tool, we construct the Jacquet Module of a representation ρ of G:

$$\mathrm{J}(V_\rho) = \{v \in V_\rho | \rho(u)(v) = v, \ \forall u \in U\}$$

Clearly,

$$\mathrm{J}\left(\bigoplus_{n=1}^{m} V_{\rho_n}\right) = \bigoplus_{n=1}^{m} \mathrm{J}(V_{\rho_n})$$

Theorem 1. *If $\hat{\rho}$ is induced from a 1-dimensional representation ρ of B, then* $\dim(\mathrm{J}(V_{\hat{\rho}})) = 2$.

The proof of this is actually surprisingly simple, and since the line of reasoning is fairly indicative of the reasoning used for most theorems pertaining to representations of $GL_2(\mathbb{F}_q)$, we present it here:

Proof. $V_{\hat{\rho}}$ is the set of all functions $f\colon G \to \mathbb{C}$ satisfying

$$f(bg) = \rho(b)f(g), \quad \forall b \in B, g \in G.$$

If a vector f is in $\mathrm{J}(V_{\hat{\rho}})$, then f also satisfies the relation

$$f(bu) = f(b)$$

Thus,

$$f(b) = \rho(b)f(1) \quad \text{and} \quad f(bwu) = \rho(b)f(w)$$

The Bruhat decomposition of $G = B\coprod(BwU)$ shows that any f is determined uniquely by its value on 1 and w. Thus, $\mathrm{J}(V_{\hat{\rho}}) = \langle f_1, f_w \rangle$, where $f_1(1) = 1 \quad f_1(w) = 0$, $f_w(1) = 0 \quad f_w(w) = 1$. ⊠

This theorem, in conjunction with the following, shows that $\mathrm{Ind}_B^G(\rho_i^j)$ has at most two irreducible components.

Theorem 2. $\mathrm{J}(V_{\hat{\rho}}) \neq 0$ *iff* ρ *is a direct summand of* $\mathrm{Ind}_B^G(\rho_i^j)$ *for some* i *and* j*.*

A simple dimension argument shows that $\mathrm{Ind}_B^G(\rho_i^j)$ has at most 2 irreducible components:

Proof. Assume $\rho = \mathrm{Ind}_B^G(\rho_i^j) = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n$. Since each ρ_i is a direct summand of ρ, we know that $\mathrm{J}\left(V_{\rho_i}\right) \neq 0$. Thus,

$$\mathrm{J}\left(V_\rho\right) = \mathrm{J}\left(V_{\rho_1}\right) \oplus \cdots \oplus \mathrm{J}\left(V_{\rho_n}\right)$$

has dimension greater than or equal to n. However, by the previous theorem, $\dim\left(\mathrm{J}\left(V_\rho\right)\right) = 2$, so $n \leq 2$ ⊠

The introduction of the description of a representation of B by an ordered pair of representations of $\mathbb{F}_q^\times$ can now be used to determine exactly when a representation has 2 irreducible components:

Theorem 3. *If* ρ *is a 1-dimensional representation of* B*, then the* $q+1$ *dimensional representation* $\mathrm{Ind}_B^G(\rho)$ *is reducible iff* $\rho = \rho_w$*. If it is reducible, then it has a 1-dimensional component and an irreducible* q*-dimensional component.*

The proof is lengthy and therefore will be omitted.

This amazing theorem not only classifies $\mathrm{Ind}_B^G(\rho)$ for all 1-dimensional representation of B but also tells us how many there are of each dimension:

$q-1$	1-dimensional representations ($\rho_i \circ \det$)
$q-1$	q-dimensional representations
$\frac{1}{2}(q-1)(q-2)$	$(q+1)$-dimensional representations ($\mathrm{Ind}_B^G(\rho) = \mathrm{Ind}_B^G(\rho_w)$).

3. Cuspidal Representations

The cuspidal representations of G are those whose Jacquet module is the zero-module. A more useful characterization is that ρ is cuspidal iff

$$\mathrm{Res}_P^G(\rho) = \phi$$

There are $\frac{1}{2}(q^2 - q)$ cuspidal representations, so simple calculation shows that if ρ is cuspidal, then

$$\dim(\rho) = q - 1$$

We will use these two facts extensively in the construction of the cuspidal representations of G.

We also, in order to construct representations easily, need the "generators and relations" description of G:

Let

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s = \omega z$$

Then

$$G = \langle B, \omega \rangle$$

subject to the relations

1. $\omega \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right] \omega$;
2. $\omega^2 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$;
3. $s^3 = 1$.

Because the cuspidal representations are difficult to describe, we must approach them via a fairly circuitous route. We describe a certain class of 1-dimensional representations of $\mathbb{F}_{q^2}^\times$, the non-decomposable representations, and find a way to construct a representation that corresponds to each of these.

3.1. Non-decomposable 1-dimensional Representations of $\mathbb{F}_{q^2}^\times$.

$\mathbb{F}_{q^2}^\times$ is a cyclic group of order $q^2 - 1$, so it has $q^2 - 1$ 1-dimensional representations ν. We say that a representation ν of $\mathbb{F}_{q^2}^\times$ is **decomposable** iff

$$\nu(\alpha) = \rho_i(\mathrm{N}(\alpha)).$$

Relatively simple computation shows that ν is decomposable if and only if $\nu(\alpha) = \nu(\bar{\alpha})$.

In order to describe the cuspidal representations, we associate to each non-decomposable representation a cuspidal representation.

Since the dimension of the cuspidal representations is $q - 1$, we may pick any $(q-1)$-dimensional space. It is convenient to use the space V of all functions from $\mathbb{F}_q^\times$ into $\mathbb{C}$. We also want the restriction of the representation to P to be ϕ. With this in mind, we begin defining a representation ρ_ν.

$$\rho_\nu \left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) (f)(x) = \psi(bx) f(ax)$$

Furthermore, it is nice if ρ_ν coincides with ν on Z:

$$\rho_\nu \left(\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \right) (f)(x) = \nu(d) f(x)$$

Thus,

$$\rho_\nu \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) (f)(x) = \nu(c) \psi(bc^{-1}x) f(ac^{-1}x)$$

Under this definition, ρ_ν is a homomorphism from B into $GL_2(V)$.

The difficulties arises when we try to define $\rho_\nu(\omega)$ in such a way that ρ_ν preserves the relations on G. We first define a function $j \colon \mathbb{F}_q^\times \to \mathbb{C}$ by

$$j(u) = \frac{1}{q} \sum_{\mathrm{N}(t)=u} \psi(t + \bar{t}) \nu(t)$$

With this defined, we can define $\rho_\nu(\omega)$ to be

$$\rho_\nu(\omega)(f)(x) = \sum_{y \in \mathbb{F}_q^\times} \nu(y^{-1}) j(xy) f(y)$$

That ρ_ν preserves the relations 1–3 is tedious, and we shall therefore only show that it satisfies relation 1.

$$\rho_\nu \left(\omega \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) f(x) = \rho_\nu(\omega)(\nu(b) f(ab^{-1}x)) =$$

$$\sum_{y \in \mathbb{F}_q} \nu(by^{-1}) j(xy) f(ab^{-1}y).$$

Making the substitution $z = ab^{-1}y$, we transform the sum into

$$\sum_{z \in \mathbb{F}_q} \nu(az^{-1}) j(ba^{-1}xz) f(z) =$$

$$\left(\rho_\nu \left(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \right) \right) \left(\sum_{z \in \mathbb{F}_q} \nu\left(z^{-1}\right) j\left(xz\right) f(z) \right) = \rho_\nu \left(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \omega \right) f(x).$$

Since we know how ρ_ν acts on B and on ω, we know how it acts on G. Just for the interested reader, we present how ρ_ν acts on an arbitrary element in $B\omega U$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b - ac^{-1}d & -a \\ 0 & -c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & c^{-1}d \\ 0 & 1 \end{bmatrix}$$

So,

$$\rho_\nu \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \sum_{y \in \mathbb{F}_q^\times} \left(-\frac{1}{q} \psi \left(\frac{ax + dy}{c} \right) \sum_{\mathrm{N}(u) = xy^{-1} \det(g)} \psi \left(-\frac{y}{c} (\mathrm{Tr}(u)) \right) \nu(u) \right)$$

By construction, $\mathrm{Res}_P^G(\rho_\nu) = \phi$, so ρ_ν is cuspidal. Routine, but messy, analysis also shows that $\rho_\nu \cong \rho_{\hat{\nu}}$ iff $\nu(\alpha) = \hat{\nu}(\bar{\alpha})$. Thus, there are exactly $\frac{1}{2}(q^2 - q)$ irreducible representations “induced” by non-decomposable representations.

4. Irreducible Representations of G

We present the irreducible representations in a clear, easy to read format. If we let ρ_i^j be the representation of B correlating to the ordered pair (ρ_i, ρ_j), then G has:

$(q-1)$	1-dimensional representations, given by $\rho_i \circ \det$;
$\frac{1}{2}(q^2 - q)$	$(q-1)$-dimensional representations ρ_ν;
$(q-1)$	q-dimensional representations σ_i, given by $\sigma_i \oplus \rho_i \circ \det = \mathrm{Ind}_B^G(\rho_i^i)$;
$\frac{1}{2}(q-1)(q-2)$	$(q+1)$-dimensional representations given by $\mathrm{Ind}_B^G(\rho_i^j)$.

References

1. Ilya Piatetski-Shapiro, *Complex representations of $GL(2, K)$ for finite fields K*, Contemporary Mathematics, vol. 16, pp. 1–43, American Mathematical Society, Providence, RI, 1980.