SCHUR FUNCTORS
OR
THE WEYL CONSTRUCTION
MATH 126 FINAL PAPER

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1. Introduction

We will consider a group G with a representation on a vector space V. The vector space should be a finite dimensional complex vector space. The group need not be finite; later in this paper, the most important example will be $G = GL(V)$. The representation, which will be denoted by just $g, v \mapsto gv$, need not be irreducible for what follows, although that is most important example. Essentially all of the mathematics in this paper is based on [1, pp. 44–47, 75–78, 84–86, 222, 231–232, 471–475]. In addition to these pages, the remaining parts of lectures 4, 6, and 15, and appendices A and B of [1] are also relevant to this topic.

We consider the tensor power,

$$V^{\otimes n} = V \otimes V \otimes \cdots \otimes V.$$

We have a representation of G on this space given by $g(v_1 \otimes \cdots \otimes v_n) = (gv_1) \otimes \cdots \otimes (gv_n)$. Our goal is to break this up into irreducible representations of G. Of course, not knowing anything more about G, we cannot actually do this, but we can break up $V^{\otimes n}$ into some subrepresentations, which is progress, at least; and if $G = GL(V)$ these subrepresentations, the Weyl modules, also known as Schur functors, will turn out to be irreducible.

We start by defining some well-known[1] subspaces of the tensor power. (We will see that these subspaces are invariant under S_n.)

Definition 1.1. The nth symmetric power $\operatorname{Sym}^n V$ of a vector space V is the subspace of $V^{\otimes n}$ spanned by

$$\left\{ \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \;\middle|\; v_i \in V \right\}.$$

Definition 1.2. The nth alternating power $\operatorname{Alt}^n V$ of a vector space V is the subspace of $V^{\otimes n}$ spanned by

$$\left\{ \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \;\middle|\; v_i \in V \right\}.$$

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[1] This means that I had heard of them before taking this course, and is true even though I had originally learned of them from [1, 472–475]

It follows by straightforward calculation that these are invariant subspaces of $V^{\otimes n}$. Except in trivial cases, $V^{\otimes n}$ is therefore reducible: if $n \geq 2$ and $\dim V \geq 2$, and v and w are linearly independent elements of V, then $v \otimes \cdots \otimes v \in \operatorname{Sym}^n V$, so $\operatorname{Sym}^n V$ is nontrivial, but $v \otimes w \otimes \cdots \otimes w \notin \operatorname{Sym}^n V$, so $\operatorname{Sym}^n V$ is a proper subspace of $V^{\otimes n}$.

To find a more obviously generalizable definition of $\operatorname{Sym}^n V$ and $\operatorname{Alt}^n V$, we introduce a representation of S_n:

Proposition 1.3. Defining

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \tag{1}$$

and extending linearly to the rest of $V^{\otimes n}$ yields a (right) representation of S_n. This commutes with the (left) representation of G, in the sense that $(g\omega)\sigma = g(\omega\sigma)$ for $\omega \in V^{\otimes n}$.

Proof. It is clear, by direct calculation, that $(v_1, \dots, v_n) \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ defines a multilinear map from $V \times \cdots \times V$ to $V \otimes \cdots \otimes V$. By the definition of tensor product (see [1, pp. 471–472]), there is a unique linear map from $V \otimes \cdots \otimes V$ to $V \otimes \cdots \otimes V$ satisfying (1). That $(\omega\sigma)\tau = \omega(\sigma\tau)$ for $\sigma, \tau \in S_n$ is easy calculation when $\omega = v_1 \otimes \cdots \otimes v_n$, and follows by linearity for arbitrary $\omega \in V^{\otimes n}$. The statement that the two representations commute is just as easy to verify. □

With this representation of S_n, we can usefully introduce the group algebra. Since the action of G commutes with the action of S_n, it also commutes with the action of $\mathbb{C}[S_n]$. We can consider $V^{\otimes n}$ to be a (right) $\mathbb{C}[S_n]$-module, and we are now able to redefine the symmetric and alternating powers:

With this representation of S_n in hand, we can usefully introduce the group algebra $\mathbb{C}[S_n]$. We can consider $V^{\otimes n}$ to be a (right) $\mathbb{C}[S_n]$-module. Since the action of G commutes with the action of S_n, it also commutes with the action of $\mathbb{C}[S_n]$. We are now able to redefine the symmetric and alternating powers:"

Proposition 1.4.

$$\operatorname{Sym}^n V = V^{\otimes n} \cdot \sum_{\sigma \in S_n} e_\sigma$$

$$\operatorname{Alt}^n V = V^{\otimes n} \cdot \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_\sigma$$

Proof. Since the proofs of the above two equations are so similar, we prove only the second. Let $c = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_\sigma$. It follows straight from the definition that $\operatorname{Alt}^n V = \operatorname{Span}\left\{(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})c \mid v_i \in V\right\}$. We can move the c out of the expression for the set, just by the definition of multiplying a set by an element, and we can move it out the span by linearity of the action of c. Thus, $\operatorname{Alt}^n V = \left(\operatorname{Span}\left\{v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid v_i \in V\right\}\right) c$, which is just $V^{\otimes n} c$. □

We now see how we might generalize $\operatorname{Sym}^n V$ and $\operatorname{Alt}^n V$. We choose some element c of the group algebra, other than the two used above, and consider $V^{\otimes n} c$.

Proposition 1.5. The subspace $V^{\otimes n} c$ of $V^{\otimes n}$ is invariant under the action of G, for any $c \in \mathbb{C}[S_n]$.

Proof. Any element of $V^{\otimes n} c$ can be expressed as ωc for some $\omega \in V^{\otimes n}$. Then $g(\omega c) = (g\omega)c \in V^{\otimes n} c$, since the two relevant actions commute, so $V^{\otimes n} c$ is invariant under G. □

We now have an embarrassment of riches. There are uncountably many choices for c. Many choices of different c's lead to the same $V^{\otimes n}c$, which is good; e.g., if G is finite, $V^{\otimes n}$ must break up into only finitely many irreducibles. We can now state our goal more precisely: Find finitely many c such that $V^{\otimes n}$ breaks up into a sum of the resulting subrepresentations.

2. The Construction

To clarify the logic of the upcoming construction, we introduce a new kind of tensor product, a tensor product over the algebra $\mathbb{C}[S_n]$, denoted $\otimes_{\mathbb{C}[S_n]}$. We take a right $\mathbb{C}[S_n]$-module R, such as $V^{\otimes n}$, and tensor it with a left $\mathbb{C}[S_n]$-module L, such as $\mathbb{C}[S_n]$.

Definition 2.1. Given a right $\mathbb{C}[S_n]$-module R, and a left $\mathbb{C}[S_n]$-module L, define $R \otimes_{\mathbb{C}[S_n]} L$ to be $R \otimes L$ modulo the relation $ra \otimes l \sim r \otimes al$, for $r \in R, l \in L$, and $a \in \mathbb{C}[S_n]$. (More formally, we quotient out by the relation that generates, i.e., we quotient out by the subspace spanned by $\{ra \otimes l - r \otimes al\}$.)

Of course, this can be defined over more general algebras than $\mathbb{C}[S_n]$, and many of the properties are the same, but we only need $\mathbb{C}[S_n]$.

Proposition 2.2. If R in the above definition is a (left) representation of G, and that representation commutes with the action of $\mathbb{C}[S_n]$, then $R \otimes_{\mathbb{C}[S_n]} L$ is also a (left) representation of G, defined by $g(r \otimes_{\mathbb{C}[S_n]} l) = (gr) \otimes_{\mathbb{C}[S_n]} l$.

Proof. The map $(r, l) \mapsto (gr) \otimes_{\mathbb{C}[S_n]} l$ is bilinear, so we have a well-defined linear map $r \otimes l \mapsto (gr) \otimes_{\mathbb{C}[S_n]} l$, which sends both $ra \otimes l$ and $r \otimes al$ to the same element, by the definition of $\otimes_{\mathbb{C}[S_n]}$ and by the factor the two actions commute. Therefore, the action of g given in the above proposition is well-defined. That it satisfies the other properties of a representation follows straight from the definition of the action in question. $\square$

Before we go on, we will prove one useful property of $\otimes_{\mathbb{C}[S_n]}$.

Proposition 2.3. The operation $\otimes_{\mathbb{C}[S_n]}$ distributes over $\oplus$, in that $R \otimes_{\mathbb{C}[S_n]} (L \oplus L') \cong (R \otimes_{\mathbb{C}[S_n]} L) \oplus (R \otimes_{\mathbb{C}[S_n]} L')$, not just as vector spaces but as representations of G.

Proof. To start, we show that $R \otimes_{\mathbb{C}[S_n]} (L \oplus L') \cong (R \otimes_{\mathbb{C}[S_n]} L) \oplus (R \otimes_{\mathbb{C}[S_n]} L')$, just as vector spaces. We know that $R \otimes (L \oplus L') \cong (R \otimes L) \oplus (R \otimes L')$ as vector spaces. Each of these spaces has to be quotiented out by an equivalence relation; we will write both relations in terms of the latter space, and compare. That is, $\omega \otimes a(m \oplus m') \sim \omega a \otimes (m \oplus m')$ maps to the relation $(\omega \otimes am) \oplus (\omega \otimes am') \sim (\omega a \otimes m) \oplus (\omega a \otimes m')$ on $(R \otimes L) \oplus (R \otimes L')$. This space has two relations, one for the first factor and one for the second, which generate the same relation as $\omega \otimes am \oplus \omega' \otimes a'm' \sim \omega a \otimes m \oplus \omega' a' \otimes m'$. Examining the spaces of vectors identified with zero shows these relations to be the same, so the quotient spaces are isomorphic. $\square$

Using this new kind of tensor product, we have another expression for $V^{\otimes n}c$.

Proposition 2.4. $V^{\otimes n}c \cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]c$, as representations of G.

Proof. First, we need to check that $\mathbb{C}[S_n]c$ is a left $\mathbb{C}[S_n]$-module, before the statement of the proposition even makes sense. But this is easy, since for $bc \in \mathbb{C}[S_n]c$, we have $a(bc) = (ab)c \in \mathbb{C}[S_n]c$ for any $a \in \mathbb{C}[S_n]$. That is, $\mathbb{C}[S_n]c$ is closed under left multiplication by elements of $\mathbb{C}[S_n]$ so it is a submodule of $\mathbb{C}[S_n]$ (where we are considering $\mathbb{C}[S_n]$ as a left-module over itself).

Any element of $V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]c$ can be expressed as a sum of elements of the form $\omega \otimes_{\mathbb{C}[S_n]} bc$, which is equal to $\omega b \otimes_{\mathbb{C}[S_n]} c$. A sum of elements of this form is also of the form $\omega \otimes_{\mathbb{C}[S_n]} c$. To define our isomorphism, we map this to ωc. Linearity, injectivity, and surjectivity of this map now follow straight from the definitions, provided the map is well-defined. To show this, we consider what the isomorphism does to an element written as $\omega \otimes_{\mathbb{C}[S_n]} bc$; this is equal to $\omega b \otimes_{\mathbb{C}[S_n]} c$, and so gets sent to ωbc. We can see by straightforward calculation that map from $V^{\otimes n} \times \mathbb{C}[S_n]c$ to $V^{\otimes n}c$ defined by $(\omega, bc) \mapsto \omega bc$ is bilinear, so we have a linear map $V^{\otimes n} \otimes \mathbb{C}[S_n]c \to V^{\otimes n}c$. This map sends $\omega a \otimes bc$ and $\omega \otimes abc$ to ωabc, so it is still a well-defined map when we quotient out by the defining relation of $\otimes_{\mathbb{C}[S_n]}$.

Since any element of $V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]c$ can be written in the form $\omega \otimes_{\mathbb{C}[S_n]} c$, we see that the action of g is just $g(\omega \otimes_{\mathbb{C}[S_n]} c) = (g\omega) \otimes_{\mathbb{C}[S_n]} c$. Our isomorphism clearly respects this. □

Having found one submodule of $\mathbb{C}[S_n]$ useful, we might as well look at them all. Just because S_n is a finite group, we know its group algebra breaks up into a sum of (finitely many) irreducible (left) submodules (call them M_λ, with λ ranging over some index set to be determined later), with multiplicities equal to their dimensions. Picking some nonzero $c_\lambda \in M_\lambda$, we see that $\mathbb{C}[S_n]c_\lambda$ is nonzero (since it contains $1c_\lambda$), contained in M_λ (since that's a submodule of $\mathbb{C}[S_n]$), and in fact equal to M_λ (since that's irreducible). So we now have

$$\mathbb{C}[S_n] \cong \bigoplus (\mathbb{C}[S_n]c_\lambda)^{\oplus m_\lambda} \tag{2}$$

as $\mathbb{C}[S_n]$-modules, with $m_\lambda = \dim \mathbb{C}[S_n]c_\lambda$. We are now ready to derive the main construction in one big equation. We have

$$\begin{aligned} V^{\otimes n} &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] \cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \bigoplus (\mathbb{C}[S_n]c_\lambda)^{\oplus m_\lambda} \\ &\cong \bigoplus (V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]c_\lambda)^{\oplus m_\lambda} \cong \bigoplus (V^{\otimes n}c_\lambda)^{\oplus m_\lambda}. \end{aligned} \tag{3}$$

The first step follows by Proposition 2.4 with $c = 1$. The other steps can all be justified in an obvious manner from previous propositions. We have thus proven

Theorem 2.5. Using the c_λ and m_λ that define the irreducibles of S_n as in (2), we have $V^{\otimes n} \cong \bigoplus (V^{\otimes n}c_\lambda)^{\oplus m_\lambda}$ as a sum of subrepresentations.

Note that we haven't used any particular properties of the representations of S_n. A construction of the c_λ (where λ ranges over all partitions of n) can be found in [1, 44–47].

3. $GL(V)$

We now set the group G of the previous sections to be $GL(V)$. This has an natural representation on V, given by $gv = g(v)$. This is irreducible, since given a proper nonzero subspace of V, and given nonzero v in the subspace and w in the complement of the subspace, some element of G satisfies $gv = w$, so the subspace is not invariant. We can now state the main theorem:

Theorem 3.1. The representations $V^{\otimes n}c_\lambda$ of $GL(V)$ are irreducible, unless they are zero.

We break the proof up into lemmas. Due to shortage of time, we omit the proofs.

Lemma 3.2. $V^{\otimes n}$ is a left B-module, where $B = \mathrm{Sym}^n(\mathrm{End}(V))$. This B is exactly the set of linear endomorphisms ϕ of $V^{\otimes n}$ which commute with the action of $\mathbb{C}[S_n]$, in that $\phi(\omega c) = \phi(\omega)c$ for $\omega \in V^{\otimes n}, c \in \mathbb{C}[S_n]$.

Lemma 3.3. An irreducible B-submodule of $V^{\otimes n}$ is also an irreducible $GL(V)$-subrepresentation.

Lemma 3.4. Write $V^{\otimes n} \cong \bigoplus U_i^{\oplus \nu_i}$, where the U_i are non-isomorphic irreducible representations of S_n. Then B is isomorphic (as an algebra) to $\bigoplus M_{\nu_i}(\mathbb{C})$, where $M_{\nu_i}(\mathbb{C})$ denotes the algebra of $\nu_i \times \nu_i$ matrices over $\mathbb{C}$.

Lemma 3.5. $\mathbb{C}[S_n]$ is isomorphic (as an algebra) to $\bigoplus M_{m_\lambda}(\mathbb{C})$, with $m_\lambda = \dim \mathbb{C}[S_n]c_\lambda$ being the multiplicity of $\mathbb{C}[S_n]c_\lambda$ in $\mathbb{C}[S_n]$.

Lemma 3.6. If $\mathbb{C}[S_n]c_\lambda$ is an irreducible (left) submodule of $\mathbb{C}[S_n]$, then $V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]c_\lambda$ is an irreducible (left) B-module.

Using Lemmas 3.3 and 3.6, we have proven the theorem. The result is quite good: we have constructed many irreducible representations of $GL(V)$. In fact (see [1, 4, 231–232] for a fuller statement), we have almost all of the irreducible representations: letting D_{-k} be just $\mathbb{C}$, endowed with the representation in which g acts as $(\det g)^{-k}$, any irreducible representation of $GL(V)$ can be expressed as $(V^{\otimes n}c_\lambda) \otimes D_{-k}$. (For proof, see [1, Lecture 15, especially p. 232].)

Reference

1. W. Fulton and J. Harris, *Representation theory: A first course*, Graduate Texts in Mathematics, no. 129, Springer-Verlag, New York, US, 1991.