MATH 126
SOLUTION SET 2

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1. To avoid confusion, I've changed the indices to range from 1 to n, rather than 1 to m[1].

   First part:

   - $R_i M = e_i M$: $e_i \in R_i$, so $e_i M \subset R_i M$; $r_i m = e_i r_i m$, so $R_i M \subset e_i M$
   - $M = \sum e_i M$: $m = 1m = (e_1 + e_2 + \cdots + e_n)m = e_1 m + e_2 m + \cdots + e_n m$
   - $M = \oplus e_i M$: $e_i m = m$ for $m \in e_i M$, and $e_i m = e_i e_j m = 0m = 0$ for $m \in e_j M$, $i \neq j$

   Second part: Recall that $R_i = \sum L_i$, for any left ideal L_i. Any module over a semisimple ring is semisimple, so $M = \oplus M_i$, where M_i is simple. Recall that

$$L_i M_j = \begin{cases} 0 & M_j \ncong L_i \\ L_i & M_j \cong L_i. \end{cases} \tag{2.1}$$

   Thus,

$$R_i M = \sum L_i \oplus M_j = \sum L_i M_j = \sum_{N \cong L_i} N \tag{2.2}$$

   Note: this is Theorem XVII.4.4 in Lang's Algebra.

2. Let us present $D_4 = \left\langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^{-1} \right\rangle$. By the previous solution set, the character table for D_4 is:

	1	σ^2	σ, σ^{-1}	$\tau\sigma^2$	$\tau\sigma, \tau\sigma^{-1}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Character Table 1: D_4

All elements of O either commute or anti-commute. Thus, the conjugacy classes of O are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$. Clearly, $\langle i \rangle, \langle j \rangle, \langle k \rangle$ are all normal subgroups, so we can map any of them to zero and we are left with C_2, whose representations come from sending its generator to ± 1. Quotienting by i leaves generator $j = k$, so j, k are sent to the same element. Similarly for quotienting by j, k. Omitting the repeated trivial representation, we obtain the following partial character table for O:

	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5					

Partial Character Table 1: O

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[1] Since m will be used to denote elements of M.

The last row is uniquely determined by its dimension (which must be 2), and orthogonality[2]. Since the first four rows agree with the character table for D_4 *and* the conjugacy classes have the same size, the last row must agree as well. Thus, O has character table:

	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Character Table 2: O

Last, we must verify that $D_4 \not\cong O$. This is because the former has two elements of order 4, while the latter has six. Thus, non-isomorphic groups can have the same character table.

3. The character table for S_4 is:

	1	(2)	(3)	(4)	$(2,2)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

Character Table 3: S_4

The first two representations are the trivial and sign, the fourth comes from decomposing the permutation representation: $\chi_\pi = \chi_1 \oplus \chi_4$, the fifth from tensoring the sign and fourth: $\chi_5 = \chi_1 \otimes \chi_4$, and the third follows from dimension and orthogonality.

4. The commutator of S_n is A_n, so the one dimensional representations of S_n correspond to representations of $S_n/A_n = C_2$, and are thus the trivial and sign representations.

S_1, S_2 are commutative, so they have no simple two dimensional representations, $S_3 \cong D_3$, and its simple two dimensional representation is on solution set 1.

For S_4, note that the double conjugations (the elements of the form $(ab)(cd)$) form a normal subgroup of order 4 (in fact the Klein 4-group), so quotienting out by them gives a group of order $24/4 = 6$, which is non-commutative, as we haven't quotiented out by A_n. Thus, the quotient is S_3, and composing the quotient with the simple representation of S_3 yields a simple representation of S_4. By inspecting the character table, this is the only simple two dimensional representation.

For $n > 4$, S_n has no proper normal subgroups other than A_n. Thus, the kernel of any representation must be $1, A_n, S_n$. The latter two correspond to the sign and trivial representations. Thus, any other simple representation must be faithful.

Take two disjoint 2-cycles, say $(12), (34)$. These are commuting and diagonalizable, and are thus simultaneously diagonalizable. Thus, $\rho(12) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ and similarly for (34). However, they cannot be sent to $\pm I$, as they then would commute with everything and this representation would not be faithful. In particular, if (12) is sent to $\pm I$, then $\rho(1234) = \rho\big((12)(1234)(12)\big) = \rho(2134)$ and the representation will not be faithful. The same holds for (34) and $(12)(34)$. Thus, $\rho\big((12)\big) = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and similarly for (34). But then $\rho\big((12)(34)\big) = \pm I$ (since they are simultaneously diagonalizable), which cannot happen, by the above reasoning. Thus, for $n \geq 5$, S_n has no simple two dimensional representations.

The above can be generalized. What we did was find a commutative subgroup of S_n, all of whose elements have order 2. Thus, their image is simultaneously diagonalizable, and there are thus 2^d possible elements of the image (d is the dimension of the representation). However, none of these elements can be $-I$, as this representation would then fail to be faithful. Since we can always find $r = \lfloor \frac{n}{2} \rfloor$ disjoint 2-cycles, the subgroup they generate has order 2^r elements, which must be mapped injectively into a set of size $2^d - 1$. Thus, for $n \geq 5$, simple representations of S_n (other than the trivial and sign) must have dimension $> \lfloor \frac{n}{2} \rfloor$.

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[2] Actually, it is uniquely determined by orthonormality and the need for the first entry to be positive.