MATH 126
SOLUTION SET 1

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1. This beautiful proof was given by Carina Curto and Stefan Karpinski. $M^n = I$, so $M^n - I = 0$. Thus, M satisfies the polynomial $f(x) = x^n - 1$. A matrix is diagonalizable over K iff its minimal polynomial factors into distinct linear factors over K[1]. Since $x^n - 1$ has distinct factors and the minimal polynomial for M must divide it, the minimal polynomial for M has distinct linear factors.

   For a counter-example if we drop finite order, consider $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then the characteristic polynomial for M is $(x-1)^2$, and M doesn't satisfy $M - 1 = 0$, so the minimal polynomial for M must be $(x-1)^2$, which has repeated factors, so M is not diagonalizable.
2. An equivalent definition of semisimple is:

   1.1. Definition (Semisimple)**.** M is SEMISIMPLE iff given any injection $A \overset{\varphi}{\hookrightarrow} M$, φ has a left inverse called a SPLITTING, namely $A \overset{\psi}{\dashleftarrow} M$ such that $\psi\varphi = \mathrm{Id}_A$.[2]

   With this definition, given any submodule $A \overset{\varphi}{\hookrightarrow} M$, then given any submodule of A, $B \overset{\psi}{\hookrightarrow} A$, we can compose these maps to get $B \overset{\varphi\psi}{\hookrightarrow} M$. Since M is semisimple, we get a map $B \overset{\pi}{\dashleftarrow} M$ such that $\pi(\varphi\psi) = \mathrm{Id}_B$. So by associativity (!), we get $B \overset{\pi\varphi}{\dashleftarrow} A$ such that $(\pi\varphi)\psi = \mathrm{Id}_B$, so A is semisimple.

   To prove for factor modules, follow the dual of the above argument. That is, in the above paragraph reverse the directions of all of the arrows, replace submodule by quotient module, and injection by surjection (including in the definition), and the proof proceeds identically.
3. Let us present $D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1, \sigma\tau = \tau\sigma^{-1} \rangle$.

   For n odd, the simple one dimensional representations are given by $\sigma \mapsto 1, \tau \mapsto \pm 1$.

   For n even, the simple one dimensional representations are given by $\sigma \mapsto \pm 1, \tau \mapsto \pm 1$ (any combination).

   The simple two dimensional representations are given by

   $$\sigma \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix} \qquad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   where $\zeta_n = \exp(2\pi i/n)$, $1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$. These are irreducible because the only subspaces stabilized by τ are span $(1,1)$, span $(1,-1)$, and these are not stabilized by σ. These are distinct because their characteristic polynomials are distinct. Lastly, these are all the simple representations because the sum of the squares of the dimensions add up to the order of the group.

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[1]Consider Jordan Normal Form or come to section to see why.

[2]To see that this is equivalent to the definition given in class, note that you can view any submodule $A \subset M$ as an injection $A \hookrightarrow M$, and that given a decomposition of $M = A \oplus A'$, the splitting map is just the projection onto A (quotienting out by A'). Conversely, given a splitting map, its kernel will be the desired compliment of A.