Math 126, Problem Set 3

This problem set is due October 23, 1998.

1. (20 pts.) Let G be a finite group and V a vector space of dimension d over $\mathbb{C}$. Let $\rho\colon G \to GL(V)$ be a representation of G. Let $\chi\colon G \to \mathbb{C}$ be the associated character. Prove that $|\chi(g)| \leq d$ for all $g \in G$ and that equality occurs if and only if $\rho(g)$ is a scalar multiple of the identity transform.

2. (20 pts.) Let G be a group and H a normal subgroup of G. Let $\rho\colon G/H \to GL(V)$ be a simple representation of G/H. Prove that the representation $\hat{\rho} = \rho \circ \pi$ is also simple, where $\pi\colon G \to G/H$ is the canonical projection.

3. (20 pts.) Show that the 1 dimensional representations of a group form a group under tensor product.

4. (20 pts.) Show that the sum of the entries in any row of a character table (NOT weighted by the number of elements in the conjugacy class!) is a nonnegative integer.

5. (20 pts.)

A. Let G be a finite group and let H be an index two subgroup. Show that there are always at least as many conjugacy classes in H as there are outside of H. In particular, conclude that there are at least as many conjugacy classes of even elements in S_n as there are conjugacy classes of odd elements.

B. Let G be a finite group and let H be an index three normal subgroup. Let $\{H, sH, tH\}$ be the set of left cosets of H. First, show that H, sH, and tH are unions of conjugacy classes. Next, show that sH and tH contain equal numbers of conjugacy classes. Finally, show that H contains at least as many conjugacy classes as are contained in sH (or tH).

6. (50 pts.) You find the following fragment on a piece of paper in your closet. Turns out, it's part of the character table of some group G.

$$\omega = e^{2\pi i/6}$$

1	C	ι	Δ	∇	H	H'
1	1	1	1	1	1	1
1	1	-1	$-\omega$	ω^2	ω	

A. Complete this character table.
B. How many elements are there in each conjugacy class?
C. What are the orders of the elements in each conjugacy class?
D. What is the commutator subgroup of G?
E. Show that G has a normal subgroup H of order 14.
F. Identify which group H is and describe its structure (by showing it is isomorphic to a well understood group).
G. For each prime p, determine the number of Sylow p-subgroups.
H. Decompose each irreducible representation of G into irreducibles for H.