Induced representations and homogeneous vector bundles.

The space $\mathcal{F}(G, F)^{r,(H, □)}$.

Let H be a subgroup of the group G, so that G acts transitively on M:=G/H. We identified the space $\mathcal{F}(M)$ as the subspace of $\mathcal{F}(G)$ Consisting of those elements which are fixed under the right action of H. That is, we identified $\mathcal{F}(M)$ as the subspace of $\mathcal{F}(G)$ which transform as the trivial representation under H. If we are given any representation □ of H on a vector space F, we can consider the space $\mathcal{F}(G, F)$ of all functions on G with values in F. As usual, the group G acts on this space because of its action on itself:

$$(af)(b) := f(a^{-1} b).$$

We can consider the subspace $\mathcal{F}(G, F)^{r,(H, □)}$ of $\mathcal{F}(G, F)$ consisting of those f which satisfy

$$f(bh) = □(h)f(b) \quad □\ h\ □\ H \ \text{and}\ □\ b\ □\ G.$$

Induced representations.

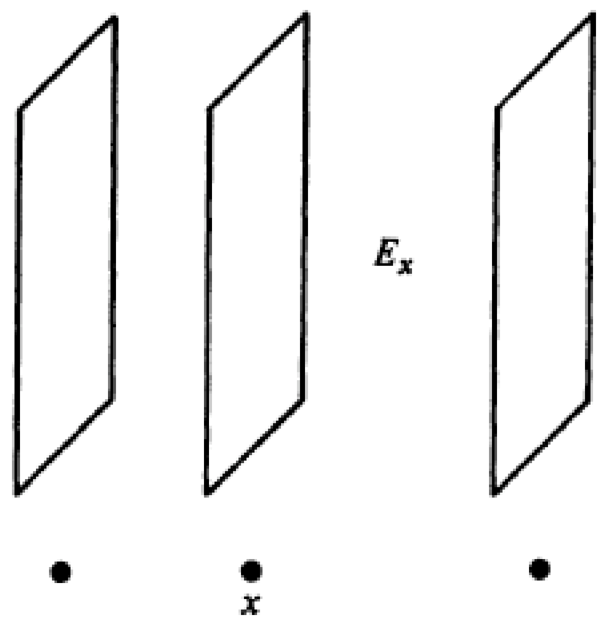
The space $\mathcal{F}(G, F)^{r,(H, □)}$ is an invariant subspace of $\mathcal{F}(G, F)$ under the left action of G since left multiplication and right multiplication commute. So we get a representation of G on $\mathcal{F}(G, F)^{r,(H, □)}$.

In other words, starting from a representation □ of the subgroup H we have obtained a representation of the group G, called the representation of G **induced** from the representation □ of H.

We want to unerstand this construction from a more geometric point of view.

Vector bundles.

Let M be a finite set. A *vector bundle* over M consists of a collection of vector spaces E_x, one vector space for each point x of M. We let $E = \bigcup_{x \in M} E_x$ (union, not direct sum) and sometimes talk of E as being the vector bundle. E is *not* a vector space. We define the map $\pi: E \to M$ by setting $\pi(\mathbf{v}) = x$ if $\mathbf{v} \in E_x$. That is, any $\mathbf{v} \in E$ belongs to one of the vector spaces E_x, and we let $\pi(\mathbf{v})$ be this particular x. The space $E_x = \pi^{-1}(x)$ is sometimes called the *fiber* over x, Fig. 3.4(a).

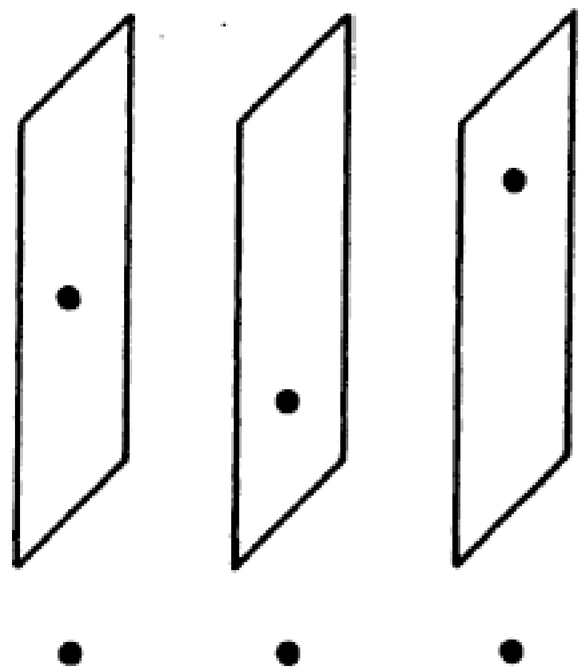


(a)

Fig. 3.4

Sections.

Let E be a vector bundle over M. A *section* of E is a function f, which assigns a vector $f(x) \in E_x$ to each $x \in M$. Thus, f is a map from M to E with the property that



The space of all sections is a vector space.

Let f_1 and f_2 be sections of the vector bundle E. Let x be a point of M. Since $f_1(x)$ and $f_2(x)$ take values in the same vector space E_x, it makes sense to add them. We thus define the sum of two sections by:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x).$$

Similarly, if c is any complex number and f is any section, we define cf by

$$(cf)(x) = cf(x).$$

In this way, *the space of all sections becomes a vector space*. We denote this vector space

by □(E).

Action of a group on a vector bundle.

Suppose that the group G acts on M and that E is a vector bundle over M. We say that G acts as a group of vector bundle morphisms on E, or that E is a *homogeneous vector bundle* under G, if

(1) G acts on E;
(2) the map $\pi: E \to M$ is a G morphism, i.e. $a\pi(v) = \pi(av)$ for all $a \in G$ and $v \in E$.

Condition (2) is the same as saying that $a: E_x \to E_{ax}$. We require also that

(3) the map $a: E_x \to E_{ax}$ is linear for each $a \in G$ and $x \in M$.

Thus, each a permutes the various vector spaces according to its action on M, and is a linear map from one vector space to another.

The representation of G on □(E).

Suppose that the group G acts on M and also acts on the vector bundle E M. We define an action of G on □(E) by setting

$$r(a)f(x) = a[f(a^{-1}x)]. \tag{2.1}$$

Notice that $f(a^{-1}x) \in E_{a^{-1}x}$ so that the right-hand side of (2.1) is an element of E_x, and thus $r(a)f$ is indeed again a section of E. It is clear that the map $f \rightsquigarrow r(a)f$ is a linear transformation, and it is easy to check that r is a representation of G on □(E). We shall denote this representation by r_E, and the corresponding character by $□_E$.

A basis of (E).

In order to evaluate χ_E we introduce a convenient basis of $\Gamma(E)$, generalizing the δ function basis we used for $F(M)$. For each vector space, E_x, we introduce a basis $\mathbf{v}_{x1}, \ldots, \mathbf{v}_{xk_x}$, where $k_x = \dim E_x$. For any $\mathbf{v} \in E_x$ we define the section $f_{\mathbf{v}}$ by

$$f_{\mathbf{v}}(y) = \begin{matrix} \mathbf{v} & \text{if } y = x \\ 0 & \text{if } y \neq x \end{matrix}.$$

It now follows, as for the case of $F(M)$, that

$$r_E(a) f_{\mathbf{v}} = f_{a\mathbf{v}}. \tag{2.2}$$

As x ranges over all of M and the $\mathbf{v}_{xi}$ over a basis of each E_x, the sections $f_{\mathbf{v}_{xi}}$ form a basis of $\Gamma(E)$:

If we use this basis, it is clear from (2.2) that the only non-zero entries on the diagonal

of the matrix representing $r_E(a)$ can come from those $\mathbf{v}_{xi}$ for which $ax = x$.

The Frobenius character fixed point formula.

For each x with $ax = x$, the element a maps E_x into itself, and

$$af_{\mathbf{v}_{xi}} = f_{a\mathbf{v}_{xi}}.$$

Now if (A_{xij}) is the matrix of the map $a: E_x \to E_x$ with respect to the basis $\mathbf{v}_{xi}$, then

$$a\mathbf{v}_{xj} = \sum_i A_{xij}\mathbf{v}_{xi}$$

and hence

$$af_{\mathbf{v}_{xj}} = \sum_i A_{xij} f_{\mathbf{v}_{xi}}.$$

Thus, for each fixed x the sum of the diagonal elements of the linear transformation $f \rightsquigarrow af$ coming from the $\mathbf{v}_{xi}$ is $\sum_i A_{xii}$, which is just the trace of the linear transformation $a: E_x \to E_x$. Summing over all the fixed points, x, of M, we obtain the celebrated

Frobenius fixed point character formula:

$$\chi_E(a) = \sum_{ax=x} [\operatorname{tr}(a: E_x \to E_x)]. \qquad (2.3)$$

This generalizes the fixed point formula that we obtained for the action of G on the space of functions on M.

Orbit decomposition.

Let $E \to M$ be a homogeneous vector bundle for the group G and let N be a subset of M. We can form $\pi^{-1}N = \bigcup_{x \in N} E_x$, which is now a vector bundle over N, which we denote by E_N. If N is mapped onto itself by all elements of G, i.e. is stable under G, then E_N is clearly a homogeneous vector bundle for G. We can identify $\Gamma(E_N)$ with a subspace of $\Gamma(E)$, namely the subspace consisting of all sections which vanish outside of N, i.e those f for which $f(y)=0$ for $y \notin N$. If $M = \cup M_i$ is a disjoint union, then we have the direct sum decomposition

$$\Gamma(E) = \Gamma(E_{M_1}) \oplus \cdots \oplus \Gamma(E_{M_k}).$$

If the M_i are invariant, in particular if they are orbits, we obtain the Mackey decomposition formula:

$$r_E = r_{E_{M_1}} \oplus \cdots \oplus r_{E_{M_k}} \tag{3.1$_r$}$$

and the corresponding character decomposition

$$\chi_E = \chi_{E_{M_1}} + \cdots + \chi_{E_{M_k}}. \tag{3.1$_\chi$}$$

The transitive case.

So we might as well look at the case where G acts transitively on M. Let m be a point of M and $H = G_m$, the isotropy group of m. Each element of H maps E_m into E_m and acts linearly. In other words, we get a representation of H on E_m. Let us denote this representation by s. Let n be some other point of M. Since G acts transitively, we can write

$$n = am$$

for some $a \in G$, and if $n = bm$, then $b = ah$ for some $h \in H$. Any vector in E_n is the image of some vector in E_m under the action of a. In other words, we can write any vector in E_n as

$$\mathbf{w} = a\mathbf{u} \quad \mathbf{u} \in E_m.$$

If also

$$\mathbf{w} = b\mathbf{v}$$

then $b\mathbf{v} = a\mathbf{u}$ or $a^{-1}b\mathbf{v} = \mathbf{u}$ or

$$\mathbf{u} = s(h)\mathbf{v}.$$

This shows that the representation s determines the vector bundle E.

Constructing the vector bundle.

We now show how to construct a homogeneous vector bundle from a representation of a subgroup.

So suppose we start with G and a subgroup H. We can reconstruct M as the coset space G/H: a point of $M = G/H$ is a coset aH. Suppose that s is a representation of the subgroup H on a vector space V. On the space $G \times V$ we introduce the equivalence relation

$$(gh, \mathbf{v}) \sim (g, s(h)\mathbf{v})$$

and let E denote the set of all equivalence classes. We denote the equivalence class of $(g, \mathbf{v}) \in G \times V$ by $[(g, \mathbf{v})]$. We write

$$E = G \underset{H}{\times} V.$$

The map sending $(g, \mathbf{v}) \rightsquigarrow gH \in M$ is clearly constant on equivalence classes, and hence defines a map from $E \to M$. Suppose that $x = gH$. Let E_x consist of all equivalence classes of elements $(g, \mathbf{v})$ as $\mathbf{v}$ ranges over V. We have an identification of the set E_x with the vector space V, which depends on the choice of g:

$$\phi_g : V \to E_x, \quad \phi_g(\mathbf{v}) = [(g, \mathbf{v})].$$

Constructing the vector bundle, continued.

If we change our choice of g by replacing g by gh, we get

$$\phi_{gh}(\mathbf{v}) = [(gh, \mathbf{v})] = [(g, s(h)\mathbf{v})] = \phi_g(s(h)\mathbf{v})$$

so that

$$\phi_{gh} = \phi_g \circ s(h). \tag{3.2}$$

This equation shows that we may use the map ϕ_g to define a vector space structure on E_x which is independent of the particular choice of g. If $\mathbf{e}_i = \phi_g(\mathbf{v}_i)$, we define

$$\mathbf{e}_1 + \mathbf{e}_2 = \phi_g(\mathbf{v}_1 + \mathbf{v}_2).$$

In view of (3.2) and the fact that $s(h)$ acts linearly on V, the value of this sum does not depend on the choice of g, and similarly for the definition of multiplication of a vector by a scalar. We have thus made E into a vector bundle over M.

The action of G.

We define the action of the group G on this vector bundle by multiplication on the left:

$$a[(g, \mathbf{v})] = [(ag, \mathbf{v})].$$

It is obvious that this definition is independent of the choice of representative. It defines a linear map from $E_x \to E_{ax}$ by the very definition of the linear structure on these fibers. Thus E is a homogeneous vector bundle for G, and we obtain a representation of G on $\Gamma(E)$. This representation was obtained from the subgroup H and the representation s of H. It is called the representation of G *induced* from the representation s, and we will denote it by $(s \uparrow G)$.

We will show that this is the same definition as we gave earlier in just a bit.

The induced character.

Let s be a representation of the subgroup H of the group G, and let σ be its character. We shall denote the character of the induced representation, $s \uparrow G$, by $\sigma \uparrow G$. The formula for $\sigma \uparrow G$ is given by (2.3). It is sometimes convenient to rewrite (2.3) in a slightly different form. The sum in (2.3) is over fixed points. To say that $x = gH$ is fixed under $a \in G$ means that $g^{-1}ag \in H$. Of course the element g is only determined by x up to right multiplication by an arbitrary element of H. The action of a on E_x can be described as follows: if $\mathbf{u} \in E_x$ is given by $\mathbf{u} = \phi_g \mathbf{v}$ for $\mathbf{v} \in V$, then

$$u = [(g,v)] \text{ so } au := [(ag,v)] = [(gg^{-1}ag,v)] = [(g, s(g^{-1}ag)v)].$$

Thus

$$\operatorname{tr}[a: E_x \to E_x] = \operatorname{tr} s_{g^{-1}ag} = \sigma(g^{-1}ag).$$

The expression on the right makes sense because $g^{-1}ag \in H$, and does not depend on which g we pick with $x = gH$. For purposes of counting, it is sometimes easier to sum over *all* g with $g^{-1}ag \in H$, instead of summing over the fixed points, but then we will have counted each fixed point $\#H$ times. We divide by $\#H$ to compensate for the overcounting, and (2.3) becomes

$$\sigma \uparrow G(a) = (1/\#H) \sum_{\substack{g \in G \\ g^{-1}ag \in H}} \sigma(g^{-1}ag). \tag{3.3}$$

The Frobenius reciprocity formula.

Let χ be a character of the group G, and let $\chi|_H$ denote the restriction of χ to H. Let us compute $(\sigma \uparrow G, \chi)_G$. We have

$$(\sigma \uparrow G, \chi)_G = \frac{1}{\#G} \sum (\sigma \uparrow G)(a)\overline{\chi(a)}$$

$$= \frac{1}{\#G}\frac{1}{\#H} \sum_{\substack{a \in G \\ h = b^{-1}ab \in H}} \sigma(b^{-1}ab)\overline{\chi(a)} \text{ by (3.3)}$$

$$= \frac{1}{\#G}\frac{1}{\#H} \sum_{\substack{h \in H \\ b \in G}} \sigma(h)\overline{\chi(bhb^{-1})}$$

$$= \frac{1}{\#H} \sum_{h \in H} \sigma(h)\overline{\chi(h)}$$

since $\chi(bhb^{-1}) = \chi(h)$

$$= (\sigma, \chi_H)_H.$$

Thus we have proved the *Frobenius reciprocity formula:*

$$(\sigma \uparrow G, \chi)_G = (\sigma, \chi|_H)_H. \tag{3.4}$$

More on Frobenius reciprocity.

Thus we have proved the *Frobenius reciprocity formula:*

$$(\sigma \uparrow G, \chi)_G = (\sigma, \chi|_H)_H. \tag{3.4}$$

If χ is the character of a representation of G on a vector space W, the left-hand side of (3.4) is just $\dim \mathrm{Hom}_G(W, \Gamma(E))$. If σ is the character of a representation of H on a vector space F (so the fiber of E over H is F), the right-hand side of (3.4) is just $\dim \mathrm{Hom}_H(W, F)$. Thus we can rewrite (3.4) as

$$\dim \mathrm{Hom}_G(W, \Gamma(E)) = \dim \mathrm{Hom}_H(W, F). \tag{3.5}$$

In fact, we can say more: that there is a natural identification of the two vector spaces $\mathrm{Hom}_G(W, \Gamma(E))$ and $\mathrm{Hom}_H(W, F)$.

The two constructions of induced representations.

Let $x \in M = G/H$. Recall that for each $b \in G$ such that $x = bH$, we have identified E_x with the set of all $(b, \mathbf{v})$, where $\mathbf{v} \in F$. If $f \in \Gamma(E)$, then $f(x) \in E_x$, so we can write

$$f(x) = [(b, \hat{f}(b))],$$

where $\hat{f}(b) \in F$. If $c \in H$, then $f(x) = f(bH) = f(bcH)$, and

$$f(bcH) = [(bc, \hat{f}(bc))].$$

Thus

$$\hat{f}(bc) = s(c)^{-1}\hat{f}(b) \quad \text{for } c \in H. \tag{3.6}$$

Conversely, any function $\hat{f}: G \to F$ satisfying (3.6) defines a section of $\Gamma(E)$. Thus *we may identify* $\Gamma(E)$ *with the space of all functions from* G *to* F *satisfying* (3.6). Let us denote this space by $\hat{\Gamma}$. We now compare the action of G on both spaces. On $\Gamma(E)$, the representation r_E is given by

$$(r_E(a)f)(x) = af(a^{-1}x)$$

while on the space of functions the representation $\hat{r}$ is given by

$$\hat{r}(a)\hat{f}(g) = \hat{f}(a^{-1}g).$$

If $x = bH$, then

$$f(x) = [(b. \hat{f}(b))]$$

and

$$\begin{aligned}(r_E(a)f)(x) &= a[(a^{-1}b, \hat{f}(a^{-1}b))]\\ &= [(b, \hat{f}(a^{-1}b))]\end{aligned}$$

so the function corresponding to $r_E(a)f$ is $\hat{r}(a)\hat{f}$ as required.