Representation theory of the symmetric groups

Construction of all the irreducible representations.

The conjugacy classes

- Every element of S_n can be written as a product of (disjoint) cycles: For example (1326)(45)(7)(8).
- If $s \in S_n$ is written in cycle form, and t is some other element of S_n , then tst^{-1} is obtained from s by replacing each integer i in the cycle form of s by t(i).
- Conversely, if s_1 and s_2 have the same cycle structure so that there is a permutation t relating the entry i in s_1 to t(i) in s_2 then $s_2 = ts_1t^{-1}$.
- A conjugacy class in S_n is thus determined by $[\square_1, \ldots \square_n]$ where $\square_1$ is the number of one cycles, $\square_2$ is the number of two cycles etc.

The number of elements in a conjugacy class.

The ν's are constrained by

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n = n.$$

The number of elements in a conjugacy class is given by $\#S_n/\#H$, where H is the isotropy subgroup of some element s in the conjugacy class, i.e. $H = \{t | tst^{-1} = s\}$. Suppose s has the cycle structure $[\nu_1, \ldots, \nu_n]$. Then t cannot interchange entries coming from cycles of different length. Within the set of cycles of a fixed length, t can act as a cyclic permutation within each cycle and can permute cycles as a whole. Thus,

considering the cycles of different length independently, we see that

$$\#H = 1^{\nu_1}\nu_1!2^{\nu_2}\nu_2!3^{\nu_3}\nu_3! \cdots n^{\nu_n}\nu_n!$$

where. for example, 3^{ν_3} is present because there are three cyclic permutations within each of the ν_3 three-cycles and $\nu_3!$ is present because there are $\nu_3!$ permutations of these three-cycles among themselves. Thus,

$$\begin{array}{l}\text{the number of elements}\\ \text{in the conjugacy class}\\ \text{given by } [\nu_1, \ldots, \nu_n]\end{array} = \frac{n!}{1^{\nu_1}\nu_1!2^{\nu_2}\nu_2! \cdots n^{\nu_n}\nu_n!}.$$

Example: the conjugacy classes of S_4 .

$$\#\{e\} = \#[4,0,0,0] = \frac{4!}{4!} = 1$$

$$\#\{(a,b)\} = \#[2,1,0,0] = \frac{4!}{2 \cdot 2!} = 6$$

$$\#\{(a,b)(c,d)\} = \#[0,2,0,0] = \frac{4!}{2^2 \cdot 2!} = 3$$

$$\#\{(abc)\} = \#[1,0,1,0] = \frac{4!}{3} = 8$$

$$\#\{(abcd)\} = \#[0,0,0,1] = \frac{4!}{4} = 6.$$

t

Partitions

Set

$$\begin{aligned}\lambda_1 &= \nu_1 + \nu_2 + \cdots + \nu_n \\ \lambda_2 &= \nu_2 + \nu_3 + \cdots + \nu_n \\ &\vdots \\ \lambda_n &= \nu_n\end{aligned}$$

Thus $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_n$, and it follows from

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n = n$$

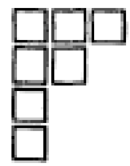
that

$$\lambda_1 + \cdots + \lambda_n = n.$$

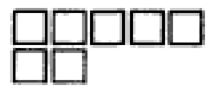
For example, the permutation $(1)(23)(45)(678) \in S_8$ has $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 1$. The set $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called a partition of n. It is conveniently represented by a *Young diagram*.

Young diagrams

We draw the diagram as an array of boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, etc. For example, if $n = 7$ then $\lambda = (3, 2, 1, 1)$ is drawn as



and similarly $(5,2) = (5,2,0,0)$ (we usually drop the zeros) is



Given $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geqslant \lambda_{i+1}$ and $\lambda_1 + \cdots + \lambda_n = n$, we recover v_i by setting

$$v_i = \lambda_i - \lambda_{i+1}.$$

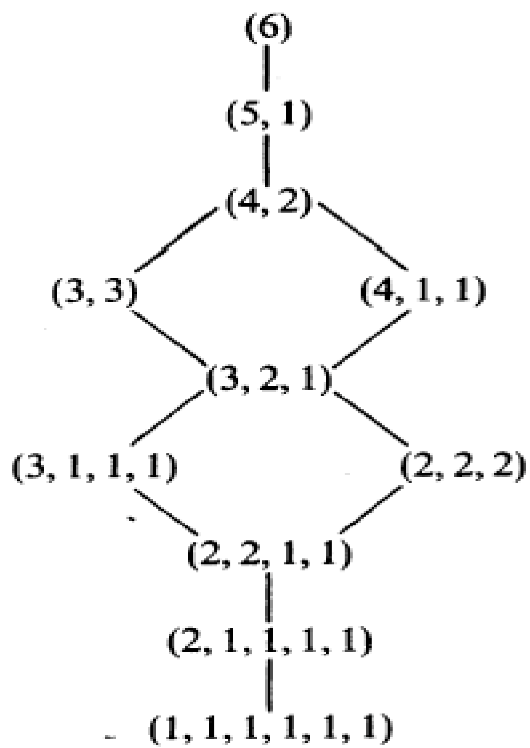
For example, the first diagram corresponds to $v_1 = 1$, $v_2 = 1$, $v_3 = 0$, $v_4 = 1$; the second to $v_1 = 3$, $v_2 = 2$. Clearly $v_1 + 2v_2 + \cdots + nv_n = n$. Thus the number of conjugacy classes of S_n, which is the same as the number of inequivalent irreducible representations of S_n, is the same as the number of Young diagrams. Our task is to attach a distinct irreducible representation of S_n to each diagram. Then we will know that we will have found all the irreducibles of S_n.

A partial order on the Young diagrams.

From now on we consider a fixed n. We put a partial order on the diagram: saying that $\lambda \geqslant \mu$ if, for all i, the total number of boxes in the first i rows of λ i: less than the total number of boxes in the first i rows of μ; i.e. if

$$\lambda_1 \geqslant \mu_1$$
$$\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2$$
$$\lambda_1 + \lambda_2 + \lambda_3 \geqslant \mu_1 + \mu_2 + \mu_3, \text{ etc.}$$

For example, the partial ordering (down is decreasing) for S_6 is given by:



Young tabloids.

By a Young *tabloid* corresponding to the diagram $\lambda=(\lambda_1,\ldots,\lambda_n)$ we mean a decomposition of the set $\{1,\ldots,n\}$ into a union of disjoint sets where the first set contains λ_1 elements, the second set contains λ_2 elements, etc. Thus

$$\{3,5,2\}\{1,7\}\{4\}\{6\} \quad \text{or} \quad \left\{\begin{array}{lll} \boxed{3} & \boxed{5} & \boxed{2} \\ \boxed{1} & \boxed{7} & \\ \boxed{4} & & \\ \boxed{6} & & \end{array}\right\}$$

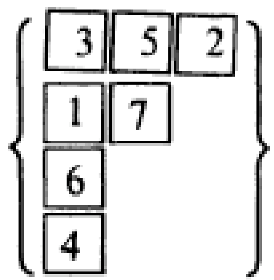
is a Young tabloid corresponding to the Young diagram $(3,2,1,1)$. The individual subsets are unordered so

$$\{2,3,5\}\{7,1\}\{4\}\{6\} \quad \text{or} \quad \left\{\begin{array}{lll} \boxed{2} & \boxed{3} & \boxed{5} \\ \boxed{7} & \boxed{1} & \\ \boxed{4} & & \\ \boxed{6} & & \end{array}\right\}$$

is the same tabloid.

Young tabloids, continued.

However, the order of the subsets is important,

$\{3,5,2\}\{1,7\}\{6\}\{4\}$ or 

is a different tabloid. We can think of a tabloid as a way of putting the number $\{1,2,\ldots,n\}$ into the boxes of a Young diagram, where the order of numbers within each row does not matter.

We let M_λ denote the set of all tabloids corresponding to a Young diagram λ.

The group S_n acts on M_λ by permuting the elements in the boxes.
This action is clearly transitive.

The number of elements in M□.

If

$\{t\}$ is a fixed tabloid corresponding to λ, the isotropy group of $\{t\}$ is clearly isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_p}$, the subgroup which permutes elements within each row of the diagram. Since S_n acts transitively on M_λ, we see that

$$\# M_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_p!}.$$

The representation of S_n on $F(M_\lambda)$.

Since S_n acts on the set M_λ, we get a representation of S_n on $\mathscr{F}(M_\lambda)$..For example, $M_{(n)}$ contains only one element,

$$\{1, \ldots, n\} \quad \text{or} \quad \{\boxed{1}\boxed{2}\boxed{3}\boxed{\cdots}\boxed{n}\}.$$

All permutations carry this tabloid into itself, so the representation of S_n on $\mathscr{F}(M_{(n)})$ is the trivial representation. An element of the set $M_{(n-1,1)}$ is of the form $\{1,\ldots,\hat{k},\ldots,n\} \cup \{k\}$, where the symbol $\hat{k}$ means that k is *missing*. So $M_{(n-1,1)}$ can be identified with the set $\{1, \ldots, n\}$, where $k \in \{1, \ldots, n\}$ corresponds to the missing $\{k\}$. For example, if $n = 3$ there are three tabloids, $\{23\}, \{1\}$; $\{13\}, \{2\}$; and $\{12\}, \{3\}$, which may be identified with 1, 2, and 3, respectively. We have seen that S_n, acting on $M_{(n-1,1)} \times M_{(n-1,1)}$, has two orbits, so that

$$\mathscr{F}(M_{(n-1,1)}) = \mathbb{C} \oplus F_{(n-1,1)},$$

where $F_{n-1,1}$ is an irreducible space of dimension $n-1$. Notice that the first component, the constant functions, is just $\mathscr{F}(M_{(n)})$

The representation of S_n on $F(M_{n-2,2})$.

The set $M_{(n-2,2)}$ $(n>3)$ can be identified with the space of all two-element subsets of $\{1,\ldots,n\}$, where we look at the entries in the second subset, so that $\{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n\}$, $\{k,l\}$ is identified with $\{k,l\}$. For example, if $n=5$, $M_{(n-2,2)}$ has ten elements. The element $\{3,4,5\}, \{1,2\}$ is associated with $\{1,2\}$, the element $\{2,4,5\}$, $\{1,3\}$ with $\{1,3\}$, and so on. A pair of two-element subsets may have either zero, one or two elements in common. Thus, S_n has three orbits when acting on $M_{(n-2,2)} \times M_{(n-2,2)}$, and so $\mathscr{F}(M_{(n-2,2)})$ breaks up into three irreducible components. We claim that two of these components are $\mathscr{F}(M_{(n)})$ and $\mathscr{F}(M_{(n-1,1)})$. Indeed, we now describe a map from $\mathscr{F}(M_{(n-1,1)})$ to $\mathscr{F}(M_{(n-2,2)})$ which commutes with the action of S_n and is injective: we must find a map, T, which goes from functions, f, on $\{1,\ldots,n\}$ to functions on two-element subsets. Take T to be given by

$$(Tf)(\{a,b\}) = f(a) + f(b).$$

It is clear that T commutes with the action of S_n. Also T (constant) = constant and $T\delta_a$ is not a constant (and, in particular, not zero). Thus T is not zero when restricted to each of the irreducible components of $\mathscr{F}(M_{(n-1,1)})$ and hence is injective. Thus

$$\mathscr{F}(M_{(n-2,2)}) = \underset{1}{\mathbb{C}} + \underset{n-1}{T(F_{(n-1,1)})} + \underset{\frac{n(n-3)}{2}}{F_{(n-2,2)}}.$$

The dimension of $F_{(n-2,2)}$ is obtained by subtracting:

$$\dim F_{(n-2,2)} = \frac{n!}{(n-2)!2!} - n = \frac{n(n-3)}{2}.$$

Goal:

We wish to prove the following: to each λ there corresponds a unique 'new' irreducible subrepresentation F_λ of $\mathscr{F}(M_\lambda)$. The space $\mathscr{F}(M_\lambda)$ decomposes into a direct sum of irreducible subrepresentations isomorphic to certain of the F_μ *with* $\mu \geqslant \lambda$ (and these may occur with multiplicity) together with the one unique new subrepresentation F_λ. Thus each Young diagram determines an irreducible representation of S_n.

Young tableaux.

By a Young *tableau* corresponding to λ we mean an assignment of the numbers $\{1, \ldots, n\}$ to each of the boxes of λ, one number to each box. In a tableau, the order in each row matters. Thus

3	5	2
1	7	
4		
6		

is a $(3, 2, 1, 1)$ tableau. Each tableau gives rise to a tabloid, by letting the entries in the first row belong to the first set, the entries of the second row correspond to the second set, etc. Two different tableaux, which differ by a permutation of the entries of their rows, give rise to the same tabloid. If t is a tableau, the corresponding tabloid will be denoted by $\{t\}$. Thus if t is the above tableau, then $\{t\} = \{3, 5, 2\}\{1, 7\}\{4\}\{6\}$.

The column group of a tableau.

Let t be a tableau. Let C_t denote the subgroup of S_n

We now describe the F_λ. Let t be a tableau. Let C_t denote the subgroup of S_n consisting of those π which permute the numbers in the various columns of t among themselves. Thus, if

$$t = \begin{array}{|c|c|c|} \hline 3 & 5 & 2 \\ \hline 1 & 7 & \\ \cline{1-2} 4 & & \\ \cline{1-1} 6 & & \\ \cline{1-1} \end{array}$$

then

$$C_t = S_{\{3,1,4,6\}} \times S_{\{5,7\}}$$

where $S_{\{3,1,4,6\}}$ are the permutations of $\{3, 1, 4, 6\}$, etc.

Let t be a tableau of shape λ. Recall that C_t denotes the column group of t. We shall associate to t an element of $\mathcal{F}(M_\lambda)$ by defining

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{t\}}.$$

Notice that the e_t depends on t and not just the tabloid $\{t\}$

Since $\sigma\delta_{\{t\}} = \delta_{\sigma\{t\}}$, we have

$$\sigma e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\sigma\pi\{t\}}.$$

We can write this as

$$\sigma e_t = \sum_{\pi \in C_t} \operatorname{sign}(\sigma\pi\sigma^{-1})\delta_{\sigma\pi\sigma^{-1}\sigma\{t\}}.$$

But $C_{\sigma t} = \sigma C_t \sigma^{-1}$ so we can write the last sum as

$$\sum_{\rho \in C_{\sigma t}} \operatorname{sgn}(\rho)\delta_{\rho\{\sigma t\}} = e_{\sigma t}.$$

We conclude that

$$\sigma e_t = e_{\sigma t}.$$

The Specht modules F_λ .

Since $\sigma e_t = e_{\sigma t}$ we see that the space spanned by the e_t is invariant under S_n .

We define this space to be

F_λ. Thus

$$F_\lambda = \{\text{linear span of all the } e_t\}$$

as t ranges over all tableaux of shape λ .

Example, the sign representation.

if $\lambda = (1, 1, \ldots, 1)$, then $C_t = S_n$ and, up to sign, there is only

one e_t and it is

$$e_t = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \delta_{\pi t}$$

where we may take

$$t = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \cdot \\ \hline \cdot \\ \hline \cdot \\ \hline n \\ \hline \end{array} \; .$$

The representation in this case is the one-dimensional sign representation. Also in this case $S_{\{t\}} = \{e\}$, so we may identify $M = S_n/S_{\{t\}}$ with S_n and thus $\mathscr{F}(M)$ with $\mathscr{F}(S_n)$. We know that the regular representation contains any irreducible representation with multiplicity equal to its dimension and hence contains $F_{(1,\ldots,1)}$ once. Also $(1, \ldots, 1)$ is the last diagram on our list. This supports the contention of the theorem.

Lemmas about tableaux,1.

(1) Let λ and μ be diagrams and let t be a λ tableau and s be a μ tableau. Suppose that for every i, the numbers from the ith row of s belong to different columns of t. Then $\lambda \geqslant \mu$.

Proof The numbers in the first row of s all lie in different columns of t. Hence, λ ias at least μ_1 columns (i.e. $\lambda_1 \geqslant \mu_1$).

We can apply an element of C_t to t so as to arrange that all the elements of the first row of s lie in the first row of t. There may be some extra boxes in this row. Since all the elements of the secor row of s lie in different columns of t, we can apply an element of C_t to t so as to arrange that all the elements of the first two rows of s lie in first two rows of t, implying that $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, the hypothesis implies that we can apply an element of C_t to t so as to arrange that all the elements of the first i rows of s Lie in the first I rows of t proving that $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$.

Lemmas about tableaux, 2.

(2) With the same notations as in (1), suppose that

$$\sum_{\pi\in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} \neq 0.$$

Then $\lambda \geqslant \mu$, and if $\lambda = \mu$ then

$$\sum_{\pi\in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} = \pm e_t = \sum_{\pi\in C_t} \operatorname{sgn}(\sigma\pi)\delta_{\pi\{t\}},$$

where $\sigma\in C_t$ and $\{s\} = \sigma\{t\}$.

Proof Let

$$A_t = \sum_{\pi\in C_t} \operatorname{sgn}(\pi)\pi$$

We are assuming that
$$\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} \neq 0$$

So if we define the operator
$$A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi$$

we want to prove that

$$A_t\delta_{\{s\}} = 0 \quad \text{if} \quad \mu \not\leq \lambda$$

and if $\mu = \lambda$ then

$$A_t\delta_{\{s\}} = \begin{cases} 0 & \text{if } \{s\} \neq \sigma\{t\} \text{ for some } \sigma \in C_t \\ \operatorname{sgn}(\sigma)e_t & \text{if } \{s\} = \sigma\{t\} \text{ for } \sigma \in C_t. \end{cases}$$

Suppose that

$$A_t\delta_{\{s\}} \neq 0.$$

Under this hypothesis, we claim that two numbers x and y which lie in the same row of s cannot lie in the same column of t. For to say that x and y lie in the same row of s implies that $(xy)\{s\} = \{s\}$. If x and y lie in the same column of t, then $(xy) \in C_t$ and, since $\operatorname{sgn}(xy) = -1$, we could have

$$A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi = \sum_{\pi \in C_t} \operatorname{sgn}(\pi \cdot (xy))\pi \cdot (xy) = -\sum \operatorname{sgn}(\pi)\pi(xy) = -A_t(xy).$$

Thus

$$A_t\delta_{\{s\}} = -A_t(xy)\delta_{\{s\}} = -A_t\delta_{(xy)\{s\}} = -A_t\delta_{\{s\}}$$

contradicting the assumption that $A_t\delta_{\{s\}} \neq 0$.

Thus $A_t\delta_{\{s\}} \neq 0$.

implies that $\lambda \geqslant \mu$. If $\lambda = \mu$, all numbers in the first row of s occur in differen

columns of t. So we can find some $\pi \in C_t$ such that πt has the same first row as s.

elements of the second row of s occur in different columns of πt and below the first row. So we can find a $\pi' \in C_{\pi t} = C_t$, leaving the numbers in the first row of s fixed with $\pi'\pi t$ having the same first two rows, as s, etc. This shows that $\{s\} = \{\sigma t\}$ for some $\sigma \in C_t$. But then $A_t\delta_{\{s\}} = \operatorname{sgn}(\sigma)e_t$. Thus we have proved (2). We conclude that for any $\{s\}$ whatsoever in M_λ, we have

$$A_t\delta_{\{s\}} = \begin{cases} e_t & \text{if } \{s\} = \sigma\{t\} \quad \operatorname{sgn}(\sigma) = 1 \\ 0 & \text{if } \{s\} \neq \sigma\{t\} \\ -e_t & \text{if } \{s\} = \sigma\{t\} \quad \operatorname{sgn}(\sigma) = -1. \end{cases}$$

$$A_t\delta_{\{s\}} = 0 \quad \text{if} \quad \mu \not\leq \lambda$$

(2) and if $\mu = \lambda$ then

$$A_t\delta_{\{s\}} = \begin{cases} 0 & \text{if } \{s\} \neq \sigma\{t\} \text{ for some } \sigma \in C_t \\ \operatorname{sgn}(\sigma)e_t & \text{if } \{s\} = \sigma\{t\} \text{ for } \sigma \in C_t. \end{cases}$$

Suppose that

We have proved that

for any $\{s\}$ whatsoever in M_λ, we have

$$A_t\delta_{\{s\}} = \begin{cases} e_t & \text{if } \{s\} = \sigma\{t\} \qquad \operatorname{sgn}(\sigma) = 1 \\ 0 & \text{if } \{s\} \neq \sigma\{t\} \\ -e_t & \text{if } \{s\} = \sigma\{t\} \qquad \operatorname{sgn}(\sigma) = -1. \end{cases}$$

Now every $f \in \mathscr{F}(M_\lambda)$ is a linear combination of the $\delta_{\{s\}}$ as $\{s\}$ ranges over the λ tabloids. Hence (3).

(3) For any $f \in \mathscr{F}(M_\lambda)$,

$$A_t f = c_f e_t$$

where c_f is a scalar, i.e. $A_t f$ is a multiple of e_t for any f.

Putting a scalar product on $\mathcal{F}(M_\lambda)$.

Let us put a scalar product (,) on $\mathscr{F}(M_\lambda)$ by taking the $\delta_{\{t\}}$ as an orthonormal basis. This is clearly S_n invariant. Now for any u, $v \in \mathscr{F}(M_\lambda)$,

$$(A_t u, v) = \sum_{\pi \in C_t} (\operatorname{sgn}(\pi)\pi u, v)$$

$$= \sum_{\pi \in C_t} (u, \operatorname{sgn}(\pi^{-1})\pi^{-1} v), \text{ since } \operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^{-1})$$

$$= \sum_{\pi \in C_t} (u, \operatorname{sgn}(\pi)\pi v)$$

$$= (u, A_t v).$$

In other words, the operator A_t is self-adjoint relative to (,).

Either or.

(4) Let U be an invariant subspace of $\mathscr{F}(M_\lambda)$. Then either $U \supset F_\lambda$ or $U \subset F_\lambda^\perp$. In particular, F_λ is irreducible.

Proof Let $u \in U$ and let t be a λ tableau. Then $A_t u$ is a multiple of e_t. If for some t and u this multiple is not zero, then $A_t u \in U$ and $A_t u = c_u e_t$, and since F_λ is generated by the σe_t as $\sigma \in S_n$, we see that $F_\lambda \subset U$. If these multiples are zero for all t and u, then $0 = (A_t u, \delta_{\{t\}}) = (u, A_t \delta_{\{t\}}) = (u, e_t)$ for all u and t, so $U \subset F_\lambda^\perp$.

Conclusion of proof.

(5) Let $T: \mathscr{F}(M_\lambda) \to \mathscr{F}(M_\mu)$ be any element of $\mathrm{Hom}_{S_n}(\mathscr{F}(M_\lambda), \mathscr{F}(M_\mu))$. Suppose that $F_\lambda \not\subset \ker T$. Then $\lambda \geqslant \mu$. If $\lambda = \mu$, then the restriction of T to F_λ is a scalar multiple of the identity.

Proof By (4), $\ker T \subset F_\lambda^\perp$. Let t be any λ tableau. Then $0 \neq Te_t = TA_t\delta_{\{t\}} = A_t T\delta_{\{t\}}$. But $T\delta_{\{t\}} \in \mathscr{F}(M_\mu)$ is some combination of $\delta_{\{s\}}$ for μ tabloids $\{s\}$ and $A_t\delta_{\{s\}} = 0$ unless $\lambda \geqslant \mu$. The second part follows from Schur's lemma and (2), since $A_t F_\lambda(M_\lambda) \subset F_\lambda$.

(6) $\mathrm{Hom}_{S_n}(F_\lambda, F_\mu) = 0$ unless $\lambda = \mu$. In particular, since the number of diagrams = the number of partitions = the number of conjugacy classes of S_n, the F_λ are exactly all the irreducible representations of S_n.

Proof Any $T \in \mathrm{Hom}_{S_n}(F_\lambda, F_\mu)$ can be extended to an element of $\mathrm{Hom}_{S_n}(\mathscr{F}(M_\lambda), \mathscr{F}(M_\mu))$ by setting it equal to zero on $F_\lambda^\perp$. By (5) this shows that if $T \neq 0$, then $\lambda \geqslant \mu$. Since F_λ and F_μ are irreducible, by Schur's lemma, if $T \neq 0$ then T is invertible and working with T^{-1} shows that $\mu \geqslant \lambda$, hence $\lambda = \mu$.

Standard tableaux.

The elements e_t , as t ranges over all tableaux of shape λ span F_λ , but they are far from being independent. However the following fact is true: A tableau is called **standard** if it increases as we move to the right along rows or down along columns. Then the elements e_t , as t ranges over all standard tableaux of shape λ form a basis of F_λ . We may prove this fact later.