Math 126 Lecture 8

- Summary of results
- Application to the Laplacian of the cube

Summary of key results

- Every representation is determined by its character. If □ is the character of a representation and □ is an irreducible character then (□, □) is the number of times that an irreducible with character □ occurs in the decomposition of the representation into a direct sum of irreducibles.
- The group G×G acts on G by left and right multiplication, and the corresponding representation of G×G on $\mathcal{F}$(G) decomposes as $\mathcal{F}(G) = W_1 \otimes W_1^{*}$ □ □ □ □ $W_k \otimes W_k^{*}$.

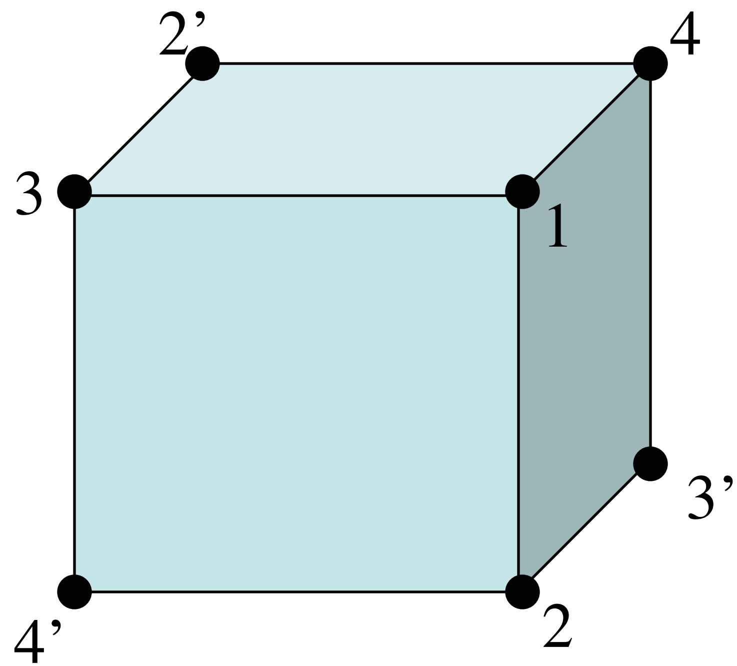
The decomposition of $\mathcal{F}(G)$.

- In the decomposition
  $\mathcal{F}(G) = W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$
  the W_i range over the irreducible representations of G. Each summand in the above decomposition is irreducible under $G \times G$, they are all distinct, and k =# of conjugacy classes of G.
- The elements of $G \times G$ of the form (a,e) act on $\mathcal{F}(G)$ by $(af)(x)=f(a^{-1}x)$. This is the (left) regular representation and will be denoted by l . On each summand the element (a,e) acts as $\rho_i(a) \otimes I$. So
- dim $\mathrm{Hom}_{l,G}(\mathcal{F}(G),\mathcal{F}(G)) = \sum n_i^2 = \#G$.

The centralizer of the left regular representation is the right regular representation.

- The elements of the form (e,b) act on $\mathcal{F}(G)$ by $((e,b)f)(x)=f(xb)$. This gives the right regular representation, denoted by r . As $((l(a)(r(b)f)))(x) = f(a^{-1}xb) = ((r(b)\,((l(a)(f)))(x)$ we see that $r(b)\,l(a) = l(a)\,r(b)$ for all a and b in G. So $r(b) \in \mathrm{Hom}_{l,G}(\mathcal{F}(G),\mathcal{F}(G))$. The $r(b)$ are independent and there are #G of them. They form a basis of $\mathrm{Hom}_{l,G}(\mathcal{F}(G),\mathcal{F}(G))$.
- Every element of $\mathrm{Hom}_{l,G}(\mathcal{F}(G),\mathcal{F}(G))$ can be written uniquely as $\sum c_b\, r(b)$ where the c_b are constants.

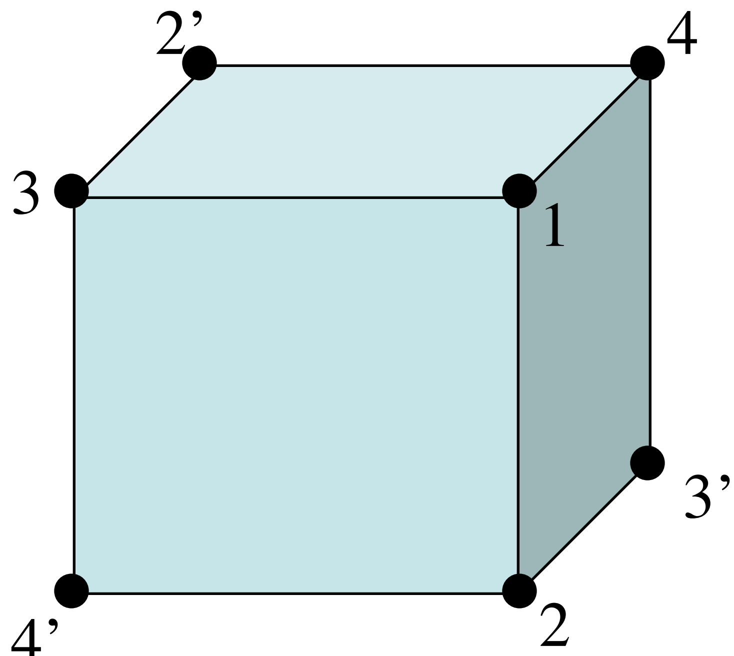
The Laplacian of the graph of the cube.



Let V denote the set of vertices of the cube. So #V = 8, and in the labeling in the figure, V={1,2,3,4,1’,2’,3’,4’}. The Laplacian L is the operator on $\mathcal{F}(V)$ where $(Lf)(x) = f(x) - \frac{1}{3} \sum f(y)$ where the sum is over all y which are connected to x by an edge. For example,

$$(Lf)(1) = f(1) - \frac{1}{3}(f(2)+f(3)+f(4)).$$

The eigenvalues of the Laplacian



We will use group theory to compute the eigenvalues of L: The group S_4 acts as rotational symmetries of the cube, and acts transitively on V.

The identity has 8 fixed points. The three cycles act as rotations through 120^o or 240^o about an axis joining diagonally opposite vertices, and so have 2 fixed points. For example (234) fixes 1 and 1’. All other elements have no fixed points. So the character χ of the representation of S_4 on $\mathcal{F}$(V) is given by $\chi(e) = 8$, $\chi((abc)) = 2$, $\chi = 0$ on all other types.

The decomposition of $\mathcal{F}$(V).

We have computed that □(e) = 8, □((abc)) =2, □ = 0 on all other types. The character table of S_4 is:

	Conjugacy class				
	1	6	8	3	6
	(1, 1, 1, 1)	(2, 1, 1)	(3, 1)	(2, 2)	(4)
Partition					
(4)	1	1	1	1	1
(3, 1)	3	1	0	− 1	− 1
(2, 2)	2	0	− 1	2	0
(2, 1, 1)	3	− 1	0	− 1	1
(1, 1, 1, 1)	1	− 1	1	1	− 1

The left hand column labels the representations as we shall explain later. We see that (□,$□_{(2,2)}$) =2□8 - 8□2 =0, while (□,□) = 1 for each of the other irreducible representations. We see that $\mathcal{F}$(V) decomposes as a direct sum of four irreducible representations of dimensions 1,3, 3, and 1.

The multiplicities of L.

Being a nearest neighbor is preserved by the action of S_4 on V.

This implies that L □ Hom $_{S_4}$ ($\mathcal{F}$(V) , $\mathcal{F}$(V)). So if f □ $\mathcal{F}$(V) is an eigenvector of L say

$$Lf = \square f,$$

Then af is an eigenvector of L with the same eigenvalue for any a □ G:

$$L(af) = a(Lf)= a(\square f) = \square(af).$$

So the set of eigenvectors of L with eigenvalue □ is an invariant Subspace and hence must be a sum of irreducibles. Thus the eigenvalues of L must occur with multiplicities 1, 3, 3 and 1.

For this argument, all we used was that L □ Hom $_{S_4}$ ($\mathcal{F}$(V) , $\mathcal{F}$(V)).

But we will in fact be able to determine the eigenvalues from group theory and the form of L.

The Laplacian and the adjacency matrix.

We can write $L = I - (1/3)A$ where A is the adjacency matrix of the graph. Here A is the 8 by 8 matrix whose rows and columns are labeled by the vertices of the graph, and an the entry in the (i,j) position is 1 if I and j are joined by an edge and 0 otherwise. In our case A =

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$

If we find the eigenvalues of A then we can derive the eigenvalues of A. We shall find that the eigenvalues of A are 3, 1, -1,-3 with multiplicities 1, 3, 3, 1.

$\mathcal{F}(V)$ as a subspace of $\mathcal{F}(G)$.

We will let $G = S_4$, and $H = \{e,(234),(243)\}$, the isotropy group of the vertex 1. Much of what we have to say will only depend on the fact that G acts transitively on V so that we can we can identify V With G/H where H is the isotropy group of a point $v \in V$. In our case v is the vertex 1. To each $f \in \mathcal{F}(V)$ assign $F \in \mathcal{F}(G)$ by the rule:

$$F(a) := f(av), \qquad a \in G.$$

Notice that if $h \in H$, then $F(ah) = f(ahv) = f(av) = F(a)$. In other words, F is invariant under the right action of H on $\mathcal{F}(G)$. In symbols, we write this as $F \in \mathcal{F}(G)^{H,r}$. Conversely, suppose that $F \in \mathcal{F}(G)^{H,r}$. Then we may define $f(w) := F(b)$ where $bv = w$. If we chose a different b' we would get the same value since $b' = bh$ for an $h \in H$. We have given a G equivalence between $\mathcal{F}(V)$ and $\mathcal{F}(G)^{H,r}$.

Extending the operator A to $\mathcal{F}(G)$.

We have identified $\mathcal{F}(V)$ with $\mathcal{F}(G)^{H,r}$ which is a subspace of $\mathcal{F}(G)$ which is invariant under the action of the subgroup $G \times H$ of $G \times G$. The operator A can be considered as an operator on this subspace $\mathcal{F}(G)^{H,r}$ and commutes with the action of $G \times H$. We can extend the operator A (in many ways) so as to get an operator B on all of $\mathcal{F}(G)$ which commutes with the action of $G \times H$ on $\mathcal{F}(G)$. (For example, we can choose an invariant complement under $G \times H$ to $\mathcal{F}(G)^{H,r}$ in $\mathcal{F}(G)$, choose some linear map F of this complement to into $\mathcal{F}(G)$ and then average over $G \times H$ so that the resulting map is a $G \times H$ morphism.) The fact that B commutes with all (a,e) implies that $B = \sum c_b\, r(b)$ where the c_b are constants. The fact that B commutes with all (e,h) implies that $\sum c_b\, r(bh) = \sum c_b\, r(hb$

i.e. that $c_{hbh^{-1}} = c_b$ for all $b \in G$ and all $h \in H$.

Using the decomposition of $\mathcal{F}$(G).

Under the decomposition $\mathcal{F}(G) = W_1 \otimes W_1^{*} \;\Box\; \Box\; \Box \;\Box\; W_k \otimes W_k^{*}$

the operator B becomes

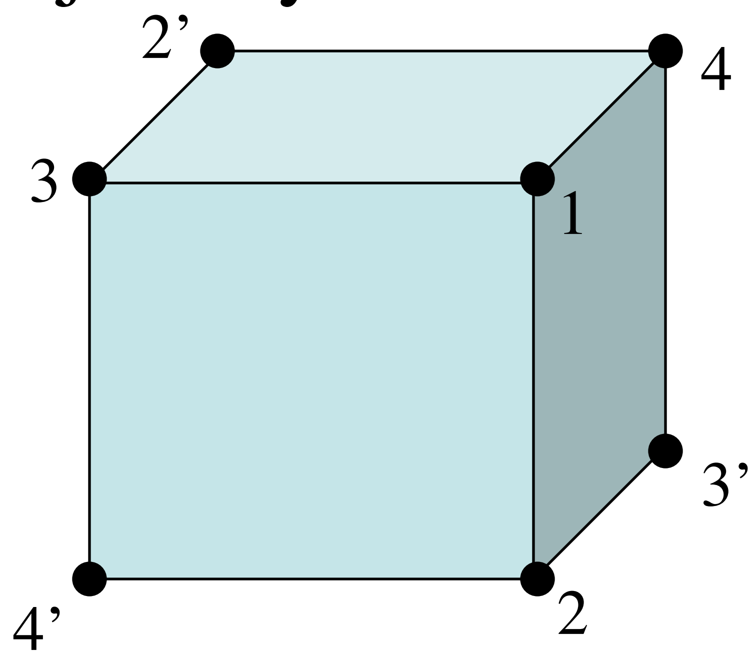
$B = I_1 \otimes (\Box c_b \Box_1(b)^{*-1}) \Box \;\Box\; \Box \;\Box\; I_k \otimes (\Box c_b \Box_k(b)^{*-1})$

and the operator A becomes the restriction of this operator to the subspace

$$W_1 \otimes W_1^{*H} \;\Box\; \Box\; \Box \;\Box\; W_k \otimes W_k^{*H}$$

where W_i^{*H} denotes the subspace of W_k^{*} consisting of those vectors which are fixed by all element of H. So if we let B_i denote the restriction of $\Box c_b \Box_1(b)^{*-1}$ to the subspace W_i^{*H}, the problem of computing the eigenvalues of A has been reduced to the problem of computing the eigenvalues of the B_i which is usually a much smaller matrix, and each such eigenvalue will occur with multiplicity $n_i = \dim W_i$.

The adjacency matrix of the cube.



For each vertex there are two cycles of degree four which map the vertex into one of the adjacent vertices. For example the cycles (1234) and (1243) map the vertex 1 into the vertex 2. So if we let

B= (1/2)*r*((1234)+(1243)+(1324) + (1342) +(1423)+(1432))

then

BF = Af.

Computing the eigenvalues for the cube.

We must compute the image of

$$(1/2)\big((1234)+(1243)+(1324) + (1342) +(1423)+(1432)\big)$$

in each irreducible representation of S_4, or rather the restriction of this image to the space of H invariant vectors.

Notice that the sum extends over an entire conjugacy class (the four cycles) and so the image in any representation commutes with the entire S_4 action. By Schur’s lemma, this image must be a scalar.

Hence there will be no algebraic equations to solve! Taking the trace shows that if the image of the above element in the i-th representation is $\square_i I$, then $n_i \square_i = 3 \square_i ((abcd))$ so we can read the desired eigenvalues from the character table. A look at the column for (abcd) in the character table of S_4 shows that

$\square_{(4)} = 3$, $\square_{(3,1)} = -1$, $\square_{(2,1,1)} = 1$ and $\square_{(1,1,1,1)} = -3$.

We still must compute the multiplicity, i.e. dim W_i^{*H} .

The multiplicities.

We want to know how many times the trivial representation of H occurs in an irreducible representation of S_4. (Since all characters are real there is no difference between W and W^*.) We do this by looking at the identity and (abc) columns of the character table of S_4.

Partition	e	2(abc)
(4)	1	1
(3, 1)	3	0
(2, 2)	2	−1
(2, 1, 1)	3	0
(1, 1, 1, 1)	1	1

Here we have taken the character table of S_4 and restricted it to H. Taking the scalar product (with respect to H) with the trivial character of H shows that the trivial representation of H does not occur in the two dimensional representation of S_4, and occurs exactly once in all the other irreducibles.