Math 126 Lecture 7

Direct products of groups and tensor products of representations.

Let G and H be groups. Their **direct product** consists of all pair $(a, b), a \in G,\ b \in H$ with the multiplication law

$$(a, b) \cdot (c, d) := (ac, bd).$$

Let r be a representation of G on the vector space U and s a representation of H on the vector space V. We can form the tensor product

$$U \otimes V$$

of these two vector spaces. From the general theory of tensor products we know that if $A \in \mathrm{Hom}(U, U)$ and $B \in \mathrm{Hom}(V, V)$ then there exists a unique element

$$A \otimes B \in \mathrm{Hom}(U \otimes V, U \otimes V)$$

such that

$$(A \otimes B)(u \otimes v) = Au \otimes Bv.$$

Also

$$(A \otimes B) \cdot (C \otimes D) = AC \otimes BD.$$

This shows that we get a representation $r \otimes s$ of $G \times H$ on $U \otimes V$ by setting

$$(r \otimes s)(a, b) := r(a) \otimes s(b).$$

The character of the tensor product of two representations is the product of the characters.

$$\operatorname{tr}(A \otimes B) = \operatorname{tr} A \cdot \operatorname{tr} B$$

so

$$\chi^{r \otimes s} = \chi^r \cdot \chi^s.$$

The norm of the character of the tensor product.

If $(\ ,\ )_{G\times H}$ denotes the scalar product on $G \times H$, and $\|\ \|_{G\times H}$ the corresponding norm, then

$$\|\chi^{r\otimes s}\|^2_{G\times H} = (\chi^{r\otimes s}, \chi^{r\otimes s})_{G\times H}$$

$$= \frac{1}{\#(G\times H)} \sum_{\substack{a\in G\\ b\in H}} \chi^{r\otimes s}(a,b)\overline{\chi^{r\otimes s}(a,b)}$$

$$= \frac{1}{\#G} \cdot \frac{1}{\#H} \left(\sum_{a\in G} \chi^r(a)\overline{\chi^r(a)} \right) \left(\sum_{b\in H} \chi^s(b)\overline{\chi^s(b)} \right)$$

$$= \|\chi^r\|^2_G \|\chi^s\|^2_H.$$

The tensor product of two irreducibles.

if $\| \chi^r \|_G^2 = \| \chi^s \|_H^2 = 1$, then $\| \chi^{r \otimes s} \|_{G \times H}^2 = 1$. Thus,

If r is an irreducible representation of G and s is an irreducible representation of H, then $r \otimes s$ is an irreducible representation of $G \times H$.

The dual of a representation.

Let r be a representation of G on W.

We can construct a representation $\hat{r}$ of G on the dual space W^* by defining $\hat{r}(a)l = r(a)^{*-1}l.$

This is a representation because

$$\begin{aligned} r(ab)^{*-1} &= (r(a)r(b))^{*-1} \\ &= (r(b)^* r(a)^*)^{-1} \\ &= r(a)^{*-1} r(b)^{*-1} \\ &= \hat{r}(a)\hat{r}(b). \end{aligned}$$

We thus get a representation of $G \times G$ on $W \otimes W^*$.

If the representation of G on W is irreducible, then so is the representation of $G \times G$ on $W \otimes W^*$.

The representation of GxG on *F(G)*.

the group $G \times G$ acts on G by right and left multiplication:

$$(a, b)c = acb^{-1}$$

and hence we get a corresponding representation $\hat{r}^G$ on $\mathscr{F}(G)$

$$[\hat{r}^G(a, b)f](c) = f(a^{-1}cb).$$

We will prove that this representation decomposes into k inequivalent irreducible representations where k is the number of conjugacy classes of G.

Let W be an irreducible representation space of G.

We have seen how to attach a function $f^l_{\mathbf{w}}$ on G to each pair $(\mathbf{w}, l)$ with $\mathbf{w} \in W$ and $l \in W^*$.

$$f^l_w(a) = \langle r(a^{-1})\mathbf{w}, l \rangle.$$

Since $f^l_{\mathbf{w}}$ depends linearly on $\mathbf{w}$ for l fixed, and linearly on l for $\mathbf{w}$ fixed, we have thus defined a map

$$W \otimes W^* \to \mathscr{F}(G)$$
$$w \otimes l \rightsquigarrow f^l_{\mathbf{w}}.$$

Notice that

$$\begin{aligned} f^{\hat{r}(b)l}_{r(a)\mathbf{w}}(c) &= \langle r(c)^{-1} r(a)\mathbf{w}, r(b)^{*-1} l \rangle \\ &= \langle r(b)^{-1} r(c)^{-1} r(a)\mathbf{w}, l \rangle \\ &= \langle r(a^{-1}cb)^{-1}\mathbf{w}, l \rangle \\ &= f^l_{\mathbf{w}}(a^{-1}cb). \end{aligned}$$

In other words, the map from $W \otimes W^*$ to $\mathscr{F}(G)$ is a morphism for the action of $G \times G$.

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decompose $\mathscr{F}(G)$ into irreducibles under the action of $G \times G$: For each

irreducible representation W_i of G, $W_i \otimes W_i^*$ occurs as an irreducible component under $G \times G$ on $\mathscr{F}(G)$.

Under G, the space $W_i \otimes W_i^*$ decomposes into a direct sum of n_i copies of W_i.

In particular, no $W_i \otimes W_i^*$ has any component in common with $W_j \otimes W_j^*$ for $i \neq j$. So

$W_i \otimes W_i^*$ and $W_j \otimes W_j^*$ are inequivalent as representations of $G \times G$.

Thus $W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$ occurs as a summand of $\mathscr{F}(G)$, where $W_1, \ldots, W_k$ are all the irreducible representations of G.

the dimension of this summand is $\sum n_i^2 = \dim \mathscr{F}(G)$. Thus

$$\mathscr{F}(G) = W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$$

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gives the decomposition of $\mathscr{F}(G)$ into irreducibles of $G \times G$, where each summand occurs once. Thus

$$\dim \mathrm{Hom}_{G \times G}(\mathscr{F}(G), \mathscr{F}(G)) = \underbrace{1^2 + \cdots + 1^2}_{k \text{ times}} = k.$$

this dimension must equal the number of orbits of $G \times G$ acting on $G \times G$ by the rule

$$(a, b)(c, d) = (acb^{-1}, adb^{-1}).$$

we can always find an element of the form (e, d) on any orbit. $(a, b)(e, d) = (ab^{-1}, adb^{-1})$ will have the same form if $b = a$. Thus (e, d) and (e, ada^{-1}) lie on the same orbit, and hence the number of orbits of $G \times G$ on $G \times G$ is equal to the number of conjugacy classes of G.

$$\dim \mathrm{Hom}_{G \times G}(\mathscr{F}(G), \mathscr{F}(G)) = k = \#\text{ of conjugacy classes.}$$

Central functions.

We have proved that

the number of distinct irreducible representations is equal to the number of conjugacy classes.

Let C denote the space of functions on G which are constant on conjugacy classes. These are called **central** functions. For example characters are central functions. Let $\chi_1, \ldots, \chi_k$ be the distinct irreducible characters.

The $\chi_i \in C$ are mutually orthogonal and have length one.

Since $k = \#$ of conjugacy classes $= \dim C$,

they form an orthonormal basis of C. Any $f \in C$ can be expanded in terms of the basis $\chi_1, \ldots, \chi_p$:

$$f = (f, \chi_1)\chi_1 + \cdots + (f, \chi_p)\chi_p.$$

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Let us apply this formula to the function f_j,

which equals one on the jth conjugacy class

and vanishes on all the others.

Then $(f, \chi_i) = (\#C_j/\#G)\overline{\chi_i(j)}$, where $\#C_j$ is the number of

elements in the jth conjugacy class, C_j,

and $\chi_i(j)$ is the value of χ_i on any element of this class. Substituting into the above formula and evaluating at a point in the jth conjugacy class, C_j,

$$1 = (\#C_j/\#G)(\chi_1(j)\overline{\chi_1(j)} + \cdots + \chi_p(j)\overline{\chi_p(j)}). \tag{6.4}$$

Evaluating at a different conjugacy class gives

$$0 = \chi_1(k)\overline{\chi_1(j)} + \cdots + \chi_p(k)\overline{\chi_p(j)} \quad \text{if } j \neq k. \tag{6.5}$$

Orthogonality relations for characters.

$$(\chi_i, \chi_k) = (1/\#G) \sum_{a \in G} \chi_i(a)\overline{\chi_k(a)} = (1/\#G) \sum_{j=1}^{p} (\#C_j)\chi_i(j)\overline{\chi_k(j)}.$$

=1 if i=k
=0 otherwise

So

$$(\#C_1)\chi_i(1)\overline{\chi_k(1)} + \cdots + (\#C_p)\chi_i(p)\overline{\chi_k(p)} = \begin{cases} \#G & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases}. \qquad (7.1)$$

Also

$$(\#C_j)[\chi_1(j)\overline{\chi_1(l)} + \cdots + \chi_p(j)\overline{\chi_p(l)}] = \begin{cases} \#G & \text{if} \quad j = l \\ 0 & j \neq l \end{cases}. \qquad (7.2)$$

Equations (7.1) and (7.2) can be summarized in the form of a table. We label the columns by the conjugacy classes, indicating, alongside C_j, the number of its elements $\#C_j$. We label the rows by the characters χ_i, and place the value $\chi_i(j)$ in the i, j position. Then (7.1) says that the 'scalar product' of two distinct rows is zero, and of a row with itself is $\#G$, provided that we weight the jth column by $\#C_j$. Similarly, (7.2) says the same thing about the scalar product of the columns, again weighting the columns by $\#C_j$. The table so obtained is called the character table of the group. In a sense, it contains all the information about the representations of the group.

Review:

Consider the group S_n acting on the n-element set $M = \{1, \ldots, n\}$. On $M \times M$ there are two orbits

$$\{(x, y) | x \neq y\} \quad \text{and} \quad \{(x, x)\}.$$

Indeed, if $x \neq y$ and $z \neq w$ we can find a permutation σ such that $\sigma(x) = z$ and $\sigma(y) = w$ Thus, the set $\{(x, y) | x \neq y\}$ is a single orbit in $M \times M$. Similarly the set $\{(x, x)\}$ is a single orbit. Thus,

$$\dim \operatorname{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) = 2 = p_1^2 + \cdots + p_k^2$$

so $k = 2$ and $p_1 = p_2 = 1$. Thus, $\mathscr{F}(M)$ is the direct sum of two irreducible representations. We already know one of them – the trivial one-dimensional representation, corresponding to the constant functions. The other must then be $n - 1$ dimensional. Thus

$$\mathscr{F}(M) = \underset{\substack{\uparrow \\ \text{one} \\ \text{dimensional}}}{V_1} + \underset{\substack{\uparrow \\ n-1 \\ \text{dimensional}}}{V_2}.$$

The character of any element *a* acting on *F(M)* is the number of fixed points of *a*. So the character of *a* acting on V_2 is the (number of fixed points of *a*)-1.

The character of the two dimensional representation of S_3 .

The number of fixed points of e is 3 so $\chi^{(2)}(e) = 2$

The number of fixed points of any two cycle is 1 so $\chi^{(2)}((ab)) = 0$

The number of fixed points of any three cycle is 0 so $\chi^{(2)}((abc)) = -1.$

The character table of S_3 .

$6S_3$	$1C_1$	$3C_2$	$2C_3$
χ_1	1	1	1
χ_2	2	0	-1
χ_3	1	-1	1

$$(\#C_1)\chi_i(1)\overline{\chi_k(1)} + \cdots + (\#C_p)\chi_i(p)\overline{\chi_k(p)} = \begin{cases} \#G & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases}. \tag{7.1}$$

$$(\#C_j)[\chi_1(j)\overline{\chi_1(l)} + \cdots + \chi_p(j)\overline{\chi_p(l)}] = \begin{cases} \#G & \text{if} \quad j = l \\ 0 & j \neq l \end{cases}. \tag{7.2}$$

Equation (7.1) for $i = k$ says

$$1^2 + 3\cdot 1^2 + 2\cdot 1^2 = 6$$

$$2^2 + 3\cdot 0^2 + 2(-1)^2 = 6$$

and

$$1^2 + 3\cdot(-1)^2 + 2\cdot 1^2 = 6.$$

For $i = 1$ and $k = 3$, equation (7.1) says

$$1\cdot 1 + 3\cdot 1\cdot(-1) + 2\cdot 1\cdot 1 = 0.$$

(7.2) for $j = l = 1$ reduces to 1+4+1=6, the sum of the squares of the dimensions of the irreducible representations equals the number of elements in the group.

A group is abelian if and only if all its irreducible represenations are one dimensional.

We have proved

$$\#G = \sum n_i^2. \tag{6.2}$$

and

the number of distinct irreducible representations is equal to the number of conjugacy classes. (6.3)

Suppose G is abelian. Then each element makes up its own conjugacy class. So each n_i must be equal to 1. Suppose that all the $n_i = 1$. Then (6.2) implies that there are #G summands, and hence (6.3) implies that each element is its own conjugacy class so G is abelian.

The character tables of the cyclic groups.

We can now write down the character table for the cyclic group C_n. Let a be a generator for C_n, so that the conjugacy classes are the various $[a^{j-1}]$, $j = 1, 2, \ldots, n$. Let ε be a primitive nth root of unity. Then the characters χ_i, determined by $\chi_i(a) = \varepsilon^{i-1}$, $i = 1, 2, \ldots, n$, are all distinct, and thus give all the characters. The character table of C_n is thus given by Table 7.

Table 7.

nC_n	$1[e]$	$1[a]$	$1[a^2]$	$\cdots$	$1[a^{n-1}]$
χ_1	1	1	1	$\cdots$	1
χ_2	1	ε	ε^2	$\cdots$	ε^{n-1}
χ_3	1	ε^2	ε^4	$\cdots$	$\varepsilon^{2(n-1)}$
.	.	.	.	$\cdots$	.
.	.	.	.	$\cdots$	.
.	.	.	.	$\cdots$	.
χ_n	1	ε^{n-1}	$\varepsilon^{2(n-1)}$	$\cdots$	$\varepsilon^{(n-1)^2}$

The character table of T .

#T =12 and T has a three dimensional representation as the symmetries of the tetrahedron. The trace of any rotation in three dimensions is given by

$$1 + 2\cos\phi.$$

Thus, for this three-dimensional representation we have

$$\chi(e) = 3, \quad \chi(R_{120^\circ}) = \chi(R_{240^\circ}) = 0 \quad \text{and} \quad \chi(R_{180^\circ}) = -1.$$

so

$$(\#G)\|\chi\|^2 = 9 + 3\cdot 1 = 12$$

since there are three rotations through 180° and four each through 120° and 240°. We see that this three-dimensional representation is irreducible. Since the sum of squares of the degrees of all irreducible representations is 12, and $3^2 = 9$, there must also be three one-dimensional representations. These can be found as follows: let H be the subgroup of T consisting of the identity and the rotations through 180°. Then H is a normal subgroup and hence any representation of the quotient group, T/H, lifts to a representation of T. But T/H is just the cycle group C_3, which has three one-dimensional representations. Thus the character table of T is given by Table 8.

The character table of T.

Table 8.

$12T$	$[e]$	$4[r_3]$	$4[r_3^2]$	$3[r^2]$
χ_1	1	1	1	1
χ_2	1	ε	ε^2	1
χ_3	1	ε^2	ε	1
χ_4	3	0	0	-1

$\varepsilon = \exp 2\pi i/3$

Groups with a large abelian subgroup.

Suppose that the group G has a commutative subgroup H. Then any irreducible representation of G has degree at most $\#G/\#H$. (7.4)

Proof Let r be an irreducible representation of G on the vector space V. Then r_H, the restriction of r to H, gives a representation of the Abelian group H on V. Let W

be an irreducible subspace of V under H, so that W is one dimensional by (7.3). The space spanned by all the $r(a)W\, a \in G$, is clearly invariant and hence must be all of V. Let G_W be the subgroup of G which stabilizes W, i.e. $G_W = \{a \in G : r(a)W \subset W\}$. Then the number of distinct subspaces, among the $r(a)W$, is given by $\#(G/G_W)$, and therefore the maximal number of linearly independent such subspaces is at most this amount. Since $H \subset G_W$, we conclude (7.4).