Review: Orthogonality relations for matrix elements of irreducible representations.

For any two functions on G we defined their scalar product by

$$(f_1, f_2) = (1/\#G) \sum_{a \in G} f_1(a)\overline{f_2(a)}.$$

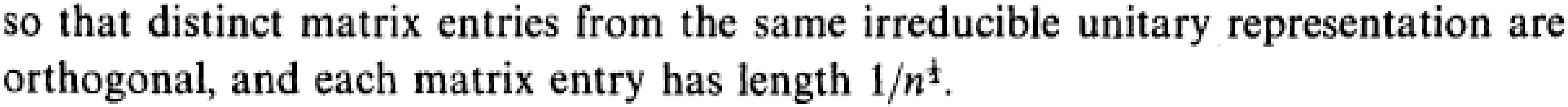
Then

If $r^1 \not\sim r^2$, then $(r^2_{kl}, r^1_{ji}) = 0$ for all i, j, k and l. (3.3)

In other words, matrix entries from two inequivalent representations are orthogonal.

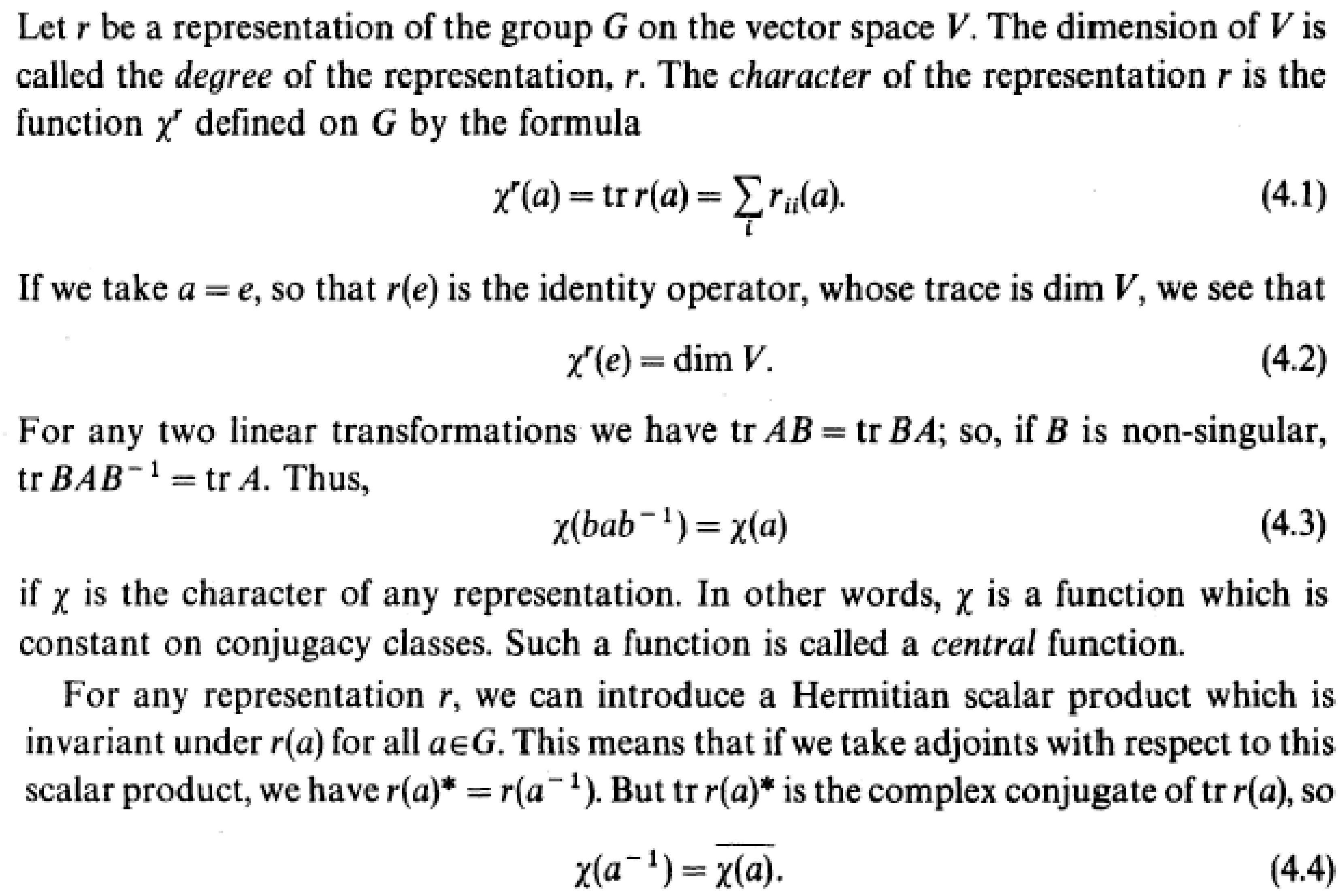
For a single irreducible $r = r^1 = r^2$ we have

$$(r^2_{kl}, r^1_{ij}) = (1/n)\delta_{ki}\delta_{lj} \tag{3.4}$$



so that distinct matrix entries from the same irreducible unitary representation are orthogonal, and each matrix entry has length $1/n^{\frac{1}{2}}$.

Characters



Let r be a representation of the group G on the vector space V. The dimension of V is called the *degree* of the representation, r. The *character* of the representation r is the function χ^r defined on G by the formula

$$\chi^r(a) = \operatorname{tr} r(a) = \sum_i r_{ii}(a). \tag{4.1}$$

If we take $a = e$, so that $r(e)$ is the identity operator, whose trace is $\dim V$, we see that

$$\chi^r(e) = \dim V. \tag{4.2}$$

For any two linear transformations we have $\operatorname{tr} AB = \operatorname{tr} BA$; so, if B is non-singular, $\operatorname{tr} BAB^{-1} = \operatorname{tr} A$. Thus,

$$\chi(bab^{-1}) = \chi(a) \tag{4.3}$$

if χ is the character of any representation. In other words, χ is a function which is constant on conjugacy classes. Such a function is called a *central* function.

For any representation r, we can introduce a Hermitian scalar product which is invariant under $r(a)$ for all $a \in G$. This means that if we take adjoints with respect to this scalar product, we have $r(a)^* = r(a^{-1})$. But $\operatorname{tr} r(a)^*$ is the complex conjugate of $\operatorname{tr} r(a)$, so

$$\chi(a^{-1}) = \overline{\chi(a)}. \tag{4.4}$$

Orthogonality relations for characters

Let r^1 and r^2 be irreducible representations of G.

If $r^1 \nsim r^2$, then $(r^2_{kl}, r^1_{ji}) = 0$ for all i, j, k and l.

(3.3)

For a single irreducible $r = r^1 = r^2$ we have

$$(r^2_{kl}, r^1_{ij}) = (1/n)\delta_{ki}\delta_{lj} \qquad (3.4)$$

The character χ^1 of the representation r^1 is given by

$$\chi^1 = \sum_i r^1_{ii}$$

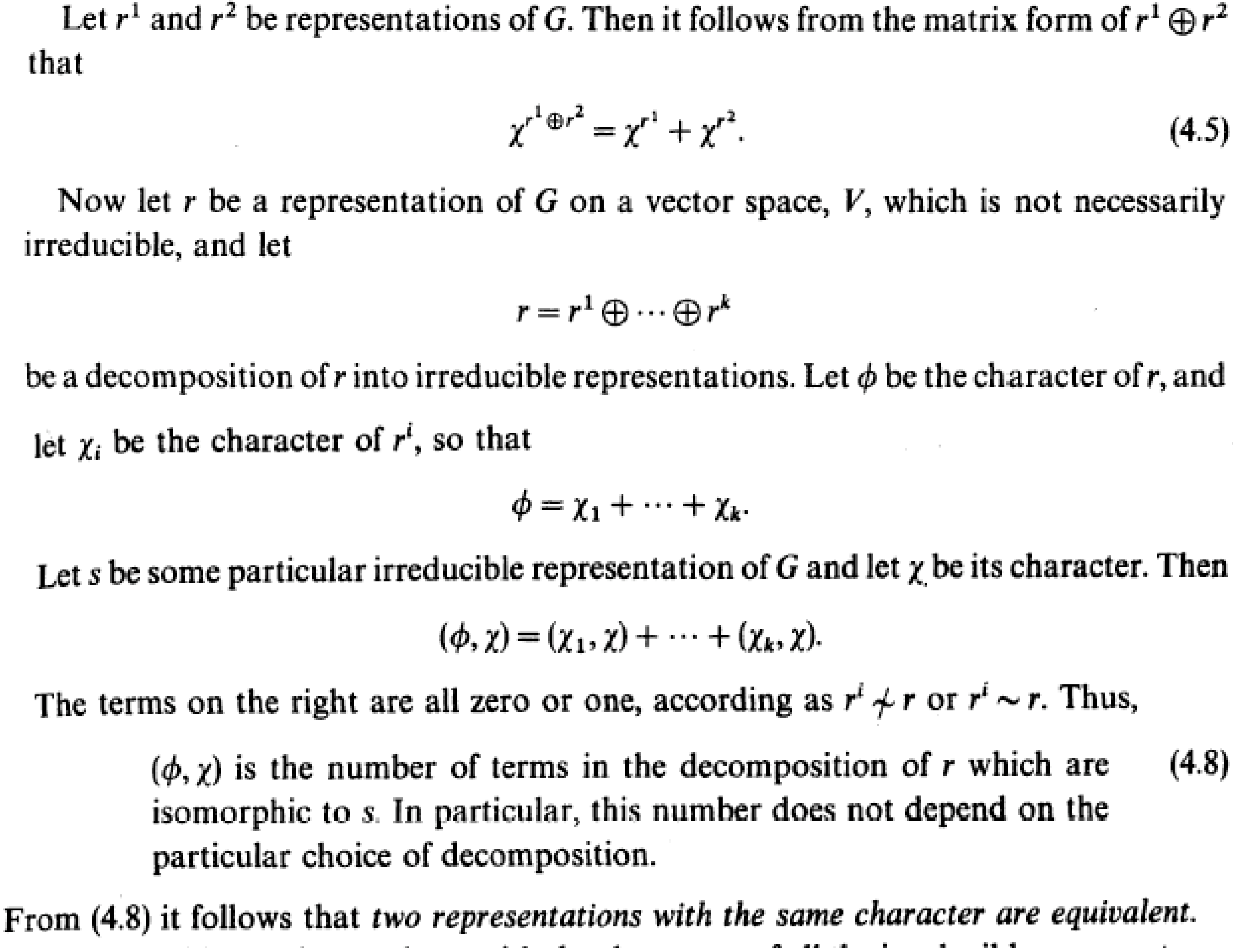
and similarly for the character χ^2 of r^2. It now follows from (3.3) and (3.4) that

$$\text{if } r^1 \nsim r^2, \quad \text{then } (\chi^1, \chi^2) = 0, \qquad (4.6)$$

and

$$(\chi, \chi) = 1, \text{ if } \chi \text{ is the character of an irreducible representation.} \qquad (4.7)$$

Characters of direct sums.



Let r^1 and r^2 be representations of G. Then it follows from the matrix form of $r^1 \oplus r^2$ that

$$\chi^{r^1 \oplus r^2} = \chi^{r^1} + \chi^{r^2}. \tag{4.5}$$

Now let r be a representation of G on a vector space, V, which is not necessarily irreducible, and let

$$r = r^1 \oplus \cdots \oplus r^k$$

be a decomposition of r into irreducible representations. Let ϕ be the character of r, and let χ_i be the character of r^i, so that

$$\phi = \chi_1 + \cdots + \chi_k.$$

Let s be some particular irreducible representation of G and let χ be its character. Then

$$(\phi, \chi) = (\chi_1, \chi) + \cdots + (\chi_k, \chi).$$

The terms on the right are all zero or one, according as $r^i \nsim r$ or $r^i \sim r$. Thus,

> (ϕ, χ) is the number of terms in the decomposition of r which are isomorphic to s. In particular, this number does not depend on the particular choice of decomposition. (4.8)

From (4.8) it follows that *two representations with the same character are equivalent.*

□ is irreducible if and only if (□ , □)=1

Proof: any character ϕ can be written as

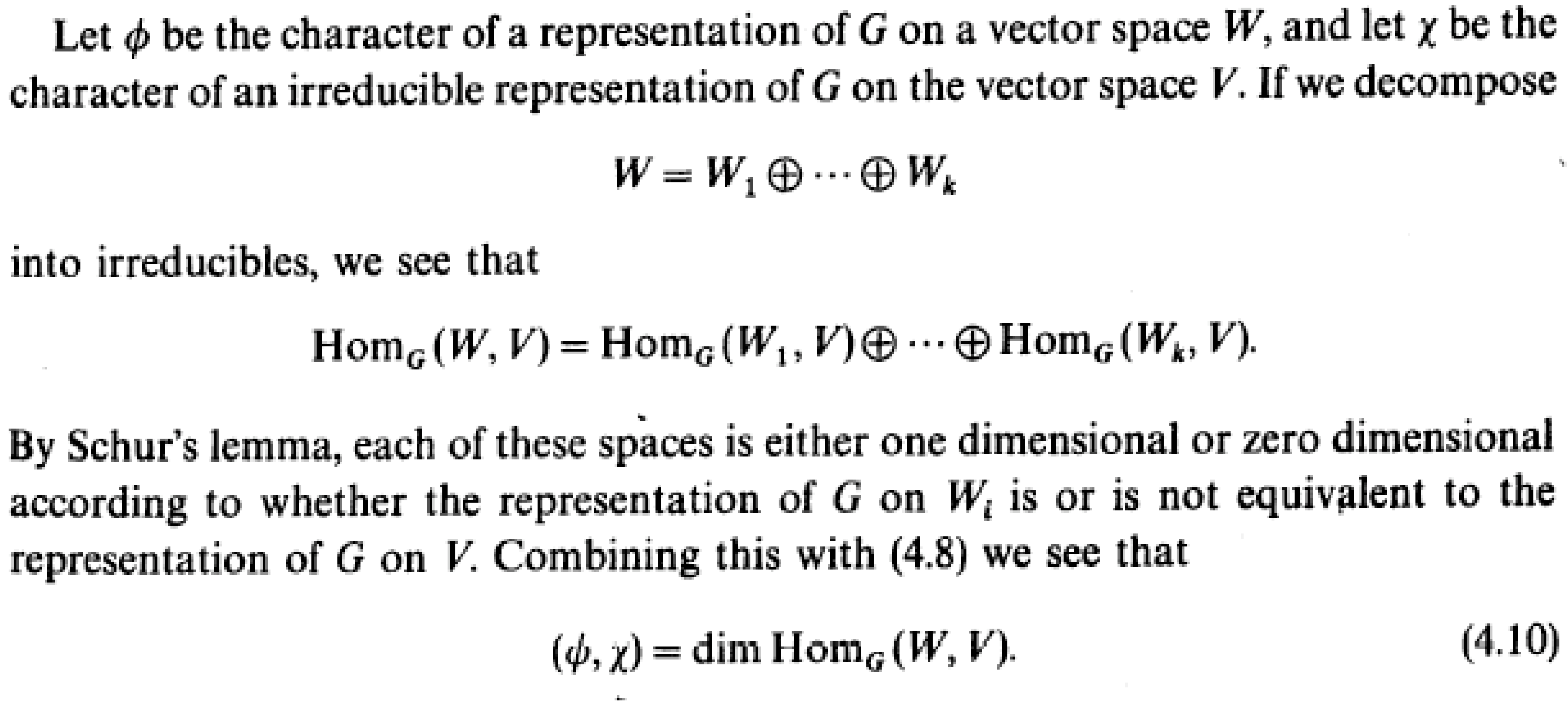
$$\phi = m_1\chi_1 + \cdots + m_p\chi_p,$$

where the χ_i are irreducible orthogonal characters. Hence

$$(\phi, \phi) = m_1^2 + \cdots + m_p^2 \qquad (4.9)$$

We already know that if □ is irreducible then (□ , □)=1. Conversely, If (□ , □)=1, then the only way that (4.9) could hold is for all the m_i but one =0, and one of them =1 which says that □ is irreducible.

dim $\mathrm{Hom}_G(W,V)$ when V is irreducible



Let ϕ be the character of a representation of G on a vector space W, and let χ be the character of an irreducible representation of G on the vector space V. If we decompose

$$W = W_1 \oplus \cdots \oplus W_k$$

into irreducibles, we see that

$$\mathrm{Hom}_G(W, V) = \mathrm{Hom}_G(W_1, V) \oplus \cdots \oplus \mathrm{Hom}_G(W_k, V).$$

By Schur's lemma, each of these spaces is either one dimensional or zero dimensional according to whether the representation of G on W_i is or is not equivalent to the representation of G on V. Combining this with (4.8) we see that

$$(\phi, \chi) = \dim \mathrm{Hom}_G(W, V). \tag{4.10}$$

dim Hom_G(U,V) in general.

Now let r_u and r_v be representations of G on U and V. We do not assume that r_u and r_v are irreducible. We wish to compute $\dim \mathrm{Hom}_G(U, V)$.

Let us first consider a special case. Suppose $U = V = W \oplus W$, where W is irreducible. We can write any vector in U as $(\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are in W. Thus, for any $T \in \mathrm{Hom}(V, V)$ we have

$$T(\mathbf{w}_1, \mathbf{w}_2) = (T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2, T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)$$

where $T_{ij} \in \mathrm{Hom}(W, W)$. So

$$\begin{aligned} T \circ r_{W\oplus W}(a)(\mathbf{w}_1, \mathbf{w}_2) &= T(r_W(a)\mathbf{w}_1, r_W(a)\mathbf{w}_2) \\ &= (T_{11}r_W(a)\mathbf{w}_1 + T_{12}r_W(a)\mathbf{w}_2, T_{21}r_W(a)\mathbf{w}_1 + T_{22}r_W(a)\mathbf{w}_2) \end{aligned}$$

while

$$\begin{aligned} & r_{W\oplus W}(a)T(\mathbf{w}_1, \mathbf{w}_2) \\ &= (r_W(a)(T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2), r_W(a)(T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)) \\ &= (r_W(a)T_{11}\mathbf{w}_1 + r_W(a)T_{12}\mathbf{w}_2, r_W(a)T_{21}\mathbf{w}_1 + r_W(a)T_{22}\mathbf{w}_2). \end{aligned}$$

So $T \in \mathrm{Hom}_G(V, V)$ if and only if each $T_{ij} \in \mathrm{Hom}_G(W, W)$. By Schur's lemma, each T_{ij} ranges over a one-dimensional space, hence $\dim \mathrm{Hom}_G(W \oplus W, W \oplus W) = 4 = 2 \times 2$.

dim $\mathrm{Hom}_G(U,V)$ in general, continued

For any representation, we may make the decomposition

$$U = (U_1 \oplus \cdots \oplus U_{p_1}) \oplus (U_{p_1+1} \oplus \cdots \oplus U_{p_1+p_2}) \oplus \cdots (\cdots U_{p_1+\cdots p_k})$$

where the first p_1 spaces are all equivalent to the irreducible representation W_1, the next p_2 spaces are all equivalent to the irreducible representation W_2 etc., and $W_1, \ldots, W_k$ are *inequivalent* irreducible representations of G. We may make the same decomposition

$$V = (V_1 \oplus \cdots \oplus V_{q_1}) \oplus (V_{q_1+1} \oplus \cdots \oplus V_{q_1+q_2}) \oplus \cdots$$

for V. By Schur's lemma, any $T \in \mathrm{Hom}_G(U, V)$ when applied to any $\mathbf{u} \in U_1 \oplus \cdots \oplus U_{p_1}$ must give $T\mathbf{u}$ lying in $V_1 \oplus \cdots \oplus V_{q_1}$. Then the same argument as in the special case shows that

$$\dim \mathrm{Hom}_G(U, V) = p_1 q_1 + p_2 q_2 + \cdots + p_k q_k. \tag{4.11}$$

In particular, if $U = V$,

$$\dim \mathrm{Hom}_G(V, V) = p_1^2 + \cdots + p_k^2,$$

where p_i is the number of times that the ith irreducible representation occurs in V.

The representation on function spaces induced from an action on a set.

Suppose that we are given an action of the group G on the set M. Let $\mathscr{F}(M)$ denote the vector space of all complex-valued functions on M. Define an action of G on $\mathscr{F}(M)$ by

$$(af)(x) = f(a^{-1}x).$$

Put another way, we define af by

$$af = f \circ a^{-1},$$

We have

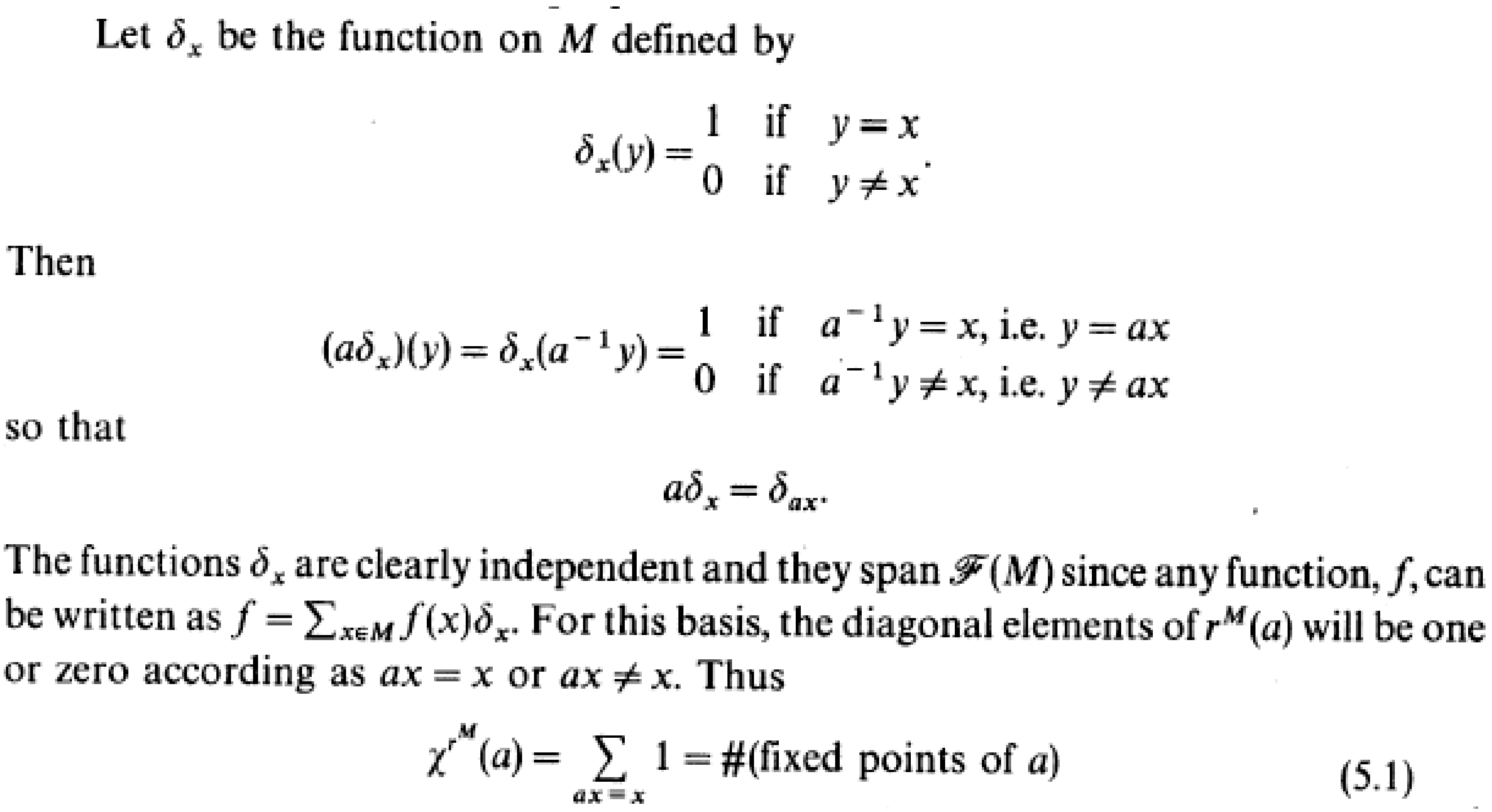
$$a(bf) = (bf)\circ a^{-1} = (f\circ b^{-1})\circ a^{-1} = f\circ(b^{-1}\circ a^{-1}) = f\circ(ab)^{-1} = (ab)f.$$

So we get a representation of G on $\mathscr{F}(M)$

Let us denote this representation by r^M .

The character fixed point formula.

Introduce the following basis of $\mathscr{F}(M)$



Let δ_x be the function on M defined by

$$\delta_x(y) = \begin{matrix} 1 & \text{if} & y = x \\ 0 & \text{if} & y \neq x \end{matrix}.$$

Then

$$(a\delta_x)(y) = \delta_x(a^{-1}y) = \begin{matrix} 1 & \text{if} & a^{-1}y = x, \text{ i.e. } y = ax \\ 0 & \text{if} & a^{-1}y \neq x, \text{ i.e. } y \neq ax \end{matrix}$$

so that

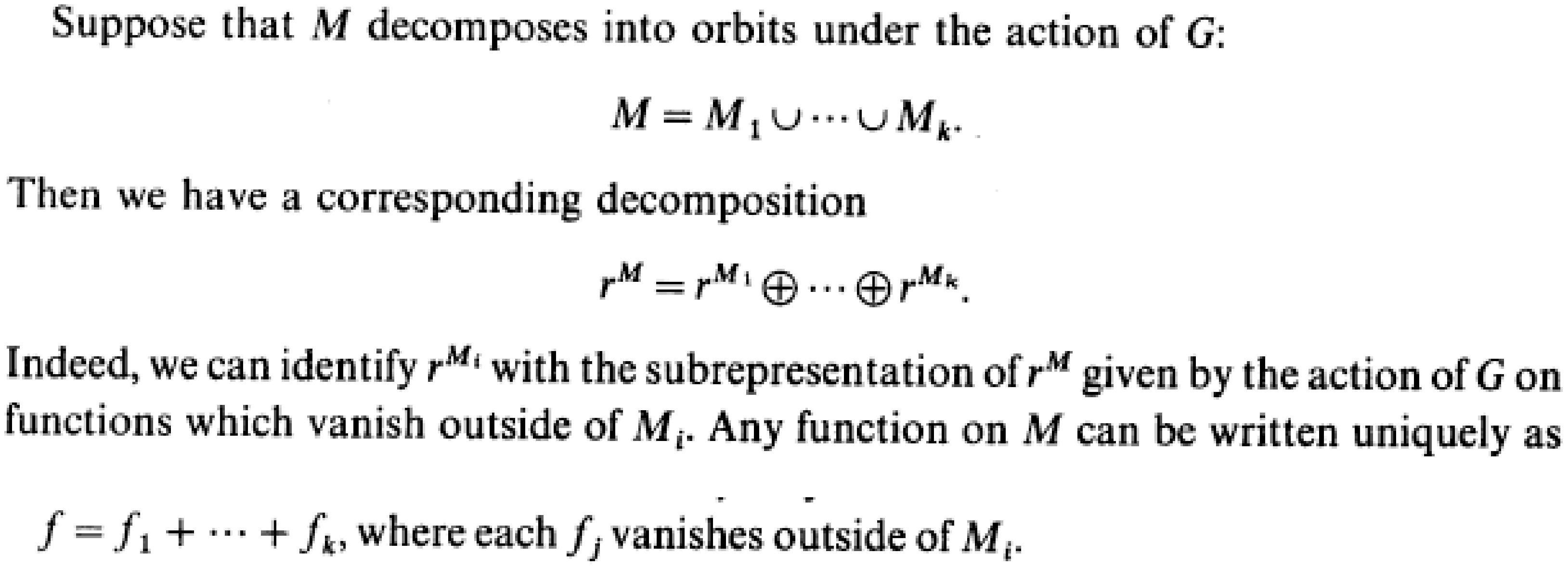
$$a\delta_x = \delta_{ax}.$$

The functions δ_x are clearly independent and they span $\mathscr{F}(M)$ since any function, f, can be written as $f = \sum_{x \in M} f(x)\delta_x$. For this basis, the diagonal elements of $r^M(a)$ will be one or zero according as $ax = x$ or $ax \neq x$. Thus

$$\chi^{r^M}(a) = \sum_{ax = x} 1 = \#(\text{fixed points of } a) \qquad (5.1)$$

This formula and its generalizations will be the key tool we will use in many computations. We may call it the **Frobenius fixed point formula**.

Orbits and direct sums



Suppose that M decomposes into orbits under the action of G:

$$M = M_1 \cup \cdots \cup M_k.$$

Then we have a corresponding decomposition

$$r^M = r^{M_1} \oplus \cdots \oplus r^{M_k}.$$

Indeed, we can identify r^{M_i} with the subrepresentation of r^M given by the action of G on functions which vanish outside of M_i. Any function on M can be written uniquely as

$f = f_1 + \cdots + f_k$, where each f_j vanishes outside of M_i.

Morphisms between function spaces.

Suppose that G acts on the two finite sets M and N. We wish to studv the space $\operatorname{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ and the action of G on it. Notice that

$$\begin{aligned}\dim \operatorname{Hom}(\mathscr{F}(M), \mathscr{F}(N)) &= \dim \mathscr{F}(M) \times \dim \mathscr{F}(N) \\ &= (\# M)\cdot(\# N) \\ &= \#(N \times M) \\ &= \dim \mathscr{F}(N \times M).\end{aligned}$$

We claim that there is a natural isomorphism between $\mathscr{F}(N \times M)$ and $\operatorname{Hom}(\mathscr{F}(M), \mathscr{F}(N))$. Indeed, given any function K on $N \times M$ define the operator $T_K : \mathscr{F}(M) \to \mathscr{F}(N)$ by

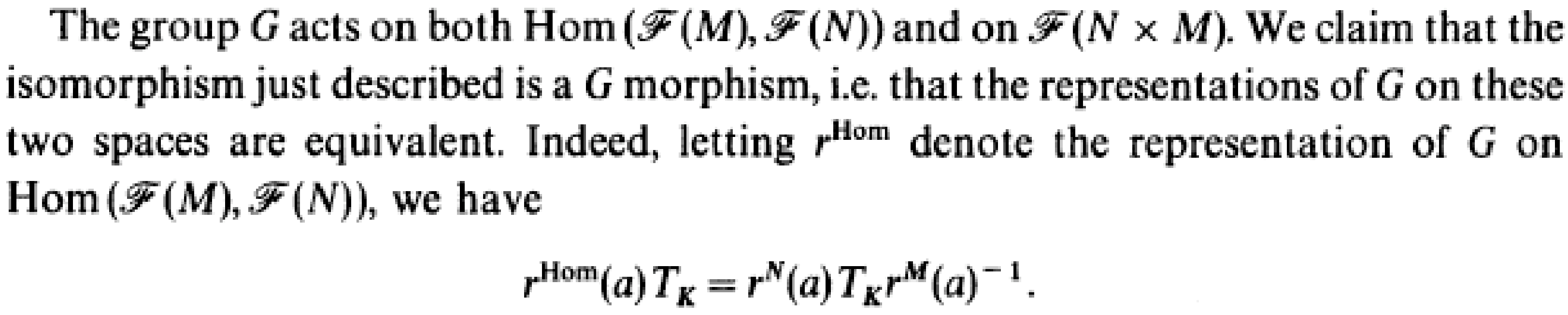
$$(T_K f)(y) = \sum_{x \in M} K(y, x) f(x).$$

The map sending K into T_K is one-to-one; indeed, for any $u \in M$

$$(T_K \delta_u)(y) = K(y, u)$$

so if $T_K = 0$, then $K(y, u) = 0$ for all y and u, i.e. $K = 0$. Since $\mathscr{F}(N \times M)$ and $\operatorname{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ have the same dimension, we conclude that the map sending K to T_K is an isomorphism of vector spaces.

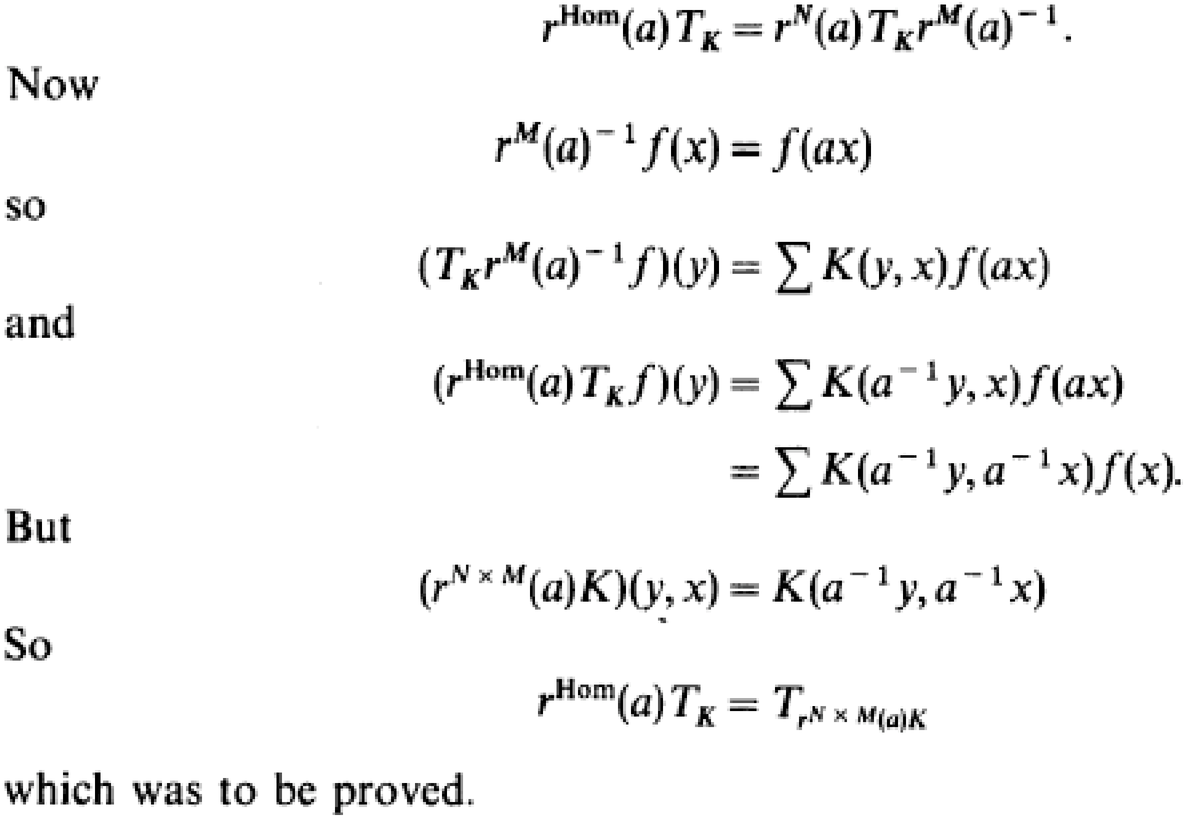
Morphisms between function spaces, continued.



The group G acts on both $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ and on $\mathscr{F}(N \times M)$. We claim that the isomorphism just described is a G morphism, i.e. that the representations of G on these two spaces are equivalent. Indeed, letting r^{Hom} denote the representation of G on $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$, we have

$$r^{\mathrm{Hom}}(a) T_{\boldsymbol{K}} = r^{N}(a) T_{\boldsymbol{K}} r^{M}(a)^{-1}.$$

Morphisms between function spaces, continued.



$$r^{\mathrm{Hom}}(a)T_K = r^N(a)T_K r^M(a)^{-1}.$$

Now

$$r^M(a)^{-1}f(x) = f(ax)$$

so

$$(T_K r^M(a)^{-1} f)(y) = \sum K(y,x)f(ax)$$

and

$$\begin{aligned}(r^{\mathrm{Hom}}(a)T_K f)(y) &= \sum K(a^{-1}y,x)f(ax)\\ &= \sum K(a^{-1}y,a^{-1}x)f(x).\end{aligned}$$

But

$$(r^{N\times M}(a)K)(y,x) = K(a^{-1}y,a^{-1}x)$$

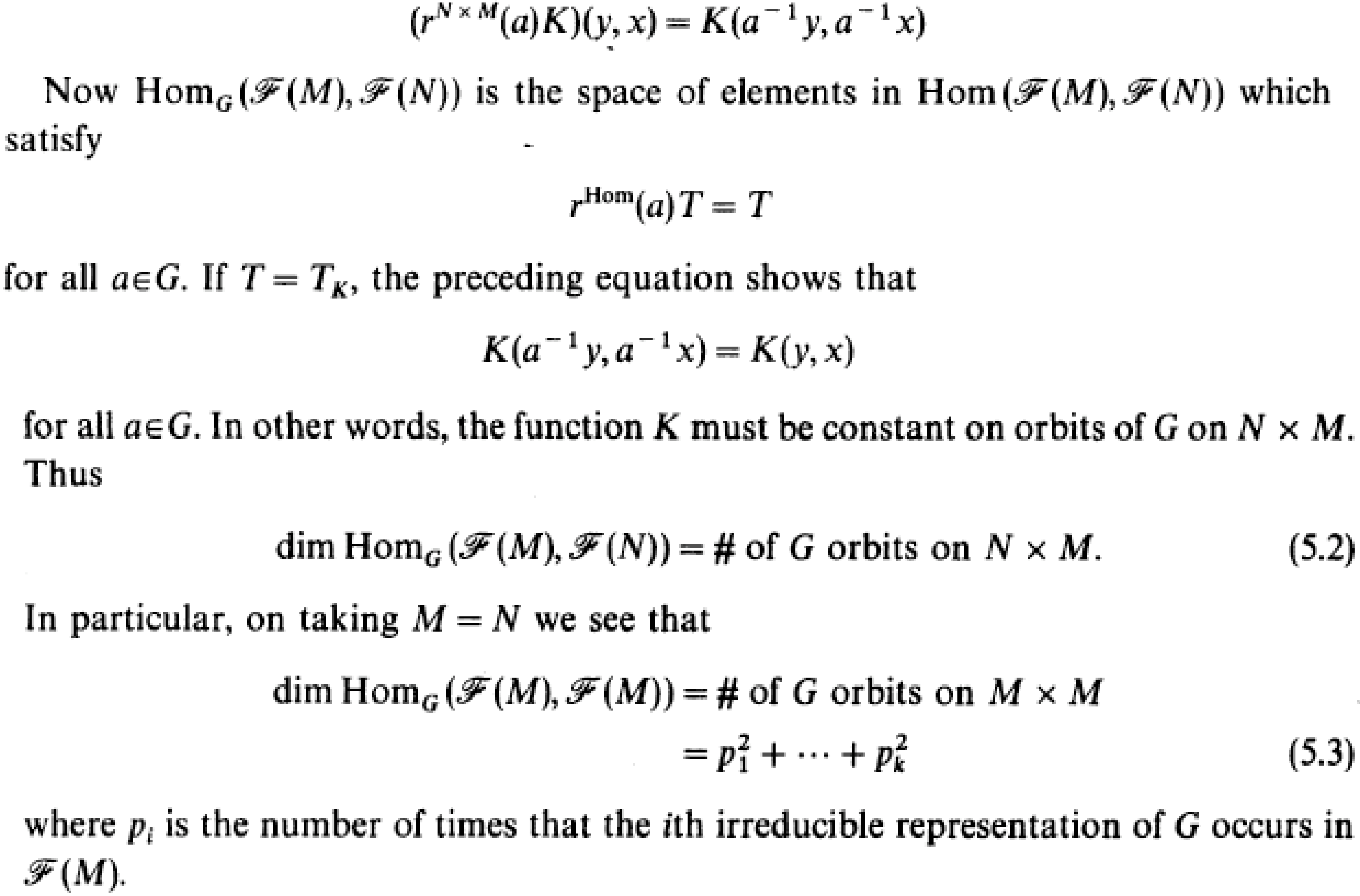
So

$$r^{\mathrm{Hom}}(a)T_K = T_{r^{N\times M}(a)K}$$

which was to be proved.

Morphisms between function spaces, continued.

We have proved that $r^{\mathrm{Hom}}(a)T_K = r^N(a)T_K r^M(a)^{-1}$ and we have



$$(r^{N\times M}(a)K)(y,x) = K(a^{-1}y, a^{-1}x)$$

Now $\mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(N))$ is the space of elements in $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ which satisfy

$$r^{\mathrm{Hom}}(a)T = T$$

for all $a \in G$. If $T = T_K$, the preceding equation shows that

$$K(a^{-1}y, a^{-1}x) = K(y,x)$$

for all $a \in G$. In other words, the function K must be constant on orbits of G on $N \times M$. Thus

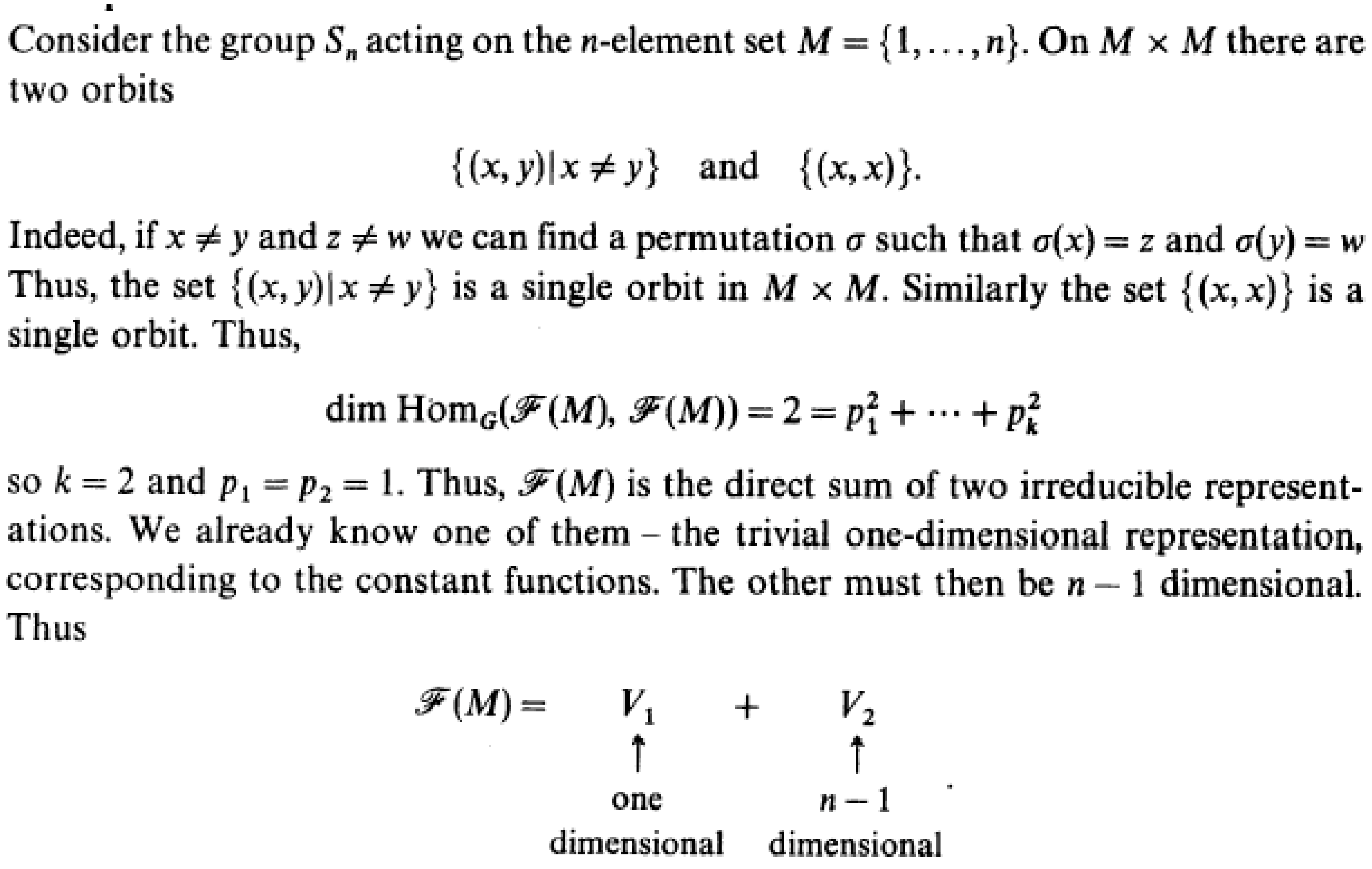
$$\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(N)) = \#\text{ of } G \text{ orbits on } N \times M. \tag{5.2}$$

In particular, on taking $M = N$ we see that

$$\begin{aligned} \dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) &= \#\text{ of } G \text{ orbits on } M \times M \\ &= p_1^2 + \cdots + p_k^2 \end{aligned} \tag{5.3}$$

where p_i is the number of times that the ith irreducible representation of G occurs in $\mathscr{F}(M)$.

Example: S_n acting on $\{1,\ldots.,n\}$.



Consider the group S_n acting on the n-element set $M = \{1, \ldots, n\}$. On $M \times M$ there are two orbits

$$\{(x, y) | x \neq y\} \quad \text{and} \quad \{(x, x)\}.$$

Indeed, if $x \neq y$ and $z \neq w$ we can find a permutation σ such that $\sigma(x) = z$ and $\sigma(y) = w$ Thus, the set $\{(x, y) | x \neq y\}$ is a single orbit in $M \times M$. Similarly the set $\{(x, x)\}$ is a single orbit. Thus,

$$\dim \operatorname{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) = 2 = p_1^2 + \cdots + p_k^2$$

so $k = 2$ and $p_1 = p_2 = 1$. Thus, $\mathscr{F}(M)$ is the direct sum of two irreducible representations. We already know one of them – the trivial one-dimensional representation, corresponding to the constant functions. The other must then be $n - 1$ dimensional. Thus

$$\mathscr{F}(M) = \underset{\substack{\text{one}\\ \text{dimensional}}}{\underset{\uparrow}{V_1}} + \underset{\substack{n-1\\ \text{dimensional}}}{\underset{\uparrow}{V_2}}.$$

The regular representation.

We apply the results of the preceding section to the special case where $M = G$ and G acts on itself by left multiplication. The corresponding representation, r^G, of G on $\mathscr{F}(G)$ is called the *regular* representation. It is defined by $[r^G(a)f](b) = f(a^{-1}b)$. We have

$$\#G = \dim \mathscr{F}(G) = \sum p_i n_i$$

where p_i is the number of times that the ith irreducible representation occurs in $\mathscr{F}(G)$, while n_i is the dimension of the ith irreducible representation. Also

$$\begin{aligned} \dim \operatorname{Hom}_G(\mathscr{F}(G), \mathscr{F}(G)) &= \sum p_i^2 \\ &= \# \text{ of } G \text{ orbits on } G \times G. \end{aligned}$$

We compute the number of orbits as follows: we can always act on (a, b) by a^{-1} to get $(e, a^{-1}b)$. Thus, each orbit of G in $G \times G$ contains a point of the form (e, c). But this is the only element of this form in its orbit, since multiplying by d sends (e, c) into (d, dc). Thus each orbit contains a unique representative of the form (e, c), and hence the number of orbits is equal to $\#G$. Thus

$$\#G = \sum p_i^2.$$

Since $\sum p_i^2 = \sum p_i n_i$ we are led to guess that $p_i = n_i$, i.e. that each irreducible representation, W, occurs in $\mathscr{F}(G)$ with a multiplicity equal to its dimension, i.e. that

$$\dim \operatorname{Hom}_G(W, \mathscr{F}(G)) = \dim W. \tag{6.1}$$

The regular representation, continued.

We wish to prove that $\dim \operatorname{Hom}_G(W, \mathscr{F}(G)) = \dim W.$ (6.1)

We shall prove this fact by constructing an isomorphism between W^*, the dual space of W, and $\operatorname{Hom}_G(W, \mathscr{F}(G))$. To each $l \in W^*$ and to each $\mathbf{w} \in W$ we assign the function $f^l_{\mathbf{w}}$ on G defined by

$$f^l_{\mathbf{w}}(a) = \langle r(a^{-1})\mathbf{w}, l \rangle.$$

Here $r(a^{-1})\mathbf{w} \in W$ and $l \in W^*$, and $\langle \mathbf{v}, l \rangle$ denotes the value of the linear function $l \in W^*$ at the element $\mathbf{v}$ of V. For fixed l the map sending $\mathbf{w}$ into $f^l_{\mathbf{w}}$ is a map from W to $\mathscr{F}(G)$. Thus each $l \in V^*$ defines an element of $\operatorname{Hom}(W, \mathscr{F}(G))$. We must show that this element lies in $\operatorname{Hom}_G(W, \mathscr{F}(G))$, i.e. that

$$f^l_{r(b)\mathbf{w}} = r^G(b) f^l_{\mathbf{w}}$$

or that

$$f^l_{r(b)\mathbf{w}}(a) = f^l_{\mathbf{w}}(b^{-1}a) \quad \text{for all } a, b \in G.$$

But

$$\begin{aligned} f^l_{r(b)\mathbf{w}}(a) &= \langle r(a)^{-1} r(b) w, l \rangle \\ &= \langle r(a^{-1}b) w, l \rangle \\ &= \langle r(b^{-1}a)^{-1} w, l \rangle \\ &= f^l_{\mathbf{w}}(b^{-1}a) \end{aligned}$$

as required.

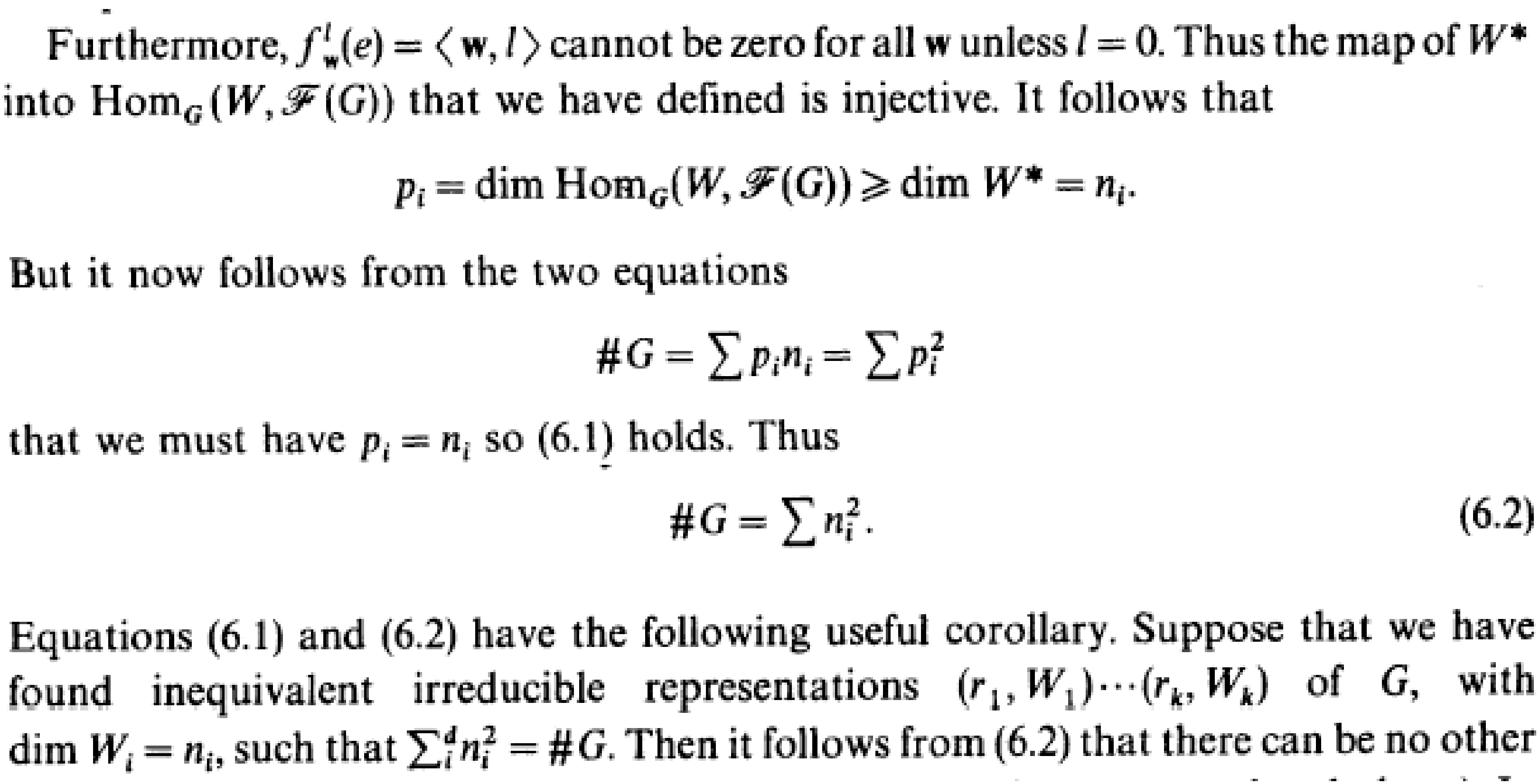
The regular representation, continued

Conclusion of the proof that

$$\dim \operatorname{Hom}_G(W, \mathscr{F}(G)) = \dim W. \tag{6.1}$$

We defined the G morphism from W* to $\mathscr{F}(G)$ by

$$f^l_{\mathbf{w}}(a) = \langle r(a^{-1})\mathbf{w}, l \rangle.$$



Furthermore, $f^l_{\mathbf{w}}(e) = \langle \mathbf{w}, l \rangle$ cannot be zero for all $\mathbf{w}$ unless $l = 0$. Thus the map of W^* into $\operatorname{Hom}_G(W, \mathscr{F}(G))$ that we have defined is injective. It follows that

$$p_i = \dim \operatorname{Hom}_G(W, \mathscr{F}(G)) \geqslant \dim W^* = n_i.$$

But it now follows from the two equations

$$\#G = \sum p_i n_i = \sum p_i^2$$

that we must have $p_i = n_i$ so (6.1) holds. Thus

$$\#G = \sum n_i^2. \tag{6.2}$$

Equations (6.1) and (6.2) have the following useful corollary. Suppose that we have found inequivalent irreducible representations $(r_1, W_1) \cdots (r_k, W_k)$ of G, with $\dim W_i = n_i$, such that $\sum_i^d n_i^2 = \#G$. Then it follows from (6.2) that there can be no other

Irreducible representation, i.e. ones not equivalent to the ones we have on our list.