Math 126 lecture 5.

First applications of Schur’s lemma.

Recalling Schur's lemma.

Let r and s be irreducible representations of G on V and W, and $T \in \mathrm{Hom}_G(V, W)$. Schur's lemma makes two assertions:

$$r \not\sim s \quad \Rightarrow T = 0, \tag{1}$$

and, if

$$r = s \quad (\text{so } V = W) \text{ then } T = zI \tag{2}$$

for some scalar z.

Proof of (1): The subspace $\ker T \subset V$ is invariant. so the alternatives are $\ker T = V$ in which case $T = 0$, or $\ker T = \{0\}$, in which case T is one to one. Also Im $(T) \subset W$ is invariant so $= \{0\}$ and so $T = 0$ or Im$T = W$ is which case T is surjective so an equivalence contrary to hypothesis.

Proof of (2): Apply (1) to $T - cI$ where c is an eigenvalue of T. We must have $T - cI = 0$ or $T = cI$. □

The representation of G on $\mathrm{Hom}(V,W)$.

For vector spaces V and W, $\mathrm{Hom}(V,W)$ denotes the space of all linear maps from V to W. So $\mathrm{Hom}(V,W)$ is itself a vector space. We know how to add two linear transformations or multiply a linear transformation by a scalar. Let r and s be representations of G on V and W. Then G acts on $\mathrm{Hom}(V,W)$ by letting $a \in G$ send $Q \in \mathrm{Hom}(V,W)$ into

$$s(a)Qr(a)^{-1}.$$

It is easy to check that this is a representation of G on $\mathrm{Hom}(V,W)$. Indeed

$$s(ab)Qr(ab)^{-1} = s(a)s(b)Q\left(r(a)r(b)\right)^{-1} = s(a)\left(s(b)Qr(b)^{-1}\right)r(a)^{-1}.$$

Also $T \in \mathrm{Hom}(V,W)$ belongs to $\mathrm{Hom}_G(V,W)$ if and only if $T \in \mathrm{Fix}(G)$. Indeed, to say that T is fixed by all elements of G means that

$$s(a)Tr(a)^{-1} = T \qquad \text{which is the same as} \qquad s(a)T = Tr(a)$$

for all elements of G.

Combining Schur's lemma with averaging.

Recall that Schur's lemma says that if r and s are irreducible representations of G then

(1) $r \not\sim s \quad \Rightarrow T = 0,$ and (2) if $r = s \quad (\text{so } V = W)$ then $T = zI$

Starting with any $Q \in \text{Hom}(V, W)$ the element

$$Q_{av} := \frac{1}{\#G} \sum_{b \in G} s(b) Q r(b)^{-1}$$

belongs to $\text{Fix}(G)$ and so to $\text{Hom}(V, W)$. It follows from (1) that if r and s are irreducible then

$$r \not\sim s \Rightarrow Q_{av} = 0, \tag{3}$$

while if $r = s$ then (2) implies that

$$Q_{av} = cI. \tag{4}$$

Averages of matrix elements.

$$Q_{av} := \frac{1}{\#G} \sum_{b \in G} s(b) Q r(b)^{-1}$$

$$r \not\sim s \Rightarrow Q_{av} = 0, \tag{3}$$

$$\text{if } r = s \text{ then } \; Q_{av} = cI. \tag{4}$$

Choose a basis of V and of W so we get square matrices $(r_{ij}(a))$ and $(s_{k\ell}(a))$. We can also write Q as a (rectangular) matrix (q_{ki}). Then Q_{av} has as its matrix entries

$$\frac{1}{\#G} \sum_{a \in G, \ell, k} s_{k\ell}(a) q_{\ell i} r_{ij}(a^{-1}).$$

If $r \not\sim s$ this must vanish for all $q_{\ell i}$ which tells us that

$$r \not\sim s \quad \Rightarrow \frac{1}{\#G} \sum_{a \in G} s_{k\ell}(a) r_{ij}(a^{-1}) = 0 \quad \forall\, i, j, k, \ell. \tag{5}$$

If $r = s$ then (4) implies that

$$\frac{1}{\#G} \sum r_{\ell k}(a) q_{ki} r_{ij}(a^{-1}) = cI$$

$$\frac{1}{\#G}\sum r_{\ell k}(a)q_{ki}r_{ij}(a^{-1}) = cI$$

and taking the trace of both sides shows that

$$c = \frac{1}{n}\operatorname{tr}(Q)$$

where $n = \dim V$. Comparing the coefficients of both sides shows that if r is irreducible then

$$\frac{1}{\#G}\sum_{a\in G} r_{k\ell}(a)r_{ij}(a^{-1}) = \frac{1}{n}\delta_{\ell i}\delta_{kj}, \quad n = \dim V. \tag{6}$$

Averaging matrix elements of unitary representations.

r and s are irreducible representations of G on V and W.

We have proved that in terms of a choice of bases we have:

$$r \not\sim s \quad \Rightarrow \frac{1}{\#G}\sum_{a\in G} s_{k\ell}(a) r_{ij}(a^{-1}) = 0 \quad \forall\, i,j,k,\ell. \qquad (5)$$

$$\frac{1}{\#G}\sum_{a\in G} r_{k\ell}(a) r_{ij}(a^{-1}) = \frac{1}{n}\delta_{\ell i}\delta_{kj}, \quad n = \dim V. \qquad (6)$$

These equations take on a more pleasant form if we restrict attention to unitary representations and use orthonormal bases so that

$$r_{ij}(a^{-1}) = \overline{r_{ji}(a)}$$

and similarly for s. Let $\mathcal{F}(G)$ denote the vector space of all complex valued functions defined on G and put the scalar product

$$(f,g) := \frac{1}{\#G}\sum_{a\in G} f(a)\overline{g(a)}$$

on $\mathcal{F}(G)$. Then (5) becomes

$$r \not\sim s \quad \Rightarrow (r_{ij}, s_{k\ell}) = 0 \quad \forall\, ijk\ell \qquad (7)$$

To repeat: Since G is finite, the space of all complex valued functions on G is a finite dimensional vector space over the complex numbers. We call this space *F*(G) and put a scalar product on this space by declaring the scalar product of two functions be given by

$$(f,g) := \frac{1}{\#G}\sum_{a\in G} f(a)\overline{g(a)}$$

If we are give a representation r of G on a vector space V and choose a basis of then for each position (ij) of the matrix describing r, we get a function on G. Suppose we have two inequivalent irreducible unitary representations r and s of G. Choose any matrix position for r and any matrix position for s. Then the functions we get are orthogonal relative to the scalar product. In symbols:

$$r \not\sim s \quad \Rightarrow (r_{ij}, s_{k\ell}) = 0 \quad \forall\ ijk\ell$$

while

$$\frac{1}{\#G}\sum_{a\in G} r_{k\ell}(a)r_{ij}(a^{-1}) = \frac{1}{n}\delta_{\ell i}\delta_{kj}, \quad n = \dim V. \text{ becomes } \quad (r_{ij}, r_{kl}) = \frac{1}{n}\delta_{ki}\delta_{\ell j}$$

In short, if we regard the matrix entries of irreducible unitary representations as functions on G, then matrix entries from inequivalent representations are orthogonal while different matrix entries from the same irreducible representation are also orthogonal, and each matrix entry has square length $1/n$ where $n = \dim V$.

The character of a representation.

Let r be a representation of the group G on the vector space V. The dimension of V is called the *degree* of the representation, r. The *character* of the representation r is the function χ^r defined on G by the formula

$$\chi^r(a) = \operatorname{tr} r(a) = \sum_i r_{ii}(a). \tag{4.1}$$

If we take $a = e$, so that $r(e)$ is the identity operator, whose trace is dim V, we see that

$$\chi^r(e) = \dim V. \tag{4.2}$$

For any two linear transformations we have $\operatorname{tr} AB = \operatorname{tr} BA$; so, if B is non-singular, $\operatorname{tr} BAB^{-1} = \operatorname{tr} A$. Thus,

$$\chi(bab^{-1}) = \chi(a) \tag{4.3}$$

if χ is the character of any representation. In other words, χ is a function which is constant on conjugacy classes. Such a function is called a *central* function.

For any representation r, we can introduce a Hermitian scalar product which is invariant under $r(a)$ for all $a \in G$. This means that if we take adjoints with respect to this scalar product, we have $r(a)^* = r(a^{-1})$. But $\operatorname{tr} r(a)^*$ is the complex conjugate of $\operatorname{tr} r(a)$, so

$$\chi(a^{-1}) = \overline{\chi(a)}. \tag{4.4}$$

The character of a direct sum is the sum of the characters.

Let r^1 and r^2 be representations of G. Then it follows from the matrix form of $r^1 \oplus r^2$ that

$$\chi^{r^1 \oplus r^2} = \chi^{r^1} + \chi^{r^2}. \tag{4.5}$$

The characters of two inequivalent irreducible representations are orthogonal.

Suppose that r and s are inequivalent irreducible representations of G.

We know that

$$(r_{ij}, s_{k\ell}) = 0 \qquad \forall\ i, j, k, \ell.$$

Since $\chi^r = \sum_i r_{ii}$ and $\chi^s = \sum_k s_{kk}$ this clearly implies that

$$(\chi^r, \chi^s) = 0.$$

The character of an irreducible representation has length one.

We know that if r is an irreducible unitary representation then

$$(r_{ij}, r_{kl}) = \frac{1}{n}\delta_{ki}\delta_{\ell j}$$

Since $\chi^r = \sum_i r_{ii}$ we have

$$(\chi^r, \chi^r) = \sum_{i,j}(r_{ii}, r_{jj}) = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

So

$$(\chi^r, \chi^r) = 1.$$

The character determines the representation.

Now let r be a representation of G on a vector space, V, which is not necessarily irreducible, and let

$$r = r^1 \oplus \cdots \oplus r^k$$

be a decomposition of r into irreducible representations. Let ϕ be the character of r, and

let χ_i be the character of r^i, so that

$$\phi = \chi_1 + \cdots + \chi_k.$$

Let s be some particular irreducible representation of G and let χ be its character. Then

$$(\phi, \chi) = (\chi_1, \chi) + \cdots + (\chi_k, \chi).$$

The terms on the right are all zero or one, according as $r^i \nsim r$ or $r^i \sim r$. Thus,

(ϕ, χ) is the number of terms in the decomposition of r which are isomorphic to s. In particular, this number does not depend on the particular choice of decomposition. (4.8)

From (4.8) it follows that *two representations with the same character are equivalent.* Indeed, by taking scalar products with the characters of all the irreducible representations, we can determine how many times each irreducible occurs in a decomposition of the given representation.

A character which has length one must be irreducible.

Notice that any character ϕ can be written as

$$\phi = m_1\chi_1 + \cdots + m_p\chi_p,$$

where the χ_i are orthogonal characters and the m_i are non-negative integers. It follows that

$$(\phi, \phi) = m_1^2 + \cdots + m_p^2 \tag{4.9}$$

and, in particular,

$$\phi \text{ is irreducible if and only if } (\phi, \phi) = 1.$$

dim $\text{Hom}_G(V,W)$ when V is irreducible.

Let ϕ be the character of a representation of G on a vector space W, and let χ be the character of an irreducible representation of G on the vector space V. If we decompose

$$W = W_1 \oplus \cdots \oplus W_k$$

into irreducibles, we see that

$$\text{Hom}_G(W, V) = \text{Hom}_G(W_1, V) \oplus \cdots \oplus \text{Hom}_G(W_k, V).$$

By Schur's lemma, each of these spaces is either one dimensional or zero dimensional according to whether the representation of G on W_i is or is not equivalent to the representation of G on V. Combining this with

(ϕ, χ) is the number of terms in the decomposition of r which are equivalent to the representation of G on V we see that

$$(\phi, \chi) = \dim \text{Hom}_G(W, V). \tag{4.10}$$

dim $\text{Hom}_G(U,V)$ in general, 1.

Now let r_u and r_v be representations of G on U and V. We do not assume that r_u and r_v are irreducible. We wish to compute $\dim \text{Hom}_G(U, V)$.

Let us first consider a special case. Suppose $U = V = W \oplus W$, where W is irreducible. We can write any vector in U as $(\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are in W. Thus, for any $T \in \text{Hom}(V, V)$ we have

$$T(\mathbf{w}_1, \mathbf{w}_2) = (T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2, T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)$$

where $T_{ij} \in \text{Hom}(W, W)$. So

$$\begin{aligned} T \circ r_{W \oplus W}(a)(\mathbf{w}_1, \mathbf{w}_2) &= T(r_W(a)\mathbf{w}_1, r_W(a)\mathbf{w}_2) \\ &= (T_{11} r_W(a)\mathbf{w}_1 + T_{12} r_W(a)\mathbf{w}_2, T_{21} r_W(a)\mathbf{w}_1 + T_{22} r_W(a)\mathbf{w}_2) \end{aligned}$$

while

$$\begin{aligned} &r_{W \oplus W}(a) T(\mathbf{w}_1, \mathbf{w}_2) \\ &= (r_W(a)(T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2), r_W(a)(T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)) \\ &= (r_W(a) T_{11}\mathbf{w}_1 + r_W(a) T_{12}\mathbf{w}_2, r_W(a) T_{21}\mathbf{w}_1 + r_W(a) T_{22}\mathbf{w}_2). \end{aligned}$$

So $T \in \text{Hom}_G(V, V)$ if and only if each $T_{ij} \in \text{Hom}_G(W, W)$. By Schur's lemma, each T_{ij} ranges over a one-dimensional space, hence $\dim \text{Hom}_G(W \oplus W, W \oplus W) = 4 = 2 \times 2$.

dim $\mathrm{Hom}_G(U,V)$ in general, 2.

For any representation, we may make the decomposition

$$U = (U_1 \oplus \cdots \oplus U_{p_1}) \oplus (U_{p_1+1} \oplus \cdots \oplus U_{p_1+p_2}) \oplus \cdots (\cdots U_{p_1+\cdots p_k})$$

where the first p_1 spaces are all equivalent to the irreducible representation W_1, the next p_2 spaces are all equivalent to the irreducible representation W_2 etc., and $W_1, \ldots, W_k$ are *inequivalent* irreducible representations of G. We may make the same decomposition

$$V = (V_1 \oplus \cdots \oplus V_{q_1}) \oplus (V_{q_1+1} \oplus \cdots \oplus V_{q_1+q_2}) \oplus \cdots$$

for V. By Schur's lemma, any $T \in \mathrm{Hom}_G(U, V)$ when applied to any $\mathbf{u} \in U_1 \oplus \cdots \oplus U_{p_1}$ must give $T\mathbf{u}$ lying in $V_1 \oplus \cdots \oplus V_{q_1}$. Then the same argument as in the special case shows that

$$\dim \mathrm{Hom}_G(U, V) = p_1 q_1 + p_2 q_2 + \cdots + p_k q_k. \tag{4.11}$$

In particular, if $U = V$,

$$\dim \mathrm{Hom}_G(V, V) = p_1^2 + \cdots + p_k^2,$$

where p_i is the number of times that the ith irreducible representation occurs in V.