Math 126 lecture 3

The finite subgroups of *SO(3)*.

Morphisms.

Let the group G act on two sets M_1 and M_2. A map $f : M_1 \to M_2$ is called a **morphism** or is said to be **equivariant** if

$$f(am_1) = af(m_1) \quad \forall m_1 \in M_1 \quad \text{and} \ \forall \ a \in G.$$

Examples.

- If the action on M_2 is trivial (i.e. $am_2 = m_2 \ \forall \ m_2 \in M_2$) then being equivariant means that f is **invariant**, i.e. $f(am_1) = f(m_1) \ \forall \ m_1 \in M_1$. This is an important special case, but the general notion will be important.

- Let M_2 be the set of all subgroups of G, and let G act on M_2 by the action induced from the conjugation action of G on itself. Let M_1 be any set on which G acts, and let $f : M_1 \to M_2$ be the map which assigns to each $m \in M_1$ its isotropy group:

  $$f(m) := G_m.$$

  The formula

  $$G_{am} = aG_m a^{-1}$$

  says that f is a morphism.

More examples of morphisms.

Here is a closely related example: Let G act on a set M, and consider the action of G on $G \times M$ by

$$a(b,m) := (aba^{-1}, am).$$

Then the projection onto the second factor:

$$\theta : G \times M \to M, \qquad \theta(b,m) := m$$

is a morphism as is the projection onto the first factor

$$\tau : G \times M \to G, \qquad \tau(b,m) := b$$

where in this second equation we are considering the conjugation action of G on itself.

Let $Z \subset G \times M$ be the subset consisting of all pairs (b,m) such that $bm = m$. The set Z is carried into itself by the action of G on $G \times M$. In other words $aZ = Z$ for all $a \in G$. Indeed, if $(b,m) \in Z$ meaning $bm = m$, then $(aba^{-1})am = am$ which says that $(aba^{-1}, am) \in Z$.

Counting fix points.

$a(b,m) := (aba^{-1}, am).$

$$\theta : G \times M \to M, \qquad \theta(b,m) := m$$

$$\tau : G \times M \to G, \qquad \tau(b,m) := b$$

$Z \subset G \times M$ is the subset consisting of all pairs (b,m) such that $bm = m.$

Let $\rho : Z \to M$ denote the restriction of θ to Z and $\sigma : Z \to G$ denote the restriction of τ.

For an $m \in M$, its inverse image $\rho^{-1}(m)$ under ρ consists of all (a,m) such that $a \in G_m$. In symbols

$$\rho^{-1}(m) = G_m \times \{m\}.$$

For $a \in G$, we have

$$\sigma^{-1}(a) = \{a\} \times \mathrm{Fix}(a).$$

Counting fixed points, 2.

let us remove the set $\{e\} \times M$ from Z, and consider the set $Y := Z \setminus \{e\} \times M$.

We have the map $\rho : Z \to M$

Let us also assume that every $a \neq e$ has only finitely many fixed points. This implies that Y is a finite set. Let $f : Y \to M$ denote the restriction of ρ to

Y, and let $g : Y \to G$ denote the restriction of σ to Y. Let $P \subset M$ denote the set of points of M which are left fixed by *some* $a \neq e$. So $P = f(Y)$, and $g(Y) \subset G \setminus \{e\}$.

We can now count the number of elements in Y in two ways: Using g we see that

$$\#Y = \sum_{a \neq e} \#\mathrm{Fix}(a).$$

Using P we see that

$$\#Y = \sum (\#G_m - 1).$$

Counting the fix points 3.

$$\#Y = \sum_{a \neq e} \#\,\mathrm{Fix}(a)$$

$$\#Y = \sum_{m \in P} (\#G_m - 1).$$

The -1 comes from the fact that we removed e. We can simplify this expression using the fact that $\#G_{ama^{-1}} = \#G_m$, so $\#G_m$ is constant on orbits of the G action on P. So if

$$P = P_1 \cup \cdots \cup P_r$$

is the decomposition of P into orbits, so that the number of points in each orbit is $\#G/\#G_m$ we have

$$\#Y = \sum_{orbits} \frac{\#G}{\#G_m} (\#G_m - 1)\,.$$

so

$$\sum_{a \neq e} \#\,\mathrm{Fix}(a) = \sum_{orbits} \frac{\#G}{\#G_m} (\#G_m - 1)\,.$$

Using the formula for finite subgroups of *SO(3)*.

$$\sum_{a \neq e} \# \operatorname{Fix}(a) = \sum_{orbits} \frac{\#G}{\#G_m} \left(\#G_m - 1\right).$$

We will use this last formula to classify the finite subgroups of $SO(3)$ up to conjugacy. Since $\# \operatorname{Fix}(a) = 2$ for any non-trivial rotation acting on the sphere, the left hand side of the formula above is just $2(\#G - 1)$. Introduce the notation

$$\begin{aligned} n &:= \#G \\ r &:= \#(\text{of orbits}) \\ n_i &:= \#G_m \quad \text{where } m \in \ i\text{-th orbit} \ . \end{aligned}$$

Thus

$$2(n-1) = \sum_{i=1}^{r} \frac{n}{n_i}(n_i - 1)$$

or, dividing by n:

$$2 - \frac{2}{n} = r - \sum_{i=1}^{r} \frac{1}{n_i}. \qquad (1)$$

Using the formula for finite subgroups of *SO(3)*, continued.

We have
$$2-\frac{2}{n}=r-\sum_{i=1}^{r}\frac{1}{n_i}.$$
where
$$\begin{aligned} n &:= \#G \\ r &:= \#(\text{of orbits}) \\ n_i &:= \#G_m \ \text{ where } m\in \ i\text{-th orbit} \end{aligned}$$

We may exclude the trivial case where $G=\{e\}$, so that P is not empty. By definition, each element of P is fixed by at least one $a\neq e$ so $\#G_m\geq 2$ and so

the right hand side of the above equation is $\geq \frac{r}{2}$. The left hand side is <2. So $r<4$. But $r=1$ is impossible since $n_i\leq n$ and so

$$2-\frac{2}{n}>1-\frac{1}{n_1}.$$

Thus the only possibilities are $r=2$ and $r=3$.

r = 2, the cyclic groups.

$$2 - \frac{2}{n} = r - \sum_{i=1}^{r} \frac{1}{n_i}. \tag{1}$$

If $r = 2$ all non-trivial rotations are about a fixed axis. Equation (1) becomes

$$\frac{2}{n} = \frac{1}{n_1} + \frac{1}{n_2}$$

and since $n_i \leq n$ this implies that $n_1 = n_2 = n$. So $G_m = G$, and the group consists of all rotations through angles $2\pi j/n$ about a fixed axis. For a given n, all such sugroups are conjugate, and the group is isomorphic to $C_n := \mathbb{Z}/n\mathbb{Z}$.

The possibilities for r = 3.

$$2 - \frac{2}{n} = r - \sum_{i=1}^{r} \frac{1}{n_i}. \tag{1}$$

Now suppose that $r = 3$ so (1) becomes

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{n}.$$

We choose notation so that $n_1 \leq n_2 \leq n_3$. Then $n_1 = 2$ for otherwise the left hand side of the above equation would be ≤ 1. Similarly, $n_2 \geq 4$ is impossible. If $n_2 = 3$ the above equation becomes

$$\frac{1}{n_3} = \frac{1}{6} + \frac{2}{n} \qquad \text{so } n_3 < 6.$$

The possibilities that we haven't excluded for (n_1, n_2, n_3) and n are:

$$\begin{array}{ll}
(2,2,k) & k \geq 2 \ \text{arbitrary}, \quad n_3 = k, \quad n = 2k, \\
(2,3,3) & n \quad = 12 \\
(2,3,4) & n \quad = 24 \\
(2,3,5) & n \quad = 60.
\end{array}$$

(2,2,k), the dihedral groups.

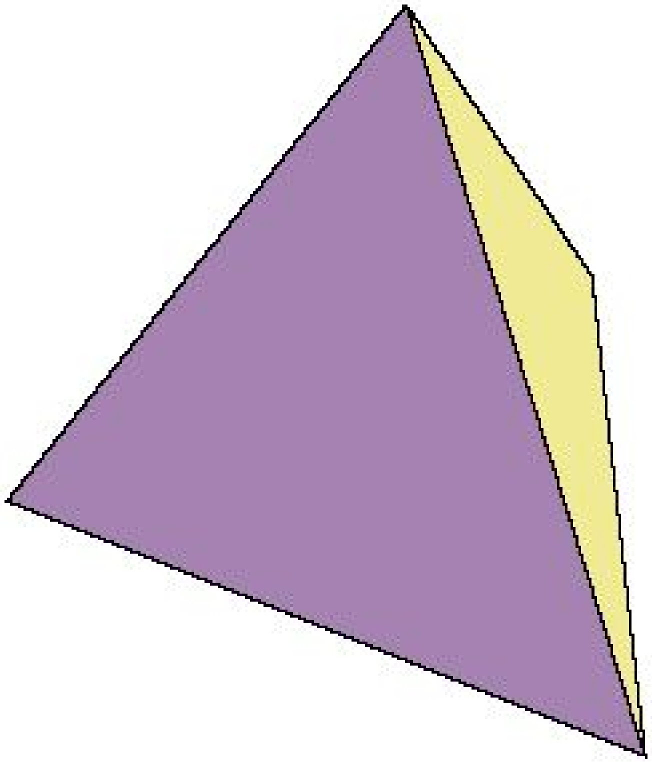
Since $n_3 = k = n/2$ the orbit P_3 has two elements, and so the subgroup G_m corresponding to each of these two elements is the same, and is the cyclic group

C_k. For any q belonging to P_1 or P_2 the group $\#G_q = 2$ and so G_q consists of the identity and rotation through 180^o . The subgroup C_k acts on P_1 and no element of C_k other than the identity fixes any point of P_1. Since $\#P_1 = k$, we see that the points of P_1 all lie in a plane and form a regular k-gon. Similarly for P_2. The case $k = 2$ is a bit degenerate in that the group is (up to conjugacy) given by the identity together with 180^o rotations about each of the coordinate axes. If $k \geq 2$ is even, there are two types of 180^o axes, those which pass through the vertices and those which bisect the sides. The two orbits P_1 and P_2 are the intersections of each of these types of axis with the sphere. If $k \geq 2$ is odd, then all 180^o axes pass through a vertex and bisect the opposite side. The vertices form one orbit, say P_1, and the other intersection of each axis with the sphere form P_2.

(2,3,3), the tetrahedral group.

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{n}.$$

So $n = 12$, $\#P_1 = 6$, $\#P_2 = \#P_3 = 4$. The group of rotational symmetries of a tetrahedron has 12 elements since a given vertex can be moved to any other, and the isotropy group of a vertex is C_3. So (2,3,3) occurs as the symmetry group of the tetrahedron. To show that up to conjugacy this is the only possibility, it suffices to show that the points of P_2 form a regular tetrahedron. Let $m \in P_2$. Then $G_m = C_3$, so the other three point3 of P_3 form an equilateral triangle. In particular they are equidistant from one another. Doing the same for each of the points in P_2 shows that they are the vertices of a regular tetrahedron.

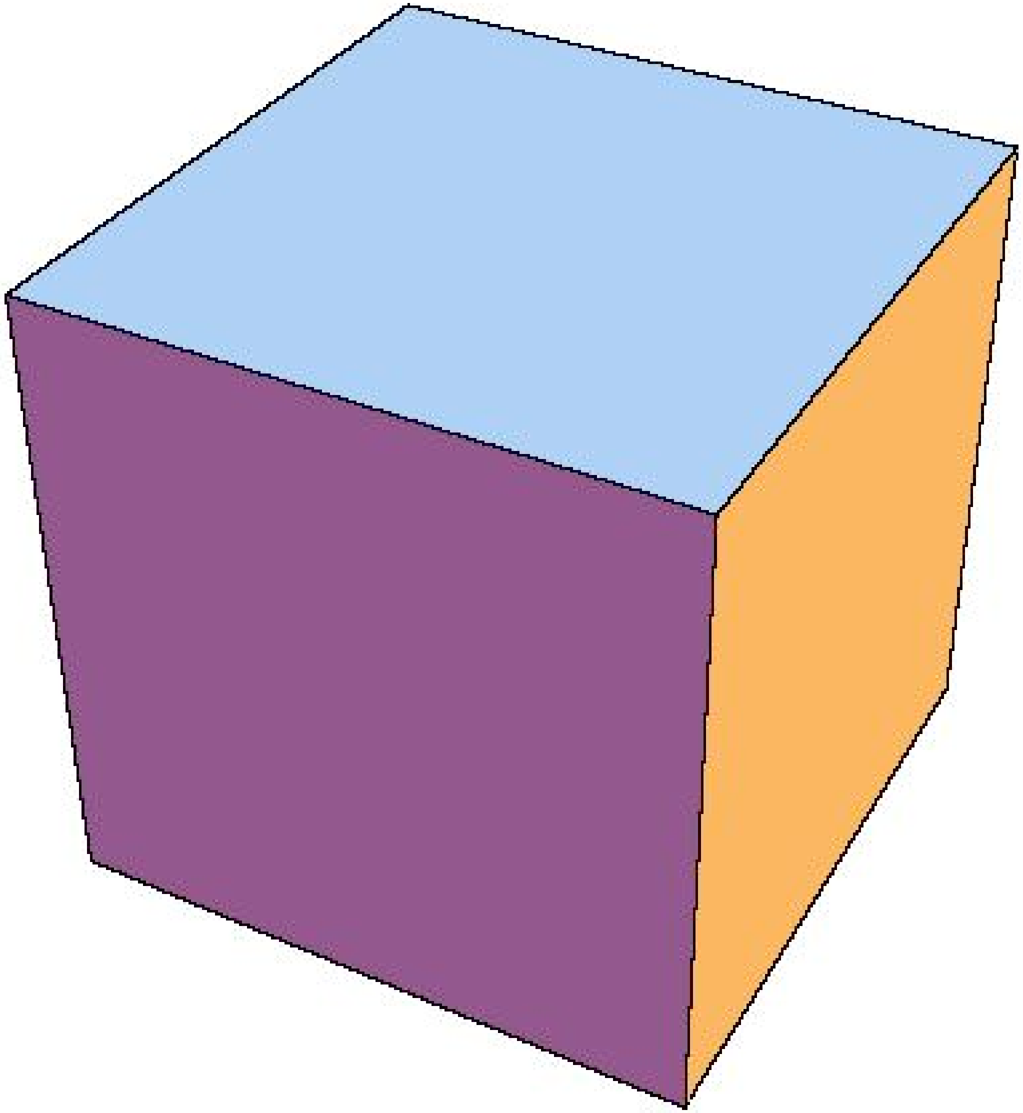


Tetrahedron

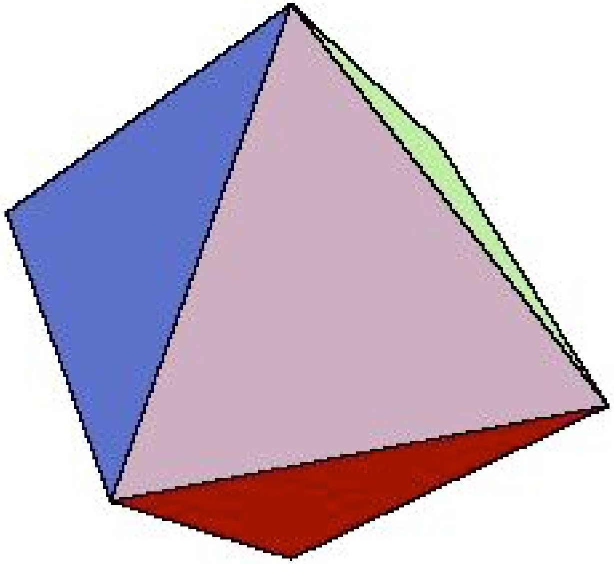
(2,3,4) the cubic group.

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{n}.$$

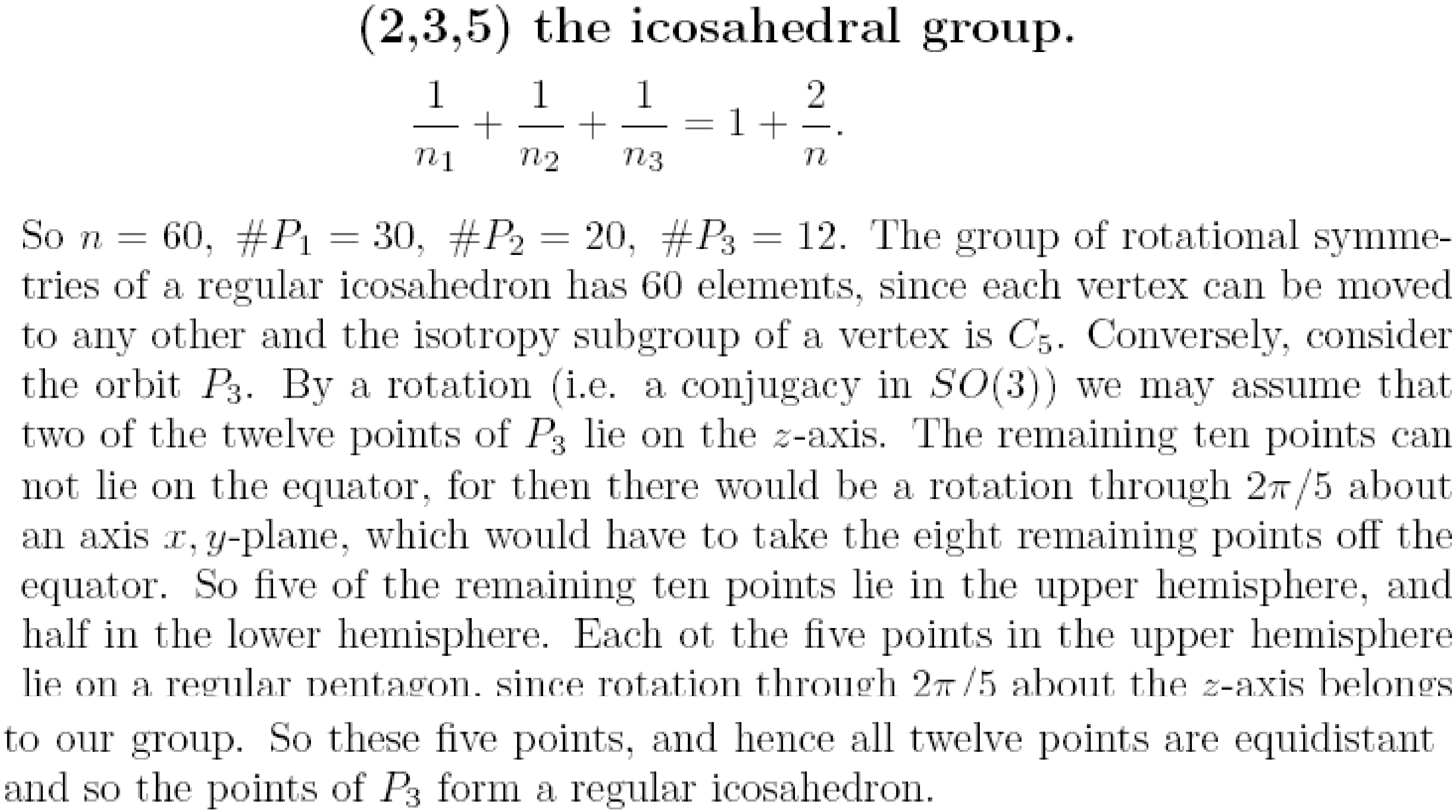
So $n = 24$, $\#P_1 = 12$, $\#P_2 = 8$, $\#P_3 = 6$. The group of rotational symmetries of a cube has 24 elements since a given vertex can be moved to any other, and the isotropy group of a vertex is C_3, as any adjacent vertex can be rotated into another. So (2,3,3) occurs as the symmetry group of the cube. Conversely, as above, the elements of P_3 form a regular octahedron (whose dual polytope is a cube).



Cube



Octahedron



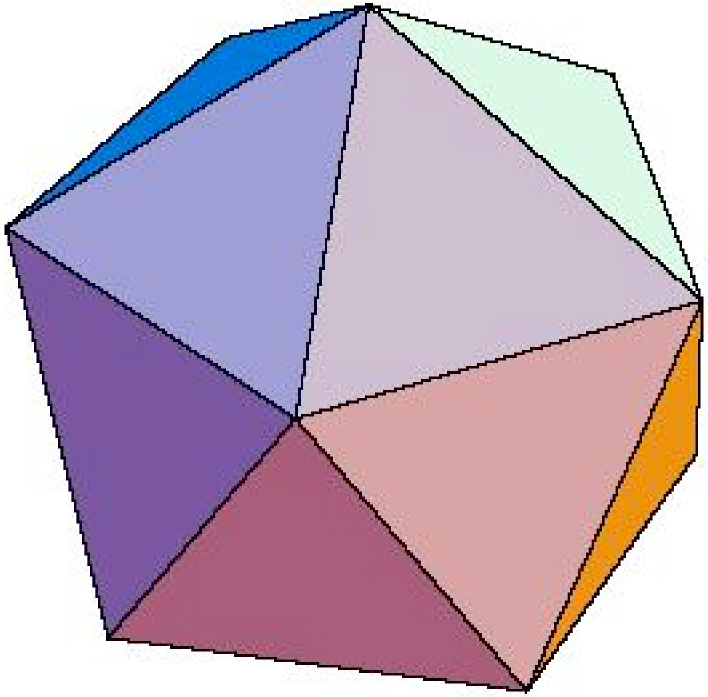
(2,3,5) the icosahedral group.

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{n}.$$

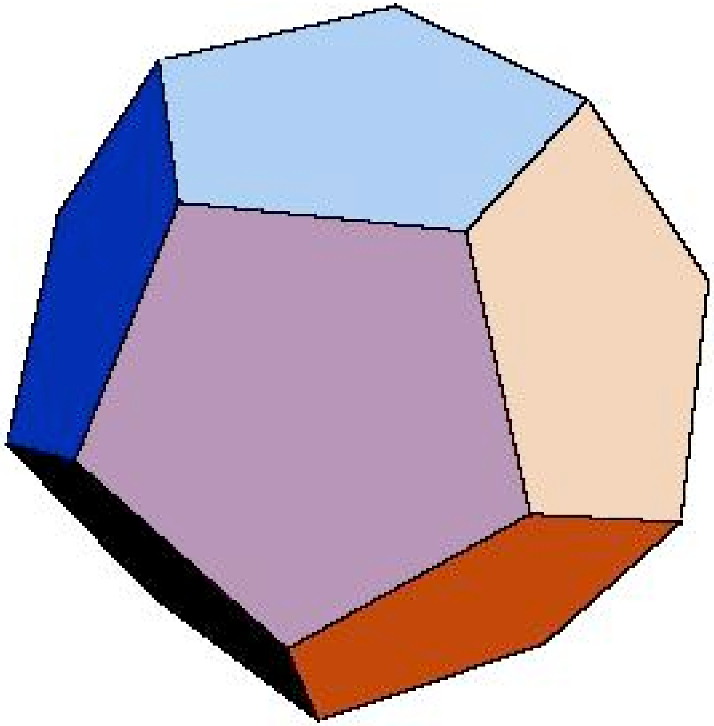
So $n = 60$, $\#P_1 = 30$, $\#P_2 = 20$, $\#P_3 = 12$. The group of rotational symmetries of a regular icosahedron has 60 elements, since each vertex can be moved to any other and the isotropy subgroup of a vertex is C_5. Conversely, consider the orbit P_3. By a rotation (i.e. a conjugacy in $SO(3)$) we may assume that two of the twelve points of P_3 lie on the z-axis. The remaining ten points can not lie on the equator, for then there would be a rotation through $2\pi/5$ about an axis x, y-plane, which would have to take the eight remaining points off the equator. So five of the remaining ten points lie in the upper hemisphere, and half in the lower hemisphere. Each ot the five points in the upper hemisphere lie on a regular pentagon, since rotation through $2\pi/5$ about the z-axis belongs to our group. So these five points, and hence all twelve points are equidistant and so the points of P_3 form a regular icosahedron.

The icosahedral group is denoted by I. It is isomorphic to the alternating group on five letters, A_5 .

To visualize this isomorphism it is best to pass to the buckyball which is obtained by truncating each vertex of the icosahedron at each vertex along a plane perpendicular to the radius at each vertex. Since each vertex is formed by the intersection of five triangles, the new faces created in this way are pentagons. The remaining portions of the original triangular faces are converted into hexagons.

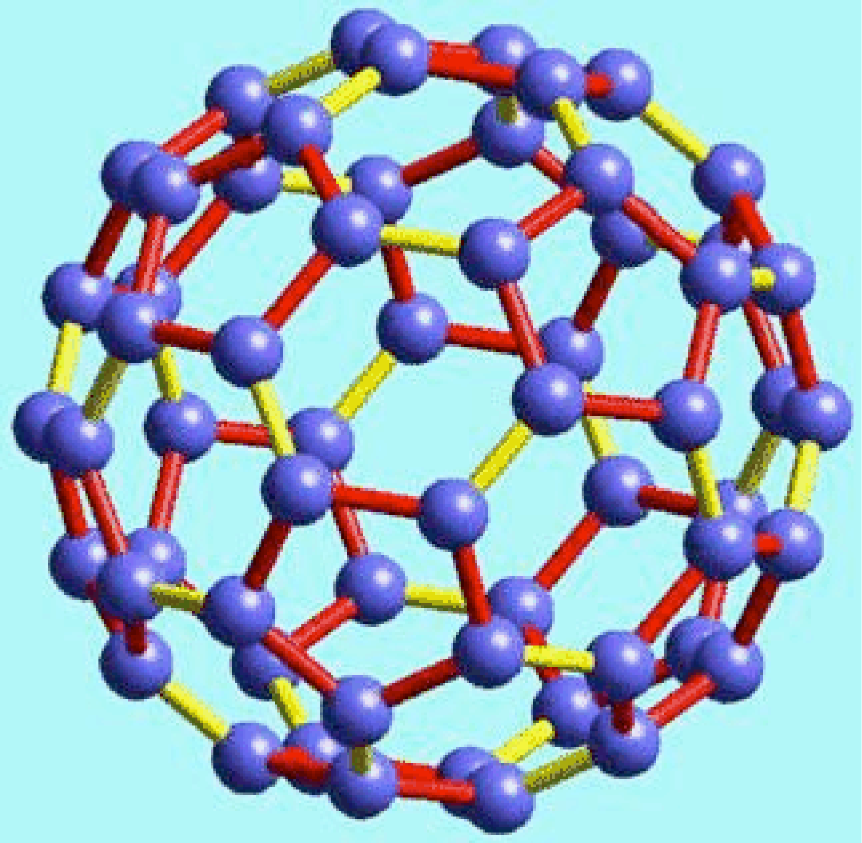


Icosahedron

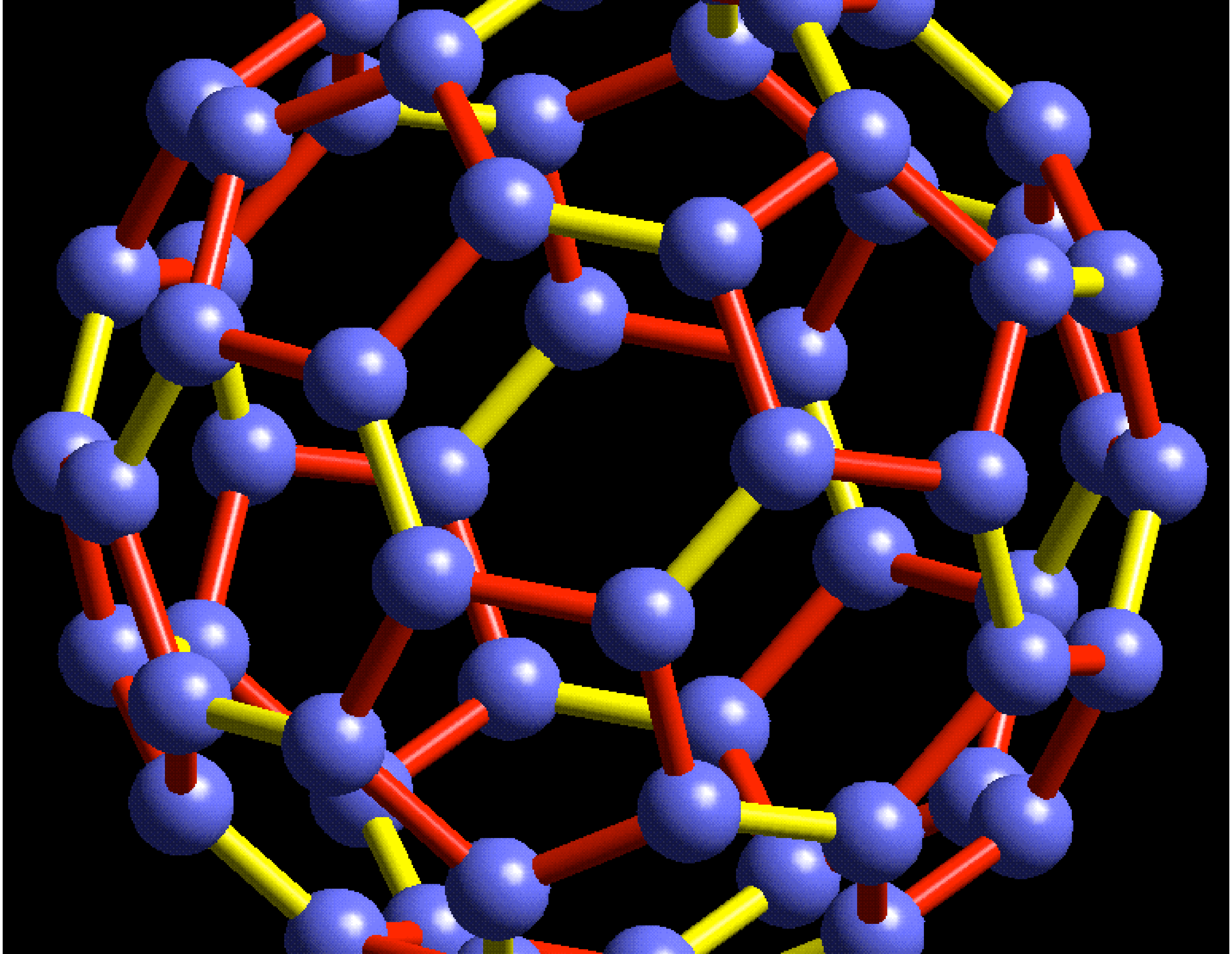


Dodecahedron

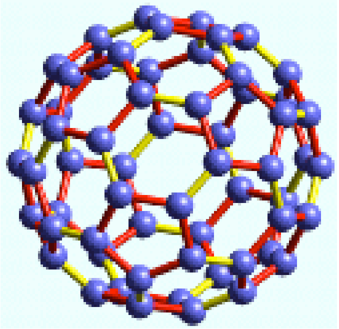
The buckyball.



if we pick a double bond, b , there will be a unique double bond, b', opposite it, in the sense that b and b' lie on a plane bisecting the buckyball. If we draw the line joining the midpoints of b and b', then a 180° rotation having this line as axis is as symmetry of the buckyball. There are thirty double bonds, and hence fifteen such pairs. Thus there are fifteen elements of “order two” (i.e. satisfying $r^2 = e$, where e denotes the identity element) in I.



Proof of the isomorphism of *I* with A_5.



There are 15 elements of degree 2. There are 10 three fold axes and so 20 elements of degree 3. There are 6 five fold axes so 24 elements of degree 5. Together with the identity element we get 1+15+20+24 =60 accounting for all the elements in *I*. The group *I* acts transitively on the set of 15 axes of degree 2. So the isotropy group of one such axis has 4 elements. Each of these four elements other than the identity must therefore be of degree 2 and so they all commute. This implies that their axes must be mutually perpendicular. So the set of 15 axes of degree 2 breaks up into 5 sets of configurations each consiting of 3 mutually perpendicular axes. So the group *I* acts transitively on a set with 5 elements and no element of I other than the identity acts trivially. So we have an isomorphism of I with a subgroup of order 60 of the permutation group on 5 objects, and the only such subgroup is the subgroup consisting of even permutations, i.e the group A_5.