2

REPRESENTATION THEORY OF FINITE GROUPS

In this chapter we present the classical theory of representations of finite groups as linear transformations on vector spaces over the complex numbers. As we have seen in Chapter 1, geometrical problems give rise to the study of the representation theory of groups over the real numbers and also over the integers. These kinds of questions are far more subtle than the theory of representations over the complex numbers. However, the complex representation theory has a beautiful structure and a highly developed theory, has an enormous range of applications in mathematics, physics, and chemistry, and is the starting point for any of the other representation theories.

In all that follows in the present chapter, unless otherwise specified, all vector spaces will be finite-dimensional complex vector spaces and all groups will be finite. (In later chapters we will have something to say about infinite-dimensional vector spaces and certain kinds of non-finite groups.)

2.1 Definitions, examples, irreducibility

We recall that a representation of a group G on a vector space V is an action of G on V, in which each $a \in G$ acts as a linear transformation: the map

$$r(a)\colon V \to V \quad \mathbf{v} \rightsquigarrow r(a)\mathbf{v}$$

is a linear transformation for each a. When we denote a representation by r, the symbol r is meant to specify the group, G, the vector space, V, and the particular action of G on V. For reasons of convenience, we will frequently write $a\mathbf{v}$ for the action of a on $\mathbf{v}$ in the representation r, so that

$$r(a)\mathbf{v} = a\mathbf{v}.$$

If we choose a basis of V, then each linear transformation, $r(a)$, has a matrix relative to the basis. We will denote the matrix associated to $r(a)$ by $(r_{ij}(a))$. For example,

$$(r_{ij}(e)) = (\delta_{ij})$$

is the identity matrix, where e is the identity element of the group. It is apparent that

$$(r_{ij}(a^{-1})) = (r_{ij}(a))^{-1}$$

and

$$r_{ij}(ab) = \sum_k r_{ik}(a) r_{kj}(b).$$

The matrices r_{ij} depend on the choice of basis for V. They are determined by the representation only up to conjugacy.

Let r and r' be representations of the same group, G, on vector spaces V and V'. We say that r and r' are *similar*, or *equivalent*, and write $r \sim r'$ if there exists a non-singular linear transformation, T, mapping V onto V' such that

$$r'(a)T = Tr(a) \quad \text{for all } a \in G.$$

In other words, T is a morphism for the group G and is an isomorphism of vector spaces.

Let r be a representation of a group G on a vector space V. A subspace W of V is said to be *invariant* if $r(a)W \subset W$ for all a in G. This means that for every $\mathbf{w}$ in W and every a in G the vector $r(a)\mathbf{w}$ is again in W. Applying $r(a^{-1}) = r(a)^{-1}$ we see that W is invariant if and only if $r(a)W = W$ for all a in G. If W is an invariant subspace, then the restriction of $r(a)$ to W, i.e. the operator $r(a)$ applied only to elements of W, is a linear transformation of W into itself, which we may denote by $r(a)|_W$. Since $r(a)|_W r(b)|_W = (r(a)r(b))|_W = r(ab)|_W$, we see that the map $r|_W$ sending $a \in G$ into $r(a)|_W$ is a representation of G on W. It is sometimes called a *subrepresentation* of the representation r.

For any r, the vector space V itself and the subspace $\{0\}$ are clearly (trivial) invariant subspaces. A representation r is called *irreducible* if these are the only invariant subspaces.

Examples

Any representation of G on a one-dimensional complex vector space is automatically irreducible, since there is no room for any subspace between $\{0\}$ and V. A linear transformation on a one-dimensional vector space is just multiplication by a complex number. So a linear representation of G on a one-dimensional vector space is a rule which assigns to each a in G a complex number, $\kappa(a)$; i.e. a representation on a one-dimensional vector space is just a complex-valued function, κ, on the group, subject to the conditions $\kappa(ab) = \kappa(a)\kappa(b)$ and $\kappa(e) = 1$. For any finite group, $a^{\#G} = e$, so $\kappa(a)^{\#G} = 1$. Thus $\kappa(a)$ is a root of unity, whose order is some divisor of $\#G$. Any two linear transformations on a one-dimensional space are equivalent if and only if they are multiplication by the same complex number. Thus the equivalence class of the one-dimensional representation is also completely determined by the function κ.

For example, consider the group C_n, and let b be a generator of C_n, i.e. an element whose powers include all elements of C_n. Then for each k with $0 \leqslant k < n$ we can define κ_k by

$$\kappa_k(b) = \exp 2\pi i k/n \quad \text{and hence} \quad \kappa_k(b^s) = \exp 2\pi i s k/n.$$

These give n distinct irreducible representations of C_n. We shall see later on that it follows from the formula

$$1^2 + 1^2 + \cdots + 1^2 = n = \#C_n, \quad n \text{ summands},$$

that these are all the irreducible representations of C_n. (Here the 1's represent the dimension of the irreducible representations, and we are summing the squares of the dimensions.)

Let us prove that if G is a finite *commutative* group, then any finite-dimensional irreducible representation of G over the complex numbers must, in fact, be one dimensional: Choose any $a \in G$. The operator $r(a)$, being a linear transformation on a finite-dimensional complex vector space, has at least one eigenvector $\mathbf{v}$ with eigenvalue λ_a. (Here is where we use the fact that we are over the complex numbers and that the vector space is finite dimensional.) Let W_1 be the subspace consisting of *all* $\mathbf{w}$ satisfying $r(a) = \lambda_a \mathbf{w}$. Then $W_1 \neq \{0\}$ since $\mathbf{v} \in W_1$. We claim that W_1 is invariant: that is, if $\mathbf{w} \in W_1$, $r(b)\mathbf{w} \in W_1$ for any b in G. The proof is straightforward:

$$\begin{aligned} r(a)(r(b)\mathbf{w}) &= [r(a)r(b)]\mathbf{w} \\ &= r(ab)\mathbf{w} \\ &= r(ba)\mathbf{w} \quad \text{(here is where we use the commutativity of } G) \\ &= r(b)r(a)\mathbf{w} \\ &= r(b)(\lambda_a \mathbf{w}) = \lambda_a(r(b)\mathbf{w}). \end{aligned}$$

Now choose some other element c in G. Inside W_1 we can find some eigenvector, $\mathbf{u}$, of $r(c)$, with eigenvalue λ_c. Let W_2 be the subspace of W_1 consisting of all vectors $\mathbf{z}$ satisfying $r(c)\mathbf{z} = \lambda_c \mathbf{z}$. Then W_2 is a non-zero invariant subspace of W_1 and every vector in W_2 satisfies

$$r(a)z = \lambda_a z \quad \text{and} \quad r(c)z = \lambda_c z.$$

Continuing this way, using all the group elements, we find a subspace $W \neq \{0\}$ and eigenvalues λ_a for every group element a such that every vector $\mathbf{v}$ in W satisfies $r(b)\mathbf{v} = \lambda_b \mathbf{v}$ for every b in G. This clearly implies that *any* one-dimensional subspace of W is invariant under G. If our original vector space V were irreducible, this one-dimensional subspace would have to be V itself. Thus, any irreducible representation of a commutative group is one dimensional.

If the group G is not commutative, it is rather hard for it to have many one-dimensional representations. In fact, we shall prove later on that if all the irreducible representations of G are one dimensional, then G must be commutative. For the moment, let us indicate partial results in this direction. If κ is any function on G which satisfies $\kappa(ab) = \kappa(a)\kappa(b)$, then

$$\kappa(bab^{-1}a^{-1}) = 1$$

for any a and b in G. Let us use this identity to determine the possible one-dimensional representations of S_3. We think of S_3 as symmetries of the triangle. If b is a reflection, then $b^2 = e$ so that $\kappa(b)^2 = 1$, i.e.

$$\kappa(b) = \pm 1 \quad \text{for reflections.}$$

If a is a rotation and b is a reflection, then $bab^{-1} = a^{-1}$ so $bab^{-1}a^{-1} = a^{-2}$. But every rotation in S_3 is of the form $c = a^{-2}$ so

$$\kappa(c) = 1 \quad \text{for rotations.}$$

Thus there are two possibilities:

$$\kappa \equiv 1 \quad \text{or} \quad \kappa(a) = \operatorname{sgn}(a) = \begin{matrix} +1 \text{ for rotations} \\ -1 \text{ for reflections.} \end{matrix}$$

(We shall see later that $\kappa \equiv 1$ and $\kappa(a) = \operatorname{sgn}(a)$ are the only one-dimensional representations of S_n for any n.)

The group S_3 also has a natural two-dimensional representation. If we think of S_3 as symmetries of the triangle, this determines an action of S_3 on the plane $\mathbb{R}^2$, i.e. a homomorphism of S_3 into $GL(2, \mathbb{R})$. We can think of $GL(2, \mathbb{R})$ as a subgroup of $GL(2, \mathbb{C})$ so the action of S_3 as symmetries of the triangle gives rise to a two-dimensional representation of S_3. (More generally, the same argument shows that any representation of any group on a real vector space gives rise to a representation of a complex vector space: in terms of a basis, just think of the real matrices $(r_{ij}(a))$ as being complex matrices.) The two-dimensional representation of S_3 that we have just constructed is irreducible. Indeed, let a be rotation through 120°. As a complex matrix $r(a)$ has two distinct eigenvalues, $\omega = \exp(2\pi i/3)$ and $\bar{\omega} = \exp(-2\pi i/3)$ with corresponding eigenvectors $\mathbf{v}$ and $\bar{\mathbf{v}}$. Since ω and $\bar{\omega}$ are distinct, the only lines in $\mathbb{C}^2$ which are left invariant by $r(a)$ are the lines through $\mathbf{v}$ and through $\bar{\mathbf{v}}$. Any non-trivial invariant subspace of $\mathbb{C}^2$ would have to be one dimensional and hence coincide with one of these lines. If b is a reflection, then $bab^{-1} = a^{-1}$, and hence $r(a)r(b^{-1})\mathbf{v} = r(b^{-1})r(a^{-1})\mathbf{v} = r(b^{-1})(\bar{\omega}\mathbf{v}) = \bar{\omega} r(b^{-1})\mathbf{v}$. In other words, $r(b)$ or $r(b^{-1})$ interchanges the lines through $\mathbf{v}$ and through $\bar{\mathbf{v}}$. Hence, neither of these lines can be invariant under S_3.

As we shall prove later, it follows from the equation

$$1^2 + 1^2 + 2^2 = 6 = \#S_3$$

(the sum of the squares of the dimensions)

that the three representations, the two one-dimensional ones and the two-dimensional one, are, up to equivalence, the only irreducible representations of S_3.

In the preceding chapter we described two inequivalent real representations of S_4, both three dimensional – the representation r_T of S_4 as all symmetries of the tetrahedron, and the representation r_O of S_4 as all rotational symmetries of the cube. Each of these gives rise to a complex three-dimensional representation, and the proof given in Chapter 1 still works to prove that these two representations are not equivalent over the complex numbers. It is not hard to show directly, and will follow from machinery to be developed later, that these representations are both irreducible.

To discover a two-dimensional representation of S_4, we note that there is a homomorphism from S_4 to S_3. One way of defining it is as follows: think of S_4 as the rotational symmetries of the cube. There are three lines which pass through the centers of opposite faces; if the cube is the one with the vertices $(\pm 1, \pm 1, \pm 1)$, then these lines are the coordinate axes. Any symmetry of the cube must permute these three lines. This gives a homomorphism ϕ from S_4 to S_3. We have already constructed a two-dimensional representation, r, of S_3. Then $r \circ \phi$, sending each a in S_4 to $r(\phi(a))$, is an

irreducible two-dimensional representation of S_4. We also have the two one-dimensional representations given by $\kappa \equiv 1$ and $\kappa(a) = \operatorname{sgn}(a)$. From

$$3^2 + 3^2 + 2^2 + 1^2 + 1^2 = 24 = \#S_4$$

we will be able to conclude that these are all the irreducible representations of S_4.

2.2 Complete reducibility

Let r^1 and r^2 be representations of the group G on the vector spaces V_1 and V_2. We obtain a representation of G on their direct sum, $V_1 \oplus V_2$ by setting

$$r(a)(\mathbf{v}_1 + \mathbf{v}_2) = r^1(a)\mathbf{v}_1 + r^2(a)\mathbf{v}_2 \quad \mathbf{v}_1 \in V_1,\ \mathbf{v}_2 \in V_2$$

for all $a \in G$. We denote the representation by $r^1 \oplus r^2$. With respect to a direct sum basis, the matrices of $r^1 \oplus r^2$ clearly have the form

$$\begin{pmatrix} (r^1_{ij}(a)) & (0) \\ (0) & (r^2_{kl}(a)) \end{pmatrix}.$$

Let r be a representation of G on V and suppose that there is an invariant subspace, W, of V. Then, by restricting each $r(a)$ to a linear transformation on W, we obtain a representation of G on W, which we have denoted by $r|_W$. We claim that there always exists a complementary subspace, W', invariant under G, so that $r \sim r_W \oplus r_{W'}$. To prove this claim, we assume that there is an Hermitian scalar product $(\ ,\ )$ defined on V which is *invariant* under G, i.e. a scalar product satisfying $(r(a)\mathbf{u}, r(a)\mathbf{v}) = (\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V and all $a \in G$. Then let W' be the subspace orthogonal to W with respect to this scalar product. We claim that W' is invariant. Indeed, if $\mathbf{u} \in W'$, then $(r(a)\mathbf{u}, \mathbf{v}) = (\mathbf{u}, r(a^{-1})\mathbf{v})$. Taking $\mathbf{v}$ in W shows that $r(a)\mathbf{u} \in W'$, which proves that W' is invariant. Thus, the existence of an invariant complement follows from the existence of an invariant scalar product. But we can always construct an invariant scalar product by the process of 'averaging over the group', as follows: we can always choose some positive definite scalar product, $(\ ,\ )_0$, on V, which need not be invariant. (Just choose some basis for V and declare it to be orthonormal.) We then define $(\ ,\ )$ by averaging over the group:

$$(\mathbf{u}, \mathbf{v}) = (1/\#G) \sum_{b \in g} (r(b)\mathbf{u}, r(b)\mathbf{v})_0.$$

then

$$\begin{aligned}(r(a)\mathbf{u}, r(a)\mathbf{v}) &= (1/\#G) \sum_{b \in G} (r(ba)\mathbf{u}, r(ba)\mathbf{v})_0 \\ &= (1/\#G) \sum_{c \in G} (r(c)u, r(c)\mathbf{v})_0 = (\mathbf{u}, \mathbf{v})\end{aligned}$$

since, as b ranges over the group, so does $c = ba$. This proves the existence of the invariant scalar product. Notice that we have used the fact that the group is finite since we summed over all group elements.

This property of representations of finite groups is known as *complete reducibility*. It need not hold for representations of infinite groups. For instance, the map $t \rightsquigarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is a representation of the additive group $\mathbb{R}$ by 2×2 matrices. The subspace $y = 0$ is invariant but has no invariant complement.

Recall that a representation r of G on V is said to be *irreducible* if there are no non-trivial subspaces invariant under G: every representation can be written as a direct sum of irreducibles. Suppose r is not irreducible. Then there is at least one proper invariant subspace, W. If there is more than one proper invariant subspace, we choose W to have the smallest possible dimension, so that it must be irreducible. The complement, W', has strictly smaller dimension than V. Either W' is irreducible, or we may again identify an irreducible subspace and take its complement. We continue the process. Eventually the complement will be irreducible, and we have then expressed r as a direct sum of irreducible representations.

Suppose that V is a vector space with a given positive definite scalar product, $(\ ,\ )$. A representation of G on V is said to be *unitary* if $(\ ,\ )$ is invariant under G. We can summarize the results of this section as saying that every unitary (finite-dimensional) representation of any group is completely reducible, and V can be decomposed into a direct sum of irreducible subspaces. We also proved that an arbitrary finite-dimensional representation by a finite group is equivalent to a unitary representation, and hence has the same complete reducibility properties.

As an example of a reducible representation, we construct the three-dimensional 'permutation representation' of S_3. We let S_3 act on the basis elements

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

of $\mathbb{R}^3$ as a three-element set. Thus, for example, the permutation (123) carries

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \text{ into } \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ into } \begin{pmatrix}0\\0\\1\end{pmatrix}, \text{ and } \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ into } \begin{pmatrix}1\\0\\0\end{pmatrix};$$

so it is represented by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly, (23) is represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly, the subspace W spanned by

$$\begin{pmatrix}1\\1\\1\end{pmatrix}$$

is invariant under all six 3×3 matrices in this representation, and on this subspace we have the identity representation of S_3. To find another invariant subspace, we note that every 3×3 matrix in the representation belongs to $O(3)$ and hence preserves the ordinary Euclidean scalar product. Therefore the subspace W' orthogonal to

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is also invariant. A convenient orthonormal basis for $\mathbb{R}^3$ is

$$\mathbf{e}_1 = \frac{1}{3^{\frac{1}{2}}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{e}_2 = \frac{1}{2^{\frac{1}{2}}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{e}_3 = \frac{1}{6^{\frac{1}{2}}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Here $\mathbf{e}_1$ spans W, while $\mathbf{e}_2$ and $\mathbf{e}_3$ span W'. With respect to this basis, each 3×3 matrix is reduced to a 1×1 matrix and a 2×2 matrix. For example, the matrix which represents (123) is

$$\begin{pmatrix} \frac{1}{3^{\frac{1}{2}}} & \frac{1}{3^{\frac{1}{2}}} & \frac{1}{3^{\frac{1}{2}}} \\ \frac{1}{2^{\frac{1}{2}}} & 0 & -\frac{1}{2^{\frac{1}{2}}} \\ -\frac{1}{6^{\frac{1}{2}}} & \frac{2}{6^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3^{\frac{1}{2}}} & \frac{1}{2^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \\ \frac{1}{3^{\frac{1}{2}}} & 0 & \frac{2}{6^{\frac{1}{2}}} \\ \frac{1}{3^{\frac{1}{2}}} & -\frac{1}{2^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3^{\frac{1}{2}}}{2} \\ 0 & \frac{3^{\frac{1}{2}}}{2} & -\frac{1}{2} \end{array}\right)$$

while the matrix which represents (23) is

$$\begin{pmatrix} \frac{1}{3^{\frac{1}{2}}} & \frac{1}{3^{\frac{1}{2}}} & \frac{1}{3^{\frac{1}{2}}} \\ \frac{1}{2^{\frac{1}{2}}} & 0 & -\frac{1}{2^{\frac{1}{2}}} \\ -\frac{1}{6^{\frac{1}{2}}} & \frac{2}{6^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3^{\frac{1}{2}}} & \frac{1}{2^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \\ \frac{1}{3^{\frac{1}{2}}} & 0 & \frac{2}{6^{\frac{1}{2}}} \\ \frac{1}{3^{\frac{1}{2}}} & -\frac{1}{2^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{1}{2} & -\frac{3^{\frac{1}{2}}}{2} \\ 0 & -\frac{3^{\frac{1}{2}}}{2} & -\frac{1}{2} \end{array}\right)$$

and the matrix which represents (12) is

$$\begin{pmatrix} \frac{1}{3^{\frac{1}{2}}} & \frac{1}{3^{\frac{1}{2}}} & \frac{1}{3^{\frac{1}{2}}} \\ \frac{1}{2^{\frac{1}{2}}} & 0 & -\frac{1}{2^{\frac{1}{2}}} \\ -\frac{1}{6^{\frac{1}{2}}} & \frac{2}{6^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3^{\frac{1}{2}}} & \frac{1}{2^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \\ \frac{1}{3^{\frac{1}{2}}} & 0 & \frac{2}{6^{\frac{1}{2}}} \\ \frac{1}{3^{\frac{1}{2}}} & -\frac{1}{2^{\frac{1}{2}}} & -\frac{1}{6^{\frac{1}{2}}} \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \\ 0 & \frac{3^{\frac{1}{2}}}{2} & -\frac{1}{2} \end{array}\right)$$

We have explicitly decomposed the three-dimensional permutation representation into the direct sum of the one-dimensional identity representation (on the subspace spanned by $\mathbf{e}_1$) and a two-dimensional irreducible representation on the subspace spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$. In fact, with the choice of basis which we made, the two-dimensional representation matrices are exactly the ones which would have been obtained by regarding S_3 as the symmetry group of the equilateral triangle and writing down the usual 2×2 matrices to represent these rotations and reflections of the plane.

2.3 Schur's lemma

Let V_1 and V_2 be vector spaces. The space $\mathrm{Hom}(V_1, V_2)$ denotes the (vector) space of all linear maps from V_1 to V_2. Suppose that we have a representation r_1 of a group G on V_1 and another representation, r_2, of G on V_2. We let $\mathrm{Hom}_G(V_1, V_2)$ denote the space of linear G morphisms from V_1 to V_2. Thus, $\mathrm{Hom}_G(V_1, V_2) \subset \mathrm{Hom}(V_1, V_2)$ consists of those linear maps T which satisfy $r_2(a)T = Tr_1(a)$ for all a in G. We can now state the main result.

Schur's lemma
Let r^1 and r^2 be irreducible representations of the group G on the vector spaces V_1 and V_2. Let T be an element of $\mathrm{Hom}_G(V_1, V_2)$, i.e. T is a linear map from V_1 to V_2 such that

$$r^2(a)T = Tr^1(a) \quad \text{for all} \quad a \in G.$$

then

(i) If $r^1 \not\sim r^2$, then $T = 0$, and

(ii) If $r^1 = r^2$ (so that $V_1 = V_2$), then T is a scalar operator i.e. $T = cI$ for some scalar c, where I is the identity operator.

Proof The subspace, $\ker T$, of V_1 is invariant, and, hence, since V_1 is irreducible, the only alternatives are $\ker T = V_1$ or $\ker T = \{0\}$. In the first case, $T = 0$, and there is nothing further to prove. If $\ker T = \{0\}$, then $T(V_1)$ is a non-trivial invariant subspace of V_2, hence coincides with V_2, showing that T is an isomorphism. This establishes (i). If $V_1 = V_2$, we can apply (i) to the operator $T - cI$. Since we are over the complex numbers, the operator T has at least one eigenvalue. Taking c to be this eigenvalue, the operator $T - cI$ is singular, and hence must be zero, by (i). This establishes (ii).

The group G acts on $\mathrm{Hom}(V_1, V_2)$ by sending $S \in \mathrm{Hom}(V_1, V_2)$ into $r^2(a)Sr^1(a)^{-1}$. It is easy to check that this defines a linear representation of G on $\mathrm{Hom}(V_1, V_2)$ which we shall denote by $\mathrm{Hom}(r^1, r^2)$. Thus

$$[\mathrm{Hom}(r^1, r^2)](a)S = r^2(a)Sr^1(a)^{-1}.$$

An element T in $\mathrm{Hom}(V_1, V_2)$ belongs to $\mathrm{Hom}_G(V_1, V_2)$, i.e. is a G morphism if and only if $r^2(a)Tr^1(a)^{-1} = T$, that is

$$[\mathrm{Hom}(r^1, r^2)](a)T = T$$

for all $a \in G$.

We now combine Schur's lemma with the method of averaging over the group:

Proposition 3.1
Let r^1 and r^2 be irreducible representations of G on V_1 and V_2. Let S_0 be any element of Hom (V_1, V_2) and set

$$S = (1/\#G) \sum_{a \in G} r^2(a) S_0 r^1(a)^{-1}$$

$$= (1/\#G) \sum_{a \in G} [\mathrm{Hom}(r^1, r^2)](a) S_0.$$

Then

(i) If $r^1 \not\sim r^2$, then $S = 0$.

(ii) If $r^1 = r^2$ (so that $V_1 = V_2$), then $S = cI$, where $c = (1/n) \operatorname{tr} S_0$ with $n = \dim V_1$.

Proof The element, S, is invariant, and hence (i) follows from part (i) of Schur's lemma, while (ii) follows from part (ii) of Schur's lemma except for the evaluation of the constant. To evaluate c, observe that $\operatorname{tr} S = \operatorname{tr} S_0$. But $\operatorname{tr} cI = nc$, finishing the proof of (ii).

If we write out the assertions of Proposition 3.1 in matrix form, we obtain the so-called orthogonality relations for the matrix elements: choose bases in V_1 and V_2 so that $r^1(a)$ has the matrix representation, $(r^1_{ij}(a))$ and $r^2(a)$ has the matrix representation $(r^2_{kl}(a))$. Suppose that S_0 has the matrix representation (s_{jk}). Then the matrix of S has, as its entries,

$$(1/\#G) \sum_{\substack{a \in G \\ l,i}} r^2_{kl}(a) s_{li} r^1_{ij}(a^{-1}).$$

In case $r^1 \not\sim r^2$, this expression is to vanish identically, no matter what the values of the s_{li} actually are. In other words, the coefficients of the s_{li} must vanish identically. Thus,

$$\text{if} \quad r^1 \not\sim r^2, \quad \text{then } (1/\#G) \sum_{a \in G} r^2_{kl}(a) r^1_{ij}(a^{-1}) = 0 \text{ for all } k, l, i, j. \tag{3.1}$$

In case $r^1 = r^2$, let us denote the matrices by r_{ij} so that (ii) of Proposition 3.1 says that

$$(1/\#G) \sum_{l,i} \sum_{a \in G} r_{kl}(a) s_{li} r_{ij}(a^{-1}) = (1/n)(\textstyle\sum s_{ii}) \delta_{kj}.$$

Comparing the coefficients of s_{li} on both sides, we get

$$(1/\#G) \sum_{a \in G} r_{kl}(a) r_{ij}(a^{-1}) = (1/n)\delta_{li}\delta_{kj} = \begin{cases} (1/n) \text{ if } l = i, k = j \\ 0 \text{ otherwise.} \end{cases} \tag{3.2}$$

Equations (3.1) and (3.2) take on a somewhat more pleasant form if we restrict attention to unitary representations, for which

$$r_{ij}(a^{-1}) = \overline{r_{ji}(a)}.$$

For any two functions, f_1 and f_2 on G, let us define their scalar product by the formula

$$(f_1, f_2) = (1/\#G) \sum_{a \in G} f_1(a) \overline{f_2(a)}.$$

Then, if we assume that we have chosen bases so that the matrices (r^1_{ij}) and (r^2_{kl}) are

Table 5.

Element	Identity	Sign	Two-dimensional
e	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(123)	1	1	$\begin{pmatrix} -\frac{1}{2} & -\frac{3^{\frac{1}{2}}}{2} \\ \frac{3^{\frac{1}{2}}}{2} & -\frac{1}{2} \end{pmatrix}$
(132)	1	1	$\begin{pmatrix} -\frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \\ -\frac{3^{\frac{1}{2}}}{2} & -\frac{1}{2} \end{pmatrix}$
(12)	1	−1	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(13)	1	−1	$\begin{pmatrix} -\frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \\ \frac{3^{\frac{1}{2}}}{2} & \frac{1}{2} \end{pmatrix}$
(23)	1	−1	$\begin{pmatrix} -\frac{1}{2} & -\frac{3^{\frac{1}{2}}}{2} \\ -\frac{3^{\frac{1}{2}}}{2} & \frac{1}{2} \end{pmatrix}$

unitary, we can write (3.1) as:

$$\text{If } r^1 \not\sim r^2, \text{ then } (r^2_{kl}, r^1_{ji}) = 0 \text{ for all } i, j, k \text{ and } l. \tag{3.3}$$

In other words, matrix entries from two inequivalent representations are orthogonal. We can write (3.2) as

$$(r^2_{kl}, r^1_{ij}) = (1/n)\delta_{ki}\delta_{lj} \tag{3.4}$$

so that distinct matrix entries from the same irreducible unitary representation are orthogonal, and each matrix entry has length $1/n^{\frac{1}{2}}$.

As an explicit example of these orthogonality relations, we write out the three representations of S_3: the identity representation, the 'sign' representation with + 1 for even permutations, − 1 for odd, and the 2×2 representations as rotations and reflections of an equilateral triangle (see Table 5).

You can check that the identity and sign representations are orthogonal to the entries in each position of the 2×2 matrices, and that these matrix entries are orthogonal to one another.

2.4 Characters and their orthogonality relations

Let r be a representation of the group G on the vector space V. The dimension of V is called the *degree* of the representation, r. The *character* of the representation r is the function χ^r defined on G by the formula

$$\chi^r(a) = \operatorname{tr} r(a) = \sum_i r_{ii}(a). \tag{4.1}$$

If we take $a = e$, so that $r(e)$ is the identity operator, whose trace is $\dim V$, we see that

$$\chi^r(e) = \dim V. \tag{4.2}$$

For any two linear transformations we have $\operatorname{tr} AB = \operatorname{tr} BA$; so, if B is non-singular, $\operatorname{tr} BAB^{-1} = \operatorname{tr} A$. Thus,

$$\chi(bab^{-1}) = \chi(a) \tag{4.3}$$

if χ is the character of any representation. In other words, χ is a function which is constant on conjugacy classes. Such a function is called a *central* function.

For any representation r, we can introduce a Hermitian scalar product which is invariant under $r(a)$ for all $a \in G$. This means that if we take adjoints with respect to this scalar product, we have $r(a)^* = r(a^{-1})$. But $\operatorname{tr} r(a)^*$ is the complex conjugate of $\operatorname{tr} r(a)$, so

$$\chi(a^{-1}) = \overline{\chi(a)}. \tag{4.4}$$

Let r^1 and r^2 be representations of G. Then it follows from the matrix form of $r^1 \oplus r^2$ that

$$\chi^{r^1 \oplus r^2} = \chi^{r^1} + \chi^{r^2}. \tag{4.5}$$

The character χ^1 of the representation r^1 is given by

$$\chi^1 = \sum_i r^1_{ii}$$

and similarly for the character χ^2 of r^2. It now follows from (3.3) and (3.2) that

$$\text{if } r^1 \not\sim r^2, \quad \text{then } (\chi^1, \chi^2) = 0, \tag{4.6}$$

and

$$(\chi, \chi) = 1, \text{ if } \chi \text{ is the character of an irreducible representation.} \tag{4.7}$$

Now let r be a representation of G on a vector space, V, which is not necessarily irreducible, and let

$$r = r^1 \oplus \cdots \oplus r^k$$

be a decomposition of r into irreducible representations. Let ϕ be the character of r, and

let χ_i be the character of r^i, so that

$$\phi = \chi_1 + \cdots + \chi_k.$$

Let s be some particular irreducible representation of G and let χ be its character. Then

$$(\phi, \chi) = (\chi_1, \chi) + \cdots + (\chi_k, \chi).$$

The terms on the right are all zero or one, according as $r^i \not\sim r$ or $r^i \sim r$. Thus,

(ϕ, χ) is the number of terms in the decomposition of r which are isomorphic to s. In particular, this number does not depend on the particular choice of decomposition. (4.8)

From (4.8) it follows that *two representations with the same character are equivalent.* Indeed, by taking scalar products with the characters of all the irreducible representations, we can determine how many times each irreducible occurs in a decomposition of the given representation.

Notice that any character ϕ can be written as

$$\phi = m_1\chi_1 + \cdots + m_p\chi_p,$$

where the χ_i are orthogonal characters and the m_i are non-negative integers. It follows that

$$(\phi, \phi) = m_1^2 + \cdots + m_p^2 \tag{4.9}$$

and, in particular,

$$\phi \text{ is irreducible if and only if } (\phi, \phi) = 1.$$

Let ϕ be the character of a representation of G on a vector space W, and let χ be the character of an irreducible representation of G on the vector space V. If we decompose

$$W = W_1 \oplus \cdots \oplus W_k$$

into irreducibles, we see that

$$\operatorname{Hom}_G(W, V) = \operatorname{Hom}_G(W_1, V) \oplus \cdots \oplus \operatorname{Hom}_G(W_k, V).$$

By Schur's lemma, each of these spaces is either one dimensional or zero dimensional according to whether the representation of G on W_i is or is not equivalent to the representation of G on V. Combining this with (4.8) we see that

$$(\phi, \chi) = \dim \operatorname{Hom}_G(W, V). \tag{4.10}$$

Now let r_u and r_v be representations of G on U and V. We do not assume that r_u and r_v are irreducible. We wish to compute $\dim \operatorname{Hom}_G(U, V)$.

Let us first consider a special case. Suppose $U = V = W \oplus W$, where W is irreducible. We can write any vector in U as $(\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are in W. Thus, for any $T \in \operatorname{Hom}(V, V)$ we have

$$T(\mathbf{w}_1, \mathbf{w}_2) = (T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2, T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)$$

where $T_{ij} \in \mathrm{Hom}(W, W)$. So

$$T \circ r_{W \oplus W}(a)(\mathbf{w}_1, \mathbf{w}_2) = T(r_W(a)\mathbf{w}_1, r_W(a)\mathbf{w}_2)$$
$$= (T_{11}r_W(a)\mathbf{w}_1 + T_{12}r_W(a)\mathbf{w}_2, T_{21}r_W(a)\mathbf{w}_1 + T_{22}r_W(a)\mathbf{w}_2)$$

while

$$r_{W \oplus W}(a)T(\mathbf{w}_1, \mathbf{w}_2)$$
$$= (r_W(a)(T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2), r_W(a)(T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2))$$
$$= (r_W(a)T_{11}\mathbf{w}_1 + r_W(a)T_{12}\mathbf{w}_2, r_W(a)T_{21}\mathbf{w}_1 + r_W(a)T_{22}\mathbf{w}_2).$$

So $T \in \mathrm{Hom}_G(V, V)$ if and only if each $T_{ij} \in \mathrm{Hom}_G(W, W)$. By Schur's lemma, each T_{ij} ranges over a one-dimensional space, hence $\dim \mathrm{Hom}_G(W \oplus W, W \oplus W) = 4 = 2 \times 2$. For any representation, we may make the decomposition

$$U = (U_1 \oplus \cdots \oplus U_{p_1}) \oplus (U_{p_1+1} \oplus \cdots \oplus U_{p_1+p_2}) \oplus \cdots (\cdots U_{p_1+\cdots p_k})$$

where the first p_1 spaces are all equivalent to the irreducible representation W_1, the next p_2 spaces are all equivalent to the irreducible representation W_2 etc., and $W_1, \ldots, W_k$ are *inequivalent* irreducible representations of G. We may make the same decomposition

$$V = (V_1 \oplus \cdots \oplus V_{q_1}) \oplus (V_{q_1+1} \oplus \cdots \oplus V_{q_1+q_2}) \oplus \cdots$$

for V. By Schur's lemma, any $T \in \mathrm{Hom}_G(U, V)$ when applied to any $\mathbf{u} \in U_1 \oplus \cdots \oplus U_{p_1}$ must give $T\mathbf{u}$ lying in $V_1 \oplus \cdots \oplus V_{q_1}$. Then the same argument as in the special case shows that

$$\dim \mathrm{Hom}_G(U, V) = p_1q_1 + p_2q_2 + \cdots + p_kq_k. \tag{4.11}$$

In particular, if $U = V$,

$$\dim \mathrm{Hom}_G(V, V) = p_1^2 + \cdots + p_k^2,$$

where p_i is the number of times that the ith irreducible representation occurs in V.

2.5 Action on function spaces

Suppose that we are given an action of the group G on the set M. Let $\mathscr{F}(M)$ denote the vector space of all complex-valued functions on M. Define an action of G on $\mathscr{F}(M)$ by

$$(af)(x) = f(a^{-1}x).$$

Put another way, we define af by

$$af = f \circ a^{-1},$$

where, on the right, we are considering the element a^{-1} as a transformation of M into itself. To verify that this is indeed a group action, observe that

$$bf = f \circ b^{-1}$$
$$\text{so } a(bf) = (bf) \circ a^{-1} = (f \circ b^{-1}) \circ a^{-1} = f \circ (b^{-1} \circ a^{-1}) = f \circ (ab)^{-1} = (ab)f.$$

We denote the representation defined by this action of G on $F(M)$ by r^M. So we may write

$$(af)(x) = [r^M(a)f](x) = f(a^{-1}x).$$

The representation r^M is not irreducible, unless M consists of a single element, because the constant functions on M always form a one-dimensional invariant subspace, on which G acts by the trivial (identity) representation.

As an illustration, we construct representations of D_4 by letting the group act on the space of functions on the vertices of the square. As a convenient notation, we denote each function by a diagram showing its value on each vertex. Thus, for example,

$$\begin{array}{cc} 4 & 3 \\ - & \cdot \\ + & - \\ 1 & 2 \end{array}$$

is the function with $f(1) = 1$, $f(2) = -1$, $f(3) = 0$, $f(4) = -1$. Then the action of a group element on a function may be determined simply by letting the element act on the diagram which represents the function. For example, if a is a 90° counterclockwise rotation, then the rotated diagram

$$\begin{array}{cc} 4 & 3 \\ \cdot & - \\ - & + \\ 1 & 2 \end{array}$$

correctly describes af. The diagram shows that $af(2) = +1$; and we check that $af(2) = f(a^{-1}2) = f(1) = +1$.

The space of functions on the vertices of the square is clearly four dimensional. One invariant subspace is spanned by

$$\begin{array}{cc} + & + \\ + & + \end{array}$$

the constant function; this gives rise to the trivial representation. Another invariant subspace is spanned by

$$\begin{array}{cc} - & + \\ + & - \end{array}$$

Clearly this gives rise to a representation in which the identity, the 180° rotation in the plane of the page, and rotations about the diagonals are represented by $+1$, while the 90° and 270° rotations and the rotations about lines parallel to the sides of the square are represented by -1. Finally, there is a two-dimensional invariant subspace, spanned by

$$f_1 = \begin{array}{cc} + & + \\ - & - \end{array} \quad \text{and} \quad f_2 = \begin{array}{cc} + & - \\ + & - \end{array}$$

The 90° rotation carries f_1 into f_2, f_2 into $-f_1$, and so is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Similarly, the reflection about the principal diagonal carries f_1 into f_2, f_2 into f_1, and is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus, we obtain the two-dimensional representation of D_4. Thus the four-dimensional space of functions in the vertices of the square breaks up into two one-dimensional and one two-dimensional representations.

There are two one-dimensional representations of D_4 which are *not* obtained by this procedure. One is the representation in which $+1$ is assigned to the four elements in the C_4 subgroup, -1 to the other four elements. The last is a representation in which $+1$ is assigned to the elements of the D_2 subgroup: the identity, and the 180° rotations about the x, y and z axes, while -1 is assigned to the 90° and 270° rotations and the 180° rotations about the diagonals. In the next section we shall see that when we let a group act on functions on the group itself, the resulting representation contains *all* irreducible representations in its decomposition.

Suppose that M decomposes into orbits under the action of G:

$$M = M_1 \cup \cdots \cup M_k.$$

Then we have a corresponding decomposition

$$r^M = r^{M_1} \oplus \cdots \oplus r^{M_k}.$$

Indeed, we can identify r^{M_i} with the subrepresentation of r^M given by the action of G on functions which vanish outside of M_i. Any function on M can be written uniquely as $f = f_1 + \cdots + f_k$, where each f_j vanishes outside of M_i. Let us compute the character of the representation r^M. For this purpose we introduce a convenient basis into the space $\mathbb{C}^M$: Let δ_x be the function on M defined by

$$\delta_x(y) = \begin{matrix} 1 & \text{if} & y = x \\ 0 & \text{if} & y \neq x \end{matrix}.$$

Then

$$(a\delta_x)(y) = \delta_x(a^{-1}y) = \begin{matrix} 1 & \text{if} & a^{-1}y = x, \text{ i.e. } y = ax \\ 0 & \text{if} & a^{-1}y \neq x, \text{ i.e. } y \neq ax \end{matrix}$$

so that

$$a\delta_x = \delta_{ax}.$$

The functions δ_x are clearly independent and they span $\mathscr{F}(M)$ since any function, f, can be written as $f = \sum_{x \in M} f(x)\delta_x$. For this basis, the diagonal elements of $r^M(a)$ will be one or zero according as $ax = x$ or $ax \neq x$. Thus

$$\begin{aligned} \chi^{r^M}(a) &= \sum_{ax = x} 1 = \#(\text{fixed points of } a) \\ &= \#FP(a). \end{aligned} \tag{5.1}$$

This formula is the prototype of all character formulas. In fact, all the character formulas that we shall present will, in a sense, be generalizations of (5.1) in one form or another. We shall return to this point later.

Suppose that G acts on the two finite sets M and N. We wish to study the space

$\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ and the action of G on it. Notice that

$$\begin{aligned} \dim \mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N)) &= \dim \mathscr{F}(M) \times \dim \mathscr{F}(N) \\ &= (\#M)\cdot(\#N) \\ &= \#(N \times M) \\ &= \dim \mathscr{F}(N \times M). \end{aligned}$$

We claim that there is a natural isomorphism between $\mathscr{F}(N \times M)$ and $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$. Indeed, given any function K on $N \times M$ define the operator $T_K : \mathscr{F}(M) \to \mathscr{F}(N)$ by

$$(T_K f)(y) = \sum_{x \in M} K(y, x) f(x).$$

The map sending K into T_K is one-to-one; indeed, for any $u \in M$

$$(T_K \delta_u)(y) = K(y, u)$$

so if $T_K = 0$, then $K(y, u) = 0$ for all y and u, i.e. $K = 0$. Since $\mathscr{F}(N \times M)$ and $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ have the same dimension, we conclude that the map sending K to T_K is an isomorphism of vector spaces.

The group G acts on both $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ and on $\mathscr{F}(N \times M)$. We claim that the isomorphism just described is a G morphism, i.e. that the representations of G on these two spaces are equivalent. Indeed, letting r^{Hom} denote the representation of G on $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$, we have

$$r^{\mathrm{Hom}}(a) T_K = r^N(a) T_K r^M(a)^{-1}.$$

Now

$$r^M(a)^{-1} f(x) = f(ax)$$

so

$$(T_K r^M(a)^{-1} f)(y) = \sum K(y, x) f(ax)$$

and

$$\begin{aligned} (r^{\mathrm{Hom}}(a) T_K f)(y) &= \sum K(a^{-1}y, x) f(ax) \\ &= \sum K(a^{-1}y, a^{-1}x) f(x). \end{aligned}$$

But

$$(r^{N \times M}(a) K)(y, x) = K(a^{-1}y, a^{-1}x)$$

So

$$r^{\mathrm{Hom}}(a) T_K = T_{r^{N \times M}(a) K}$$

which was to be proved.

Now $\mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(N))$ is the space of elements in $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ which satisfy

$$r^{\mathrm{Hom}}(a) T = T$$

for all $a \in G$. If $T = T_K$, the preceding equation shows that

$$K(a^{-1}y, a^{-1}x) = K(y, x)$$

for all $a \in G$. In other words, the function K must be constant on orbits of G on $N \times M$. Thus

$$\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(N)) = \#\text{ of } G \text{ orbits on } N \times M. \tag{5.2}$$

In particular, on taking $M = N$ we see that

$$\begin{aligned}\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) &= \#\text{ of } G \text{ orbits on } M \times M \\ &= p_1^2 + \cdots + p_k^2 \end{aligned} \tag{5.3}$$

where p_i is the number of times that the ith irreducible representation of G occurs in $\mathscr{F}(M)$.

Example

Consider the group S_n acting on the n-element set $M = \{1, \ldots, n\}$. On $M \times M$ there are two orbits

$$\{(x, y) | x \neq y\} \quad \text{and} \quad \{(x, x)\}.$$

Indeed, if $x \neq y$ and $z \neq w$ we can find a permutation σ such that $\sigma(x) = z$ and $\sigma(y) = w$ Thus, the set $\{(x, y) | x \neq y\}$ is a single orbit in $M \times M$. Similarly the set $\{(x, x)\}$ is a single orbit. Thus,

$$\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) = 2 = p_1^2 + \cdots + p_k^2$$

so $k = 2$ and $p_1 = p_2 = 1$. Thus, $\mathscr{F}(M)$ is the direct sum of two irreducible representations. We already know one of them – the trivial one-dimensional representation, corresponding to the constant functions. The other must then be $n - 1$ dimensional. Thus

$$\mathscr{F}(M) = \underset{\substack{\uparrow \\ \text{one} \\ \text{dimensional}}}{V_1} + \underset{\substack{\uparrow \\ n-1 \\ \text{dimensional}}}{V_2}$$

2.6 The regular representation

We apply the results of the preceding section to the special case where $M = G$ and G acts on itself by left multiplication. The corresponding representation, r^G, of G on $\mathscr{F}(G)$ is called the *regular* representation. It is defined by $[r^G(a)f](b) = f(a^{-1}b)$. We have

$$\#G = \dim \mathscr{F}(G) = \sum p_i n_i$$

where p_i is the number of times that the ith irreducible representation occurs in $\mathscr{F}(G)$, while n_i is the dimension of the ith irreducible representation. Also

$$\begin{aligned}\dim \mathrm{Hom}_G(\mathscr{F}(G), \mathscr{F}(G)) &= \sum p_i^2 \\ &= \#\text{ of } G \text{ orbits on } G \times G.\end{aligned}$$

We compute the number of orbits as follows: we can always act on (a, b) by a^{-1} to get $(e, a^{-1}b)$. Thus, each orbit of G in $G \times G$ contains a point of the form (e, c). But this is the only element of this form in its orbit, since multiplying by d sends (e, c) into (d, dc). Thus each orbit contains a unique representative of the form (e, c), and hence the number of orbits is equal to $\#G$. Thus

$$\#G = \sum p_i^2.$$

Since $\sum p_i^2 = \sum p_i n_i$ we are led to guess that $p_i = n_i$, i.e. that each irreducible representation, W, occurs in $\mathscr{F}(G)$ with a multiplicity equal to its dimension, i.e. that

$$\dim \mathrm{Hom}_G(W, \mathscr{F}(G)) = \dim W. \tag{6.1}$$

We shall prove this fact by constructing an isomorphism between W^*, the dual space of W, and $\mathrm{Hom}_G(W, \mathscr{F}(G))$. To each $l \in W^*$ and to each $\mathbf{w} \in W$ we assign the function $f^l_{\mathbf{w}}$ on G defined by

$$f^l_{\mathbf{w}}(a) = \langle r(a^{-1})\mathbf{w}, l \rangle.$$

Here $r(a^{-1})\mathbf{w} \in W$ and $l \in W^*$, and $\langle \mathbf{v}, l \rangle$ denotes the value of the linear function $l \in W^*$ at the element $\mathbf{v}$ of V. For fixed l the map sending $\mathbf{w}$ into $f^l_{\mathbf{w}}$ is a map from W to $\mathscr{F}(G)$. Thus each $l \in V^*$ defines an element of $\mathrm{Hom}(W, \mathscr{F}(G))$. We must show that this element lies in $\mathrm{Hom}_G(W, \mathscr{F}(G))$, i.e. that

$$f^l_{r(b)\mathbf{w}} = r^G(b) f^l_{\mathbf{w}}$$

or that

$$f^l_{r(b)\mathbf{w}}(a) = f^l_{\mathbf{w}}(b^{-1}a) \quad \text{for all } a, b \in G.$$

But

$$\begin{aligned} f^l_{r(b)\mathbf{w}}(a) &= \langle r(a)^{-1} r(b) w, l \rangle \\ &= \langle r(a^{-1}b) w, l \rangle \\ &= \langle r(b^{-1}a)^{-1} w, l \rangle \\ &= f^l_{\mathbf{w}}(b^{-1}a) \end{aligned}$$

as required.

Furthermore, $f^l_{\mathbf{w}}(e) = \langle \mathbf{w}, l \rangle$ cannot be zero for all $\mathbf{w}$ unless $l = 0$. Thus the map of W^* into $\mathrm{Hom}_G(W, \mathscr{F}(G))$ that we have defined is injective. It follows that

$$p_i = \dim \mathrm{Hom}_G(W, \mathscr{F}(G)) \geqslant \dim W^* = n_i.$$

But it now follows from the two equations

$$\#G = \sum p_i n_i = \sum p_i^2$$

that we must have $p_i = n_i$ so (6.1) holds. Thus

$$\#G = \sum n_i^2. \tag{6.2}$$

Equations (6.1) and (6.2) have the following useful corollary. Suppose that we have found inequivalent irreducible representations $(r_1, W_1) \cdots (r_k, W_k)$ of G, with $\dim W_i = n_i$, such that $\sum_1^d n_i^2 = \#G$. Then it follows from (6.2) that there can be no other irreducible representation (i.e. ones not equivalent to the ones we already have). In

other words, we have found, up to equivalence, all the irreducible representations of G. This justifies the conclusions that we drew in the examples of Section 1.

We can extract some further useful information from the proof of (6.1), but for this we must introduce some new notions. Let G and H be groups. Their direct product $G \times H$ consists of all pairs (a, b) with the multiplication law

$$(a, b)(c, d) = (ac, bd).$$

Suppose that (r, U) is a representation of G and (s, V) is a representation of H. We can form the tensor product, $U \otimes V$, of the two vector spaces U and V[†]. Recall from the theory of tensor products that if $A \in \mathrm{Hom}(U, U)$ and $B \in \mathrm{Hom}(V, V)$, then there is a unique transformation $A \otimes B$ on $U \otimes V$ such that

$$(A \otimes B)(\mathbf{u} \otimes \mathbf{v}) = A\mathbf{u} \otimes B\mathbf{v}.$$

Also

$$\mathrm{tr}(A \otimes B) = (\mathrm{tr}\, A) \cdot (\mathrm{tr}\, B).$$

Furthermore, if $C \in \mathrm{Hom}(U, U)$ and $D \in \mathrm{Hom}(V, V)$, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

This shows that we get a representation $r \otimes s$ of $G \times H$ on $U \otimes V$ by setting

$$(r \otimes s)(a, b) = r(a) \otimes s(b),$$

and that

$$\chi^{r \otimes s}(a, b) = \chi^r(a)\chi^s(b).$$

If $(\ ,\ )_{G \times H}$ denotes the scalar product on $G \times H$, and $\|\ \|_{G \times H}$ the corresponding norm, then

$$\begin{aligned} \|\chi^{r \otimes s}\|^2_{G \times H} &= (\chi^{r \otimes s}, \chi^{r \otimes s})_{G \times H} \\ &= \frac{1}{\#(G \times H)} \sum_{\substack{a \in G \\ b \in H}} \chi^{r \otimes s}(a, b)\overline{\chi^{r \otimes s}(a, b)} \\ &= \frac{1}{\#G} \cdot \frac{1}{\#H} \left(\sum_{a \in G} \chi^r(a)\overline{\chi^r(a)} \right) \left(\sum_{b \in H} \chi^s(b)\overline{\chi^s(b)} \right) \\ &= \|\chi^r\|^2_G \|\chi^s\|^2_H. \end{aligned}$$

In particular, if $\|\chi^r\|^2_G = \|\chi^s\|^2_H = 1$, then $\|\chi^{r \otimes s}\|^2_{G \times H} = 1$. Thus,

> If r is an irreducible representation of G and s is an irreducible representation of H, then $r \otimes s$ is an irreducible representation of $G \times H$.

Let r be a representation of G on W. We can construct a representation $\hat{r}$ of G on the dual space W^* by defining

$$\hat{r}(a)l = r(a)^{*-1}l.$$

[†] See Appendix B for a summary and presentation of the basic facts about tensor products.

This is a representation because

$$\begin{aligned} r(ab)^{*-1} &= (r(a)r(b))^{*-1} \\ &= (r(b)^* r(a)^*)^{-1} \\ &= r(a)^{*-1} r(b)^{*-1} \\ &= \hat{r}(a)\hat{r}(b). \end{aligned}$$

We thus get a representation of $G \times G$ on $W \otimes W^*$. If the representation of G on W is irreducible, then so is the representation of $G \times G$ on $W \otimes W^*$.

Now let us go back to the proof of (6.1). We have seen how to attach a function $f^l_{\mathbf{w}}$ on G to each pair $(\mathbf{w}, l)$ with $\mathbf{w} \in W$ and $l \in W^*$. Since $f^l_{\mathbf{w}}$ depends linearly on $\mathbf{w}$ for l fixed, and linearly on l for $\mathbf{w}$ fixed, we have thus defined a map

$$\begin{aligned} W \otimes W^* &\to \mathscr{F}(G) \\ w \otimes l &\rightsquigarrow f^l_{\mathbf{w}}. \end{aligned}$$

Now the group $G \times G$ acts on G by right and left multiplication:

$$(a, b)c = acb^{-1}$$

and hence we get a corresponding representation $\hat{r}^G$ on $\mathscr{F}(G)$

$$[\hat{r}^G(a, b)f](c) = f(a^{-1}cb).$$

Notice that

$$\begin{aligned} f^{\hat{r}(b)l}_{r(a)\mathbf{w}}(c) &= \langle r(c)^{-1} r(a)\mathbf{w}, r(b)^{*-1} l \rangle \\ &= \langle r(b)^{-1} r(c)^{-1} r(a)\mathbf{w}, l \rangle \\ &= \langle r(a^{-1}cb)^{-1}\mathbf{w}, l \rangle \\ &= f^l_{\mathbf{w}}(a^{-1}cb). \end{aligned}$$

In other words, the map from $W \otimes W^*$ to $\mathscr{F}(G)$ is a morphism for the action of $G \times G$.

Now decompose $\mathscr{F}(G)$ into irreducibles under the action of $G \times G$: For each irreducible representation W_i of G, we know that $W_i \otimes W_i^*$ occurs as an irreducible component under $G \times G$ on $\mathscr{F}(G)$. Under G, the space $W_i \otimes W_i^*$ decomposes into a direct sum of n_i copies of W_i. In particular, no $W_i \otimes W_i^*$ has any component in common with $W_j \otimes W_j^*$ for $i \neq j$. *A fortiori*, $W_i \otimes W_i^*$ and $W_j \otimes W_j^*$ are inequivalent as irreducible representations of $G \times G$. Thus $W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$ occurs as a summand of $\mathscr{F}(G)$, where $W_1, \ldots, W_k$ are all the irreducible representations of G. But the dimension of this summand is $\sum n_i^2 = \dim \mathscr{F}(G)$. Thus

$$\mathscr{F}(G) = W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^*$$

gives the decomposition of $\mathscr{F}(G)$ into irreducibles of $G \times G$, where each summand occurs once.

Thus

$$\dim \mathrm{Hom}_{G \times G}(\mathscr{F}(G), \mathscr{F}(G)) = \underbrace{1^2 + \cdots + 1^2}_{k \text{ times}} = k.$$

We know that this dimension must equal the number of orbits of $G \times G$ acting on

$G \times G$ by the rule

$$(a,b)(c,d) = (acb^{-1}, adb^{-1}).$$

As before, we can always find an element of the form (e,d) on any orbit. But now $(a,b)(e,d) = (ab^{-1}, adb^{-1})$ will have the same form if $b = a$. Thus (e,d) and (e, ada^{-1}) lie on the same orbit, and hence

> the number of orbits of $G \times G$ on $G \times G$ is equal to the number of conjugacy classes of G.

Thus $\dim \mathrm{Hom}_{G\times G}(\mathscr{F}(G), \mathscr{F}(G)) = k = \#$ of conjugacy classes.

We have proved that

> the number of distinct irreducible representations is equal to the number of conjugacy classes. (6.3)

Let C denote the space of functions which are constant on conjugacy classes, and let $\chi_1, \ldots, \chi_k$ be the distinct irreducible characters. We already know that the functions $\chi_i \in C$ are mutually orthogonal and have length one. Since $k = \#$ of conjugacy classes $= \dim C$, they form an orthonormal basis of C. Any $f \in C$ can be expanded in terms of the basis $\chi_1, \ldots, \chi_p$:

$$f = (f, \chi_1)\chi_1 + \cdots + (f, \chi_p)\chi_p.$$

Let us apply this formula to the function f_j, which equals one on the jth conjugacy class and vanishes on all the others. Then $(f, \chi_i) = (\#C_j/\#G)\overline{\chi_i(j)}$, where $\#C_j$ is the number of elements in the jth conjugacy class, C_j, and $\chi_i(j)$ is the value of χ_i on any element of this class. Substituting into the above formula and evaluating at a point in the jth conjugacy class, C_j, we get

$$1 = (\#C_j/\#G)(\chi_1(j)\overline{\chi_1(j)} + \cdots + \chi_p(j)\overline{\chi_p(j)}). \tag{6.4}$$

Evaluating at a different conjugacy class gives

$$0 = \chi_1(k)\overline{\chi_1(j)} + \cdots + \chi_p(k)\overline{\chi_p(j)} \quad \text{if } j \neq k. \tag{6.5}$$

There is one further useful piece of information that we can extract from the above analysis. We have proved that for any irreducible W, the map of $W \otimes W^* \to \mathscr{F}(G)$ sending $u \otimes l \to f^l_u \in \mathscr{F}(G)$ is injective. This means that the transpose of this map is surjective. Let us write down what this transpose is. Let us use the trace on $W \otimes W^* = \mathrm{Hom}(W, W)$ to identify any $A \in W \otimes W^*$ with a linear function on $\mathrm{Hom}(W, W)$. So we think of A as the linear function which assigns to every B the number $\mathrm{tr}\, AB$. For each function f on $\mathscr{F}(G)$, and for a representation, ρ, of G on W, let us define $\hat{\rho}(f) \in \mathrm{Hom}(W, W)$ by

$$\hat{\rho}(f)u = \sum_{a\in G} f(a)\rho(a)u$$

or, more symbolically,

$$\hat{\rho}(f) = \sum_{a\in G} f(a)\rho(a).$$

Then

$$\begin{aligned}\mathrm{tr}\, \hat{\rho}(f)(u \otimes l) &= \sum f(a)\, \mathrm{tr}\, \rho(a)(u \otimes l) \\ &= \sum f(a)\langle au, l\rangle \\ &= \sum f(a) f^l_u(a^{-1}).\end{aligned}$$

We may identify $\mathscr{F}(G)$ with $\mathscr{F}(G)^*$ if we consider $h \in \mathscr{F}(G)$ as a linear function on $\mathscr{F}(G)$ sending f into $\sum f(a)h(a^{-1})$. Then the above equation says

$$\hat{\rho}: \mathscr{F}(G) \to W \otimes W^* \text{ is the transpose of the map } W \otimes W^* \to \mathscr{F}(G)$$

which we know to be injective. But this means that $\hat{\rho}$ is surjective. But the image of $\hat{\rho}$ consists of linear combinations of the $\rho(a)$. We have thus proved the important

> if (ρ, W) is an irreducible representation, then the $\rho(a)$, $a \in G$ span all of $\mathrm{Hom}(W, W)$. (6.6)

In fact, we have proved more: let $W_1, \ldots, W_k$ be any family of inequivalent irreducible representations. Then the map of $W_1 \otimes W_1^* \oplus \cdots \oplus W_k \otimes W_k^* \to \mathscr{F}(G)$ is injective, so the transpose is surjective. This means that

> given $T_1 \in \mathrm{Hom}(W_1, W_1), T_2 \in \mathrm{Hom}(W_2, W_2), \ldots, T_k \in \mathrm{Hom}(W_k, W_k)$, there is an $f \in \mathscr{F}(G)$ such that (6.7)
>
> $$\hat{\rho}_1(f) = T_1, \ldots, \hat{\rho}_k(f) = T_k.$$

2.7 Character tables

Let $\chi_1, \ldots, \chi_p$ be the distinct irreducible characters of the group G, and let $C_1, \ldots, C_p$ denote the distinct conjugacy classes. We denote by $\chi_i(j)$ the (constant) value of the character χ_i on any element of the conjugacy class C_j. Then

$$(\chi_i, \chi_k) = (1/\#G)\sum_{a\in G} \chi_i(a)\overline{\chi_k(a)} = (1/\#G)\sum_{j=1}^{p} (\#C_j)\chi_i(j)\overline{\chi_k(j)}.$$

We can thus write the orthogonality relations (4.6) and (4.7) as

$$(\#C_1)\chi_i(1)\overline{\chi_k(1)} + \cdots + (\#C_p)\chi_i(p)\overline{\chi_k(p)} = \begin{cases} \#G & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}. \tag{7.1}$$

We can write the orthogonality relations (6.4) and (6.5) as

$$(\#C_j)[\chi_1(j)\overline{\chi_1(l)} + \cdots + \chi_p(j)\overline{\chi_p(l)}] = \begin{cases} \#G & \text{if } j = l \\ 0 & j \neq l \end{cases}. \tag{7.2}$$

Equations (7.1) and (7.2) can be summarized in the form of a table. We label the columns by the conjugacy classes, indicating, alongside C_j, the number of its elements $\#C_j$. We label the rows by the characters χ_i, and place the value $\chi_i(j)$ in the i,j position. Then (7.1)

Table 6.

$6S_3$	$1C_1$	$3C_2$	$2C_3$
χ_1	1	1	1
χ_2	2	0	−1
χ_3	1	−1	1

says that the 'scalar product' of two distinct rows is zero, and of a row with itself is $\#G$, provided that we weight the jth column by $\#C_j$. Similarly, (7.2) says the same thing about the scalar product of the columns, again weighting the columns by $\#C_j$. The table so obtained is called the character table of the group. In a sense, it contains all the information about the representations of the group. The first conjugacy class, C_1, is usually taken to be the one-element conjugacy class $[e]$. Thus, the elements of the first column consist of the values $\chi_i(e) = n_i$, the degree of the ith irreducible representation. The character χ_1 is usually taken to be the trivial representation, so that the entries of the first row are all ones.

As a first illustration, let us work out the character table for the symmetric group S_4. In the case, $\#S_3 = 6$. There are three conjugacy classes: $C_1 = e$ with $\#C_1 = 1$; C_2, consisting of all two cycles (reflections), with $\#C_2 = 3$; and C_3, consisting of all three cycles (rotations through 120° or 240°), with $\#C_3 = 2$. We let χ_1 be the trivial representation. For our second representation, we take the representation of S_3 on the plane, coming from the action of S_3 as symmetries of an equilateral triangle. We have seen that this representation is irreducible. The degree of this representation is two. The elements of C_2 act as reflections about a line, and so have trace zero. Thus $\chi_2(2) = 0$. The elements of C_3 act as rotations through angle $\pm 2\pi/3$, and hence have trace $2\cos(2\pi/3) = -1$. Thus the entries of the second row are $2, 0, -1$. Since $1^2 + 2^2 = 5$, we know from (6.2) that the one remaining is one dimensional and is, of course, the sign representation. Thus $\chi_3(1) = 1, \chi_3(2) = -1$ and $\chi_3(3) = 1$. The character table is as in Table 6.

Equation (7.1) for $i = k$ says

$$1^2 + 3\cdot 1^2 + 2\cdot 1^2 = 6$$

$$2^2 + 3\cdot 0^2 + 2(-1)^2 = 6$$

and

$$1^2 + 3\cdot(-1)^2 + 2\cdot 1^2 = 6.$$

For $i = 1$ and $k = 3$, equation (7.1) says

$$1\cdot 1 + 3\cdot 1\cdot(-1) + 2\cdot 1\cdot 1 = 0.$$

Equation (7.2) for $j = l = 1$ reduces to (6.2). This is true for any character table. For $j = 2, l = 3$ it says

$$3[1\cdot 1 + 0\cdot(-1) + (-1)\cdot 1] = 0.$$

Table 7.

nC_n	$1[e]$	$1[a]$	$1[a^2]$	⋯	$1[a^{n-1}]$
χ_1	1	1	1	⋯	1
χ_2	1	ε	ε^2	⋯	ε^{n-1}
χ_3	1	ε^2	ε^4	⋯	$\varepsilon^{2(n-1)}$
.	.	.	.	⋯	.
.	.	.	.	⋯	.
.	.	.	.	⋯	.
χ_n	1	ε^{n-1}	$\varepsilon^{2(n-1)}$	⋯	$\varepsilon^{(n-1)^2}$

For $j = l = 2$, and $j = l = 3$ (7.2) says, respectively,

$$3[1^2 + 0^2 + (-1)^2] = 6$$

and

$$2[1^2 + (-1)^2 + 1^2] = 6.$$

Notice that all the entries of the character table for S_3 happen to be integers. This is not true for the general finite group, but happens to be true for all the symmetric groups. We will discuss the representations of the general symmetric group, S_n, in the next section and in Appendix C.

The purpose of this section is to derive the character table for many interesting groups. We begin with some general remarks. Suppose that the group G is Abelian. Then each element makes up its own conjugacy class, so that there are $\#G$ conjugacy classes in all. Then (6.2) and (6.3) imply that all the $n_i = 1$, so that all irreducible representations of G are one dimensional. Conversely, suppose that all the $n_i = 1$. Then (6.2) can hold only if $k = \#G$. This means that there are $\#G$ distinct conjugacy classes. Thus, each conjugacy class contains exactly one element, i.e. G is Abelian. We have thus proved

> G is Abelian if and only if all its irreducible representations are one dimensional. (7.3)

We can now write down the character table for the cyclic group C_n. Let a be a generator for C_n, so that the conjugacy classes are the various $[a^{j-1}]$, $j = 1, 2, \ldots, n$. Let ε be a primitive nth root of unity. Then the characters χ_i, determined by $\chi_i(a) = \varepsilon^{i-1}$, $i = 1, 2, \ldots, n$, are all distinct, and thus give all the characters. The character table of C_n is thus given by Table 7.

Before proceeding further, we record the following useful fact:

> Suppose that the group G has a commutative subgroup H. Then any irreducible representation of G has degree at most $\#G/\#H$. (7.4)

Proof Let r be an irreducible representation of G on the vector space V. Then r_H, the restriction of r to H, gives a representation of the Abelian group H on V. Let W

Table 8.

$12T$	$[e]$	$4[r_3]$	$4[r_3^2]$	$3[r^2]$
χ_1	1	1	1	1
χ_2	1	ε	ε^2	1
χ_3	1	ε^2	ε	1
χ_4	3	0	0	-1

$\varepsilon = \exp 2\pi i/3$

be an irreducible subspace of V under H, so that W is one dimensional by (7.3). The space spanned by all the $r(a)Wa \in G$, is clearly invariant and hence must be all of V. Let G_W be the subgroup of G which stabilizes W, i.e. $G_W = \{a \in G : r(a)W \subset W\}$. Then the number of distinct subspaces, among the $r(a)W$, is given by $\#(G/G_W)$, and therefore the maximal number of linearly independent such subspaces is at most this amount. Since $H \subset G_W$, we conclude (7.4).

Let us now compute the character table of the group T. This group is of order 12, and has a three-dimensional representation as the symmetries of the tetrahedron. The trace of any rotation through angle ϕ in $\mathbb{R}^3$ (or $\mathbb{C}^3$) is $1 + 2\cos\phi$. Thus, for this three-dimensional representation we have

$$\chi(e) = 3, \quad \chi(R_{120°}) = \chi(R_{240°}) = 0 \quad \text{and} \quad \chi(R_{180°}) = -1.$$

Thus

$$(\#G)\,\|\chi\|^2 = 9 + 3 \cdot 1 = 12$$

since there are three rotations through 180° and four each through 120° and 240°. We see that this three-dimensional representation is irreducible. Since the sum of squares of the degrees of all irreducible representations is 12, and $3^2 = 9$, there must also be three one-dimensional representations. These can be found as follows: let H be the subgroup of T consisting of the identity and the rotations through 180°. Then H is a normal subgroup and hence any representation of the quotient group, T/H, lifts to a representation of T. But T/H is just the cycle group C_3, which has three one-dimensional representations. Thus the character table of T is given by Table 8.

For the group O, observe again that the subgroup H consisting of the identity and the 180° rotations is a normal subgroup. This time, the quotient group is S_3, whose character table we have already computed. The representations of the quotient group S_3 give rise to two one-dimensional representations and one two-dimensional representation. Since $24 - 6 = 18$, and there are five conjugacy classes altogether, we conclude that there are also two three-dimensional representations. One of these is the representation of O as the group of symmetries of the cube in three-space. The other is the representation of O as the group T_d on three-space. The character table of O is thus given as in Table 9. Notice that all the entries in the table are integers. This is because $O \sim S_4$.

Table 9.

$24O$	$[e]$	$6[r_4]$	$3[r_4^2]$	$8[r_3]$	$6[r_2]$
χ_1	1	1	1	1	1
χ_2	2	0	2	-1	0
χ_3	1	-1	1	1	-1
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Table 10.

$4D_2$	$[e]$	$[r_x]$	$[r_y]$	$[r_z]$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Table 11.

$8D_4$	$[e]$	$2[r_4]$	$[r_4^2]$	$2[r_2+]$	$2[r_2\times]$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	-1	1
χ_4	1	-1	1	1	-1
χ_5	2	0	-2	0	0

To complete our discussion of character tables for the crystallographic rotation groups, we need only consider the dihedral groups D_2, D_4 and D_6.

D_2 is Abelian: it has four elements, four conjugacy classes, and four one-dimensional representations. In each representation, other than the trivial representation, one of the 180° rotations is represented by $+1$, the other two by -1. The character table is therefore as in Table 10.

We have worked out the irreducible representations of D_4 in Section 2.5; there are four one-dimensional representations and one two-dimensional representation. The characters of the two-dimensional representation may be found from the traces of the 2×2 matrices which describe rotation and reflection symmetries of the square: 0 for the 90° and 270° rotations and for all the reflections, 2 for the identity, -2 for the 180° rotation. The complete character table is therefore as in Table 11.

Table 12.

$12D_6$	$[e]$	$2[r_6]$	$2[r_6^2]$	$[r_6^3]$	$3[r_2]$(solid line)	$3[r_2]$(broken line)
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1
χ_3	1	−1	1	−1	1	−1
χ_4	1	−1	1	−1	−1	1
χ_5	2	1	−1	−2	0	0
χ_6	2	−1	−1	2	0	0

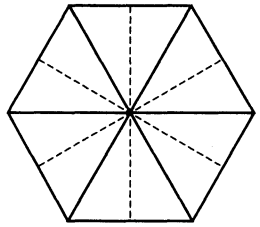


Fig. 2.1

Note that representation χ_2 has $+1$ for elements of the normal subgroup C_4, while χ_3 and χ_4 has $+1$ for elements of the two non-conjugate normal subgroups D_2.

For D_6 there are four one-dimensional representations and *two* two-dimensional representations (note that $4\cdot1^2+2\cdot2^2=12$). As with D_4, we may obtain the one-dimensional representations other than the trivial one by assigning $+1$ to each of the three normal subgroups of index 2: C_6 (rotations in plane of page), D_3 (0°, 120°, and 240° rotations in plane of page and 180° rotations about solid lines in Fig. 2.1), D_3 (0°, 120°, and 240° rotations in plane of page and 180° rotations about broken lines in Fig. 2.1). One two-dimensional representation with character χ_5 is obviously obtained by writing down the 2×2 rotation and reflection matrices. The character of this representation is 0 for reflection, $2\cos\phi$ for rotations. The other two-dimensional representation is difficult to discover by inspection (we shall construct it in the next chapter) but its character χ_6 is easily written down by orthogonality. The complete character table is shown in Table 12.

We have now computed the character tables for all the groups in the first column of Table 4. The groups in the third column are isomorphic as abstract groups to those in the first column, so we have their character tables as well. The groups in the second column are all direct products of the groups in the first column with Z_2. We get their character tables as a result of the following:

Table 13.

$12D_{3d}$	$[e]$	$2[r_3]$	$3[r_2]$	$[-I]$	$2[s_6]$	$3[\tau_d]$
χ_1	1	1	1	1	1	1
χ_2	2	−1	0	2	−1	0
χ_3	1	1	−1	1	1	−1
χ_4	1	1	1	−1	−1	−1
χ_5	2	−1	0	−2	1	0
χ_6	1	1	−1	−1	−1	1

Proposition 7.1
Let r^1 and r^2 be irreducible representations of the groups G_1 and G_2 on the vector spaces V_1 and V_2. Then the representation $r^1\otimes r^2$ of $G_1\times G_2$ on $V_1\otimes V_2$ given by

$$(r^1\otimes r^2)_{(g_1,g_2)}(v_1\otimes v_2)=r^1_{g_1}v_1\otimes r^2_{g_2}v_2,$$

is irreducible and has character $\chi^1(g_1)\chi^2(g_2)$, where χ^i is the character of r^i on G_i. Furthermore, every irreducible representation of $G_1\times G_2$ is such a tensor product.

We have already proved the first two assertions. The last follows from the fact that the sums of the squares of the degrees of these representations equals $\#(G_1\times G_2)$ and hence these are all the irreducibles.

Thus the character tables for the groups in the second column of Table 4 are obtained by 'doubling' the character table for the corresponding group in the first column. For instance, the character table for D_{3d} is given by Table 13, where the upper left-hand corner is the character table for D_3.

The character tables are useful in the explicit computation of the reduction of a reducible representation into its irreducible components. If ϕ is the character of the representation then, letting $\phi(k)$ denote the value of ϕ on any element in the kth conjugacy class, we have

$$(\phi,\chi_i)=(1/\#G)(\#C_1\phi(1)\overline{\chi_i(1)}+\cdots+\#C_p\phi(p)\overline{\chi_i(p)}),$$

and the values of the $\chi_i(j)$ can be read from the table. We shall make use of this technique in the study of molecular vibrations in the next chapter.

As an application, we determine the reduction of the representation of D_6 obtained by letting it act on the space of functions on the vectors of a hexagon. The character ϕ of this six-dimensional representation is easily determined from (5.1): it is six for the identity, which has six fixed points, two for the three 180° rotations about axes through opposite vertices, zero for all other elements. Using the character table for

D_6 we find:

$$(\phi, \chi_1) = \tfrac{1}{12}(6\cdot 1 + 3\cdot 2\cdot 1) = 1$$
$$(\phi, \chi_2) = \tfrac{1}{12}(6\cdot 1 + 3\cdot 2(-1)) = 0$$
$$(\phi, \chi_3) = \tfrac{1}{12}(6\cdot 1 + 3\cdot 2\cdot 1) = 1$$
$$(\phi, \chi_4) = \tfrac{1}{12}(6\cdot 1 + 3\cdot 2(-1)) = 0$$
$$(\phi, \chi_5) = \tfrac{1}{12}(6\cdot 2) = 1$$
$$(\phi, \chi_6) = \tfrac{1}{12}(6\cdot 2) = 1.$$

So each of the two-dimensional representations occurs once, the trivial representation occurs once, and the representation with character χ_3 occurs once. The invariant subspace for the representation with character χ_3 is clearly spanned by the function represented by the diagram

$$\begin{array}{ccc} + & & - \\ & & \\ - & & + \\ & & \\ + & & - \end{array},$$

which is carried into itself by 120° and 240° rotations and by 180° rotations about axes through opposite vertices, but into its negative by the other six group elements.

2.8 The representations of the symmetric group

In this section we show how the methods of Section 2.4 and 2.5 allow us to determine, at least in principle, all the irreducible representations of the symmetric group S_n. We insert this material here since it depends only on the theory developed in this chapter. We shall need these results for physical and mathematical applications in Chapter 5, and the reader might prefer to postpone reading this section until then.

The symmetric group S_n is the group of all permutations, that is of all one-to-one transformations of the set $\{1,\ldots,n\}$ of n elements. There are various convenient notations for writing an element s of S_n. One is to write it out as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ s(1) & s(2) & s(3) & \cdots & s(n) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} 12345 \\ 53124 \end{pmatrix}$$

is the element of S_5 which sends 1 to 5 and 2 to 3, etc. Another convenient notation is the 'cycle notation'. We start with any number, write its image to the right, and keep going until a cycle is completed. We then pick some number not in the first cycle, write its image to its right, and continue until a second cycle is completed. This procedure is repeated until the entire permutation has been described. Thus the above

element of S_5 would be written as

$$(15423)$$

since 1 goes to 5, 5 goes to 4, etc. Similarly the element

$$\begin{pmatrix} 12345 \\ 31254 \end{pmatrix}$$

would be written in terms of two cycles as

$$(132)(45).$$

It is the product of (45) with (132).

Let $s = (132)(45)$ and let t be some other element of S_5. Then

$$(tst^{-1})(t(1)) = t(s(1)) = t(3),$$
$$(tst^{-1})(t(3)) = t(s(3)) = t(2), \text{etc.}$$

The general rule is that $tst^{-1}(t(i)) = t(s(i))$, so we may write tst^{-1} in terms of cycles as

$$tst^{-1} = (t(1)t(3)t(2))(t(4)t(5)).$$

The same argument shows that, in general, for a permutations $s \in S_n$ which is written in cycle form, the element tst^{-1} is obtained by replacing each integer i in the cycle form of s by $t(i)$. Conversely, if s_1 and s_2 have the same cycle structure, so that there is a permutation t relating the entry i in s_1 to $t(i)$ in s_2, then $s_2 = ts_1t^{-1}$. Thus, two elements of S_n are conjugate if and only if they have the same cycle structure. Thus, for example, the five conjugacy classes of S_4 are

$$\{e\}$$
$$\{(ab)\} = \{(12),(13),(14),(23),(24),(34)\}$$
$$\{(ab)(cd)\} = \{(12)(34),(13)(24),(14)(23)\}$$
$$\{(abc)\} = \{(123),(124),(132),(134),(142),(143),(234),(243)\}$$
$$\{(abcd)\} = \{(1234),(1243),(1324),(1342),(1423),(1432)\}.$$

Notice that any *cyclic* permutation of the entries of a cycle does not change the permutation: (123) and (231) are the same permutation, as follows from the definition of cycle. Also, interchanging two disjoint cycles gives the same permutation. (12)(34) and (34)(12) are the same. A conjugacy class of S_n is thus determined by $[\nu_1,\ldots,\nu_n]$, where ν_1 is the number of one-cycles, ν_2 is the number of two-cycles, etc. Of course, the ν's are constrained by

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n = n.$$

The number of elements in a conjugacy class is given by $\#S_n/\#H$, where H is the isotropy subgroup of some element s in the conjugacy class, i.e. $H = \{t | tst^{-1} = s\}$. Suppose s has the cycle structure $[\nu_1,\ldots,\nu_n]$. Then t cannot interchange entries coming from cycles of different length. Within the set of cycles of a fixed length, t can act as a cyclic permutation within each cycle and can permute cycles as a whole. Thus,

considering the cycles of different length independently, we see that

$$\#H = 1^{\nu_1}\nu_1!2^{\nu_2}\nu_2!3^{\nu_3}\nu_3!\cdots n^{\nu_n}\nu_n!$$

where. for example, 3^{ν_3} is present because there are three cyclic permutations within each of the ν_3 three-cycles and $\nu_3!$ is present because there are $\nu_3!$ permutations of these three-cycles among themselves. Thus,

$$\begin{array}{l}\text{the number of elements}\\ \text{in the conjugacy class}\\ \text{given by } [\nu_1,\ldots,\nu_n]\end{array} = \frac{n!}{1^{\nu_1}\nu_1!2^{\nu_2}\nu_2!\cdots n^{\nu_n}\nu_n!}.$$

So, for S_4

$$\#\{e\} = \#[4,0,0,0] = \frac{4!}{4!} = 1$$

$$\#\{(a,b)\} = \#[2,1,0,0] = \frac{4!}{2\cdot 2!} = 6$$

$$\#\{(a,b)(c,d)\} = \#[0,2,0,0] = \frac{4!}{2^2\cdot 2!} = 3$$

$$\#\{(abc)\} = \#[1,0,1,0] = \frac{4!}{3} = 8$$

$$\#\{(abcd)\} = \#[0,0,0,1] = \frac{4!}{4} = 6.$$

We can also set

$$\begin{aligned}\lambda_1 &= \nu_1 + \nu_2 + \cdots + \nu_n\\ \lambda_2 &= \nu_2 + \nu_3 + \cdots + \nu_n\\ &\vdots\\ \lambda_n &= \nu_n\end{aligned}.$$

Thus $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_n$, and it follows from

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n = n$$

that

$$\lambda_1 + \cdots + \lambda_n = n.$$

For example, the permutation $(1)(23)(45)(678) \in S_8$ has $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 1$. The set $\lambda = (\lambda_1,\ldots,\lambda_n)$ is called a partition of n. It is conveniently represented by a *Young diagram.*

We draw the diagram as an array of boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, etc. For example, if $n = 7$ then $\lambda = (3,2,1,1)$ is drawn as

and similarly $(5,2) = (5,2,0,0)$ (we usually drop the zeros) is

Given $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\lambda_i \geqslant \lambda_{i+1}$ and $\lambda_1 + \cdots + \lambda_n = n$, we recover ν_i by setting

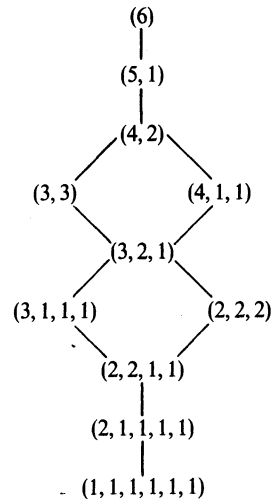
$$\nu_i = \lambda_i - \lambda_{i+1}.$$

For example, the first diagram corresponds to $\nu_1 = 1$, $\nu_2 = 1$, $\nu_3 = 0$, $\nu_4 = 1$; the second to $\nu_1 = 3$, $\nu_2 = 2$. Clearly $\nu_1 + 2\nu_2 + \cdots + n\nu_n = n$. Thus the number of conjugacy classes of S_n, which is the same as the number of inequivalent irreducible representations of S_n, is the same as the number of Young diagrams. Our task is to attach a distinct irreducible representation on each Young diagram.

From now on we consider a fixed n. We put a partial order on the diagrams by saying that $\lambda \geqslant \mu$ if, for all i, the total number of boxes in the first i rows of λ is no less than the total number of boxes in the first i rows of μ; i.e. if

$$\begin{aligned}\lambda_1 &\geqslant \mu_1\\ \lambda_1 + \lambda_2 &\geqslant \mu_1 + \mu_2\\ \lambda_1 + \lambda_2 + \lambda_3 &\geqslant \mu_1 + \mu_2 + \mu_3, \text{ etc.}\end{aligned}$$

For example, the partial ordering (down is decreasing) for S_6 is given by:



By a Young *tabloid* corresponding to the diagram $\lambda = (\lambda_1,\ldots,\lambda_n)$ we mean a decomposition of the set $\{1,\ldots,n\}$ into a union of disjoint sets where the first set

contains λ_1 elements, the second set contains λ_2 elements, etc. Thus

$$\{3,5,2\}\{1,7\}\{4\}\{6\} \quad \text{or} \quad \left\{ \begin{array}{|c|c|c|}\hline 3&5&2\\\hline\end{array} \right.$$
$$\left. \begin{array}{|c|c|}\hline 1&7\\\hline\end{array} \right\}$$
$$\begin{array}{|c|}\hline 4\\\hline\end{array}$$
$$\begin{array}{|c|}\hline 6\\\hline\end{array}$$

is a Young tabloid corresponding to the Young diagram $(3,2,1,1)$. The individual subsets are unordered so

$$\{2,3,5\}\{7,1\}\{4\}\{6\} \quad \text{or} \quad \left\{ \begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array} \right.$$
$$\left. \begin{array}{|c|c|}\hline 7&1\\\hline\end{array} \right\}$$
$$\begin{array}{|c|}\hline 4\\\hline\end{array}$$
$$\begin{array}{|c|}\hline 6\\\hline\end{array}$$

is the same tabloid. However, the order of the subsets is important,

$$\{3,5,2\}\{1,7\}\{6\}\{4\} \quad \text{or} \quad \left\{ \begin{array}{|c|c|c|}\hline 3&5&2\\\hline\end{array} \right.$$
$$\left. \begin{array}{|c|c|}\hline 1&7\\\hline\end{array} \right\}$$
$$\begin{array}{|c|}\hline 6\\\hline\end{array}$$
$$\begin{array}{|c|}\hline 4\\\hline\end{array}$$

is a different tabloid. We can think of a tabloid as a way of putting the number $\{1,2,\ldots,n\}$ into the boxes of a Young diagram, where the order of numbers within each row does not matter.

We let M_λ denote the set of all tabloids corresponding to a Young diagram λ. If $\{t\}$ is a fixed tabloid corresponding to λ, the isotropy group of $\{t\}$ is clearly isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_p}$, the subgroup which permutes elements within each row of the diagram. Since S_n acts transitively on M_λ, we see that

$$\#M_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_p!}.$$

Since S_n acts on the set M_λ, we get a representation of S_n on $\mathscr{F}(M_\lambda)$. For example, $M_{(n)}$ contains only one element,

$$\{1,\ldots,n\} \quad \text{or} \quad \{\boxed{1}\boxed{2}\boxed{3}\boxed{\cdots}\boxed{n}\}.$$

All permutations carry this tabloid into itself, so the representation of S_n on $\mathscr{F}(M_{(n)})$ is the trivial representation. An element of the set $M_{(n-1,1)}$ is of the form $\{1,\ldots,\hat{k},\ldots,n\} \cup \{k\}$, where the symbol $\hat{k}$ means that k is *missing*. So $M_{(n-1,1)}$ can be identified with the set $\{1,\ldots,n\}$, where $k \in \{1,\ldots,n\}$ corresponds to the missing $\{k\}$. For example, if $n=3$ there are three tabloids, $\{23\},\{1\}$; $\{13\},\{2\}$; and $\{12\},\{3\}$, which may be identified with 1, 2, and 3, respectively. We have seen that S_n, acting on $M_{(n-1,1)} \times M_{(n-1,1)}$, has two orbits, so that

$$\mathscr{F}(M_{(n-1,1)}) = \mathbb{C} \oplus F_{(n-1,1)},$$

where $F_{n-1,1}$ is an irreducible space of dimension $n-1$. Notice that the first component, the constant functions, is just $\mathscr{F}(M_{(n)})$.

The set $M_{(n-2,2)}$ $(n>3)$ can be identified with the space of all two-element subsets of $\{1,\ldots,n\}$, where we look at the entries in the second subset, so that $\{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n\}$, $\{k,l\}$ is identified with $\{k,l\}$. For example, if $n=5$, $M_{(n-2,2)}$ has ten elements. The element $\{3,4,5\},\{1,2\}$ is associated with $\{1,2\}$, the element $\{2,4,5\}$, $\{1,3\}$ with $\{1,3\}$, and so on. A pair of two-element subsets may have either zero, one or two elements in common. Thus, S_n has three orbits when acting on $M_{(n-2,2)} \times M_{(n-2,2)}$, and so $\mathscr{F}(M_{(n-2,2)})$ breaks up into three irreducible components. We claim that two of these components are $\mathscr{F}(M_{(n)})$ and $\mathscr{F}(M_{(n-1,1)})$. Indeed, we now describe a map from $\mathscr{F}(M_{(n-1,1)})$ to $\mathscr{F}(M_{(n-2,2)})$ which commutes with the action of S_n and is injective: we must find a map, T, which goes from functions, f, on $\{1,\ldots,n\}$ to functions on two-element subsets. Take T to be given by

$$(Tf)(\{a,b\}) = f(a) + f(b).$$

It is clear that T commutes with the action of S_n. Also T (constant) = constant and $T\delta_a$ is not a constant (and, in particular, not zero). Thus T is not zero when restricted to each of the irreducible components of $\mathscr{F}(M_{(n-1,1)})$ and hence is injective. Thus

$$\mathscr{F}(M_{(n-2,2)}) = \mathbb{C} + T(F_{(n-1,1)}) + F_{(n-2,2)}$$
$$1 \qquad n-1 \qquad \frac{n(n-3)}{2}.$$

The dimension of $F_{(n-2,2)}$ is obtained by subtracting:

$$\dim F_{(n-2,2)} = \frac{n!}{(n-2)!2!} - n = \frac{n(n-3)}{2}.$$

We wish to prove the following: to each λ there corresponds a unique 'new' irreducible subrepresentation F_λ of $\mathscr{F}(M_\lambda)$. The space $\mathscr{F}(M_\lambda)$ decomposes into a direct sum of irreducible subrepresentations isomorphic to certain of the F_μ *with* $\mu \geqslant \lambda$ (and these may occur with multiplicity) together with the one unique new subrepresentation F_λ. Thus each Young diagram determines an irreducible representation of S_n.

By a Young *tableau* corresponding to λ we mean an assignment of the numbers $\{1,\ldots,n\}$ to each of the boxes of λ, one number to each box. In a tableau, the order in each row matters. Thus

$$\begin{array}{|c|c|c|}\hline 3&5&2\\\hline\end{array}$$
$$\begin{array}{|c|c|}\hline 1&7\\\hline\end{array}$$
$$\begin{array}{|c|}\hline 4\\\hline\end{array}$$
$$\begin{array}{|c|}\hline 6\\\hline\end{array}$$

is a $(3,2,1,1)$ tableau. Each tableau gives rise to a tabloid, by letting the entries in the first row belong to the first set, the entries of the second row correspond to the second set, etc. Two different tableaux, which differ by a permutation of the entries

of their rows, give rise to the same tabloid. If t is a tableau, the corresponding tabloid will be denoted by $\{t\}$. Thus if t is the above tableau, then $\{t\} = \{3,5,2\}\{1,7\}\{4\}\{6\}$.

We now describe the F_λ. Let t be a tableau. Let C_t denote the subgroup of S_n consisting of those π which permute the numbers in the various columns of t among themselves. Thus, if

$$t = \begin{array}{|c|c|c|}\hline 3 & 5 & 2 \\ \hline 1 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

then

$$C_t = S_{\{3,1,4,6\}} \times S_{\{5,7\}}$$

where $S_{\{3,1,4,6\}}$ are the permutations of $\{3,1,4,6\}$, etc. Let

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{t\}}.$$

Notice that the e_t depends on t and not just the tabloid $\{t\}$

As an example we construct the e_t for the case of S_3, with the Young diagram

$$\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}.$$

There are three tabloids in this case,

$\{2,3\}\{1\}$ (associated with 1)
$\{1,3\}\{2\}$ (associated with 2)

and

$\{1,2\}\{3\}$ (associated with 3)

and the space $\mathscr{F}(M_{(2,1)})$ is spanned by $\delta_1, \delta_2, \delta_3$. A typical tableau is

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

The subgroup C_t consists of the identity, whose sign is $+1$, and the permutation (13), whose sign is -1. We must determine what tabloid results from the action of each element of C_t on the tableau t. Clearly the identity gives the tabloid $\{1,2\}$, $\{3\}$, while (13) to converts t to

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array},$$

which corresponds to the tabloid $\{2,3\}$, $\{1\}$. Hence

$$e_t = \delta_3 - \delta_1.$$

If we start with a different tableau, say

$$t' = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array},$$

then $C_{t'}$ consists of the identity and (32), and $e_{t'} = \delta_2 - \delta_3$. Since the space spanned by δ_1, δ_2 and δ_3 has a one-dimensional irreducible subspace spanned by the constant function $\delta_1 + \delta_2 + \delta_3$, the elements e_t and $e_{t'}$ which we have constructed span the complementary two-dimensional subspace F_λ. There are four other tableaux for which we could construct e_t; they would just lead to linear combinations of what we have already found.

Since $\sigma\delta_{\{t\}} = \delta_{\sigma\{t\}}$, we have

$$\sigma e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\sigma\pi\{t\}}.$$

We can write this as

$$\sigma e_t = \sum_{\pi \in C_t} \operatorname{sign}(\sigma\pi\sigma^{-1})\delta_{\sigma\pi\sigma^{-1}\sigma\{t\}}.$$

But $C_{\sigma t} = \sigma C_t \sigma^{-1}$ so we can write the last sum as

$$\sum_{\rho \in C_{\sigma t}} \operatorname{sgn}(\rho)\delta_{\rho\{\sigma t\}} = e_{\sigma t}.$$

We conclude that

$$\sigma e_t = e_{\sigma t}.$$

Thus the space spanned by the e_t is invariant under S_n. We define this space to be F_λ. Thus

$$F_\lambda = \{\text{linear span of all the } e_t\}.$$

Consider, for example, the action of $\sigma = (23)$ on each of the two elements e_t and $e_{t'}$ which we constructed earlier. For

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad e_t = \delta_3 - \delta_1,$$

while for

$$t' = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad e_{t'} = \delta_2 - \delta_3.$$

Now $\sigma e_t = e_{\sigma t}$, where $\sigma t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. We find $e_{\sigma t} = \delta_2 - \delta_1 = e_{t'} + e_t$.

Also, $\sigma e_{t'} = e_{\sigma t'}$ where $\sigma t' = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$, so we have $e_{\sigma t'} = \delta_3 - \delta_2 = -e_{t'}$.

Thus, relative to the basis $\{e_t, e_{t'}\}$, (23) is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

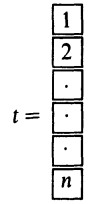
Continuing in this manner, we can construct the entire two-dimensional representation of S_3.

As another example, if $\lambda = (1,1,\ldots,1)$, then $C_t = S_n$ and, up to sign, there is only

one e_t and it is

$$e_t = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\delta_{\pi t}$$

where we may take

$t =$ 

The representation in this case is the one-dimensional sign representation. Also in this case $S_{\{t\}} = \{e\}$, so we may identify $M = S_n/S_{\{t\}}$ with S_n and thus $\mathscr{F}(M)$ with $\mathscr{F}(S_n)$. We know that the regular representation contains any irreducible representation with multiplicity equal to its dimension and hence contains $F_{(1,\ldots,1)}$ once. Also $(1,\ldots,1)$ is the last diagram on our list. This supports the contention of the theorem.

We now prove the following results about the Young tableaux and the representations of S_n.

(1) Let λ and μ be diagrams and let t be a λ tableau and s be a μ tableau. Suppose that for every i, the numbers from the ith row of s belong to different columns of t. Then $\lambda \geqslant \mu$.

Proof The numbers in the first row of s all lie in different columns of t. Hence, λ has at least μ_1 columns (i.e. $\lambda_1 \geqslant \mu_1$). Suppose $\lambda_1 = \mu_1$. The numbers in the second row of s all lie in different columns of t. We see that $\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2$, etc. QED.

(2) With the same notations as in (1), suppose that

$$\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} \neq 0.$$

Then $\lambda \geqslant \mu$, and if $\lambda = \mu$ then

$$\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{s\}} = \pm e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\sigma\pi)\delta_{\pi\{t\}},$$

where $\sigma \in C_t$ and $\{s\} = \sigma\{t\}$.

Proof Let

$$A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi$$

as an operator, so that, for example,

$$A_t\delta_{\{t\}} = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\delta_{\pi\{t\}} = e_t.$$

Thus we can rewrite (2) as saying that

$$A_t\delta_{\{s\}} = 0 \quad \text{if} \quad \mu \not\leqslant \lambda$$

and if $\mu = \lambda$ then

$$A_t\delta_{\{s\}} = \begin{cases} 0 & \text{if } \{s\} \neq \sigma\{t\} \text{ for some } \sigma \in C_t \\ \operatorname{sgn}(\sigma)e_t & \text{if } \{s\} = \sigma\{t\} \text{ for } \sigma \in C_t. \end{cases}$$

Suppose that

$$A_t\delta_{\{s\}} \neq 0.$$

Under this hypothesis, we claim that two numbers x and y which lie in the same row of s cannot lie in the same column of t. For to say that x and y lie in the same row of s implies that $(xy)\{s\} = \{s\}$. If x and y lie in the same column of t, then $(xy) \in C_t$ and, since $\operatorname{sgn}(xy) = -1$, we could have

$$A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi = \sum_{\pi \in C_t} \operatorname{sgn}(\pi\cdot(xy))\pi\cdot(xy) = -\sum \operatorname{sgn}(\pi)\pi(xy) = -A_t(xy).$$

Thus

$$A_t\delta_{\{s\}} = -A_t(xy)\delta_{\{s\}} = -A_t\delta_{(xy)\{s\}} = -A_t\delta_{\{s\}}$$

contradicting the assumption that $A_t\delta_{\{s\}} \neq 0$.

This implies that $\lambda \geqslant \mu$. If $\lambda = \mu$, all numbers in the first row of s occur in different columns of t. So we can find some $\pi \in C_t$ such that πt has the same first row as s. All elements of the second row of s occur in different columns of πt and below the first row. So we can find a $\pi' \in C_{\pi t} = C_t$, leaving the numbers in the first row of s fixed with $\pi'\pi t$ having the same first two rows, as s, etc. This shows that $\{s\} = \{\sigma t\}$ for some $\sigma \in C_t$. But then $A_t\delta_{\{s\}} = \operatorname{sgn}(\sigma)e_t$. Thus we have proved (2). We conclude that for any $\{s\}$ whatsoever in M_λ, we have

$$A_t\delta_{\{s\}} = \begin{cases} e_t & \text{if } \{s\} = \sigma\{t\} \quad \operatorname{sgn}(\sigma) = 1 \\ 0 & \text{if } \{s\} \neq \sigma\{t\} \\ -e_t & \text{if } \{s\} = \sigma\{t\} \quad \operatorname{sgn}(\sigma) = -1. \end{cases}$$

Now every $f \in \mathscr{F}(M_\lambda)$ is a linear combination of the $\delta_{\{s\}}$ as $\{s\}$ ranges over the λ tabloids. Hence (3).

(3) For any $f \in \mathscr{F}(M_\lambda)$,

$$A_t f = c_f e_t$$

where c_f is a scalar, i.e. $A_t f$ is a multiple of e_t for any f.

Let us put a scalar product $(,)$ on $\mathscr{F}(M_\lambda)$ by taking the $\delta_{\{t\}}$ as an orthonormal basis. This is clearly S_n invariant. Now for any $u, v \in \mathscr{F}(M_\lambda)$,

$$(A_t u, v) = \sum_{\pi \in C_t} (\operatorname{sgn}(\pi)\pi u, v)$$

$$= \sum_{\pi \in C_t} (u, \operatorname{sgn}(\pi^{-1})\pi^{-1}v), \text{ since } \operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^{-1})$$

$$= \sum_{\pi \in C_t} (u, \operatorname{sgn}(\pi)\pi v)$$
$$= (u, A_t v).$$

(4) Let U be an invariant subspace of $\mathscr{F}(M_\lambda)$. Then either $U \supset F_\lambda$ or $U \subset F_\lambda^\perp$. In particular, F_λ is irreducible.

Proof Let $u \in U$ and let t be a λ tableau. Then $A_t u$ is a multiple of e_t. If for some t and u this multiple is not zero, then $A_t u \in U$ and $A_t u = c_u e_t$, and since F_λ is generated by the σe_t as $\sigma \in S_n$, we see that $F_\lambda \subset U$. If these multiples are zero for all t and u, then $0 = (A_t u, \delta_{\{t\}}) = (u, A_t \delta_{\{t\}}) = (u, e_t)$ for all u and t, so $U \subset F_\lambda^\perp$.

(5) Let $T: \mathscr{F}(M_\lambda) \to \mathscr{F}(M_\mu)$ be any element of $\operatorname{Hom}_{S_n}(\mathscr{F}(M_\lambda), \mathscr{F}(M_\mu))$. Suppose that $F_\lambda \not\subset \ker T$. Then $\lambda \geqslant \mu$. If $\lambda = \mu$, then the restriction of T to F_λ is a scalar multiple of the identity.

Proof By (4), $\ker T \subset F_\lambda^\perp$. Let t be any λ tableau. Then $0 \neq Te_t = TA_t \delta_{\{t\}} = A_t T \delta_{\{t\}}$. But $T\delta_{\{t\}} \in \mathscr{F}(M_\mu)$ is some combination of $\delta_{\{s\}}$ for μ tabloids $\{s\}$ and $A_t \delta_{\{s\}} = 0$ unless $\lambda \geqslant \mu$. The second part follows from Schur's lemma and (2), since $A_t F_\lambda(M_\lambda) \subset F_\lambda$.

(6) $\operatorname{Hom}_{S_n}(F_\lambda, F_\mu) = 0$ unless $\lambda = \mu$. In particular, since the number of diagrams = the number of partitions = the number of conjugacy classes of S_n, the F_λ are exactly all the irreducible representations of S_n.

Proof Any $T \in \operatorname{Hom}_{S_n}(F_\lambda, F_\mu)$ can be extended to an element of $\operatorname{Hom}_{S_n}(\mathscr{F}(M_\lambda), \mathscr{F}(M_\mu))$ by setting it equal to zero on $F_\lambda^\perp$. By (5) this shows that if $T \neq 0$, then $\lambda \geqslant \mu$. Since F_λ and F_μ are irreducible, by Schur's lemma, if $T \neq 0$ then T is invertible and working with T^{-1} shows that $\mu \geqslant \lambda$, hence $\lambda = \mu$.

If we now decompose $\mathscr{F}(M_\mu)$ into irreducibles, we see that the only possible irreducible components, by (5), are the F with $\lambda \geqslant \mu$. This completes the proof of our assertions.

As an example of the procedure just described, we construct all the irreducible representations of S_4.

The first Young diagram is

$$\lambda = (4) = \square\square\square\square.$$

In this case, there is only one tabloid, $\{1,2,3,4\}$, and we denote the unit function on this tabloid by δ. To construct a basis, we choose any tableau,

$$t = \boxed{1}\boxed{2}\boxed{3}\boxed{4}.$$

The subgroup C_t consists only of the identity, and so

$$e_t = \delta.$$

Since $\sigma\{t\} = \{t\}$ for any $\sigma \in S_4$, we have $\sigma e_t = e_{\sigma t} = e_t$: we have constructed the trivial one-dimensional identity representation.

The next Young diagram in order is (3, 1) or

$$\begin{array}{|c|c|c|c|}\hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

There are four tabloids, $\{2,3,4\}\{1\}$, $\{1,3,4\}\{2\}$, $\{1,2,4\}\{3\}$, and $\{1,2,3\}\{4\}$. We denote the basis elements of the space of functions on these tabloids by $\delta_1, \delta_2, \delta_3, \delta_4$. To construct a basis for the new irreducible representation of S_4 corresponding to this diagram, we choose three tableaux:

$$t_1 = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}; \quad e_{t_1} = \delta_1 - \delta_2$$

$$t_2 = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}; \quad e_{t_2} = \delta_1 - \delta_3$$

$$t_3 = \begin{array}{|c|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}; \quad e_{t_3} = \delta_1 - \delta_4.$$

Clearly $\{e_{t_1}, e_{t_2}, e_{t_3}\}$ forms a basis for a three-dimensional invariant subspace of $\mathscr{F}(M_{(3,1)})$. The other invariant subspace is one dimensional, spanned by $\delta_1 + \delta_2 + \delta_3 + \delta_4$; on it we obtain the identity representation which we have already found.

The next Young diagram is (2, 2) or

$$\begin{array}{|c|c|}\hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline\end{array}.$$

Now there are six tabloids, which we label by the numbers in the second row. Thus, for example, δ_{12} is the function which is one on $\{3,4\}\{1,2\}$ but zero on other tabloids such as $\{2,4\}\{1,3\}$. To construct a new two-dimensional irreducible representation on an invariant subspace $F_{(2,2)}$ of the six-dimensional space spanned by $\delta_{12}, \delta_{13}, \delta_{14}, \delta_{23}, \delta_{24}, \delta_{34}$, we choose two tableaux:

$$t_1 = \begin{array}{|c|c|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}; \quad e_{t_1} = \delta_{12} - \delta_{23} - \delta_{14} + \delta_{34}$$

$$t_2 = \begin{array}{|c|c|c|c|}\hline 4 & 3 \\ \hline 1 & 2 \\ \hline\end{array}; \quad e_{t_2} = \delta_{12} - \delta_{24} - \delta_{13} + \delta_{34}.$$

Other tableaux would lead to linear combinations of these two basis elements: for example

$$t_3 = \begin{array}{|c|c|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}; \quad e_{t_3} = \delta_{13} - \delta_{23} - \delta_{14} + \delta_{24} = e_{t_1} - e_{t_2}$$

In the six-dimensional space $\mathscr{F}(M_{(2,2)})$ there are two other invariant subspaces, corresponding to the one-dimensional and three-dimensional representations that we have already found.

The next Young diagram is (2, 1, 1) or

Corresponding to this diagram are 12 tabloids, which we may label by the next-to-lowest and lowest rows. For example, $\delta_{1,2}$ is the function which is one on the tabloid $\{3,4\}\{1\}\{2\}$, zero on all other tabloids such as $\{3,4\}\{2\}\{1\}$ or $\{1,4\}\{2\}\{3\}$.

Again, to construct a new irreducible representation, this time a three-dimensional one, we choose three tableaux:

$$t_1 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad : \quad e_{t_1} = \delta_{1,2} - \delta_{2,1} + \delta_{3,1} - \delta_{1,3} + \delta_{2,3} - \delta_{3,2}$$

$$t_2 = \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad : \quad e_{t_2} = \delta_{1,2} - \delta_{2,1} + \delta_{4,1} - \delta_{1,4} + \delta_{2,4} - \delta_{4,2}$$

$$t_3 = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad : \quad e_{t_3} = \delta_{1,3} - \delta_{3,1} + \delta_{4,1} - \delta_{1,4} + \delta_{3,4} - \delta_{4,3}.$$

The elements e_{t_1}, e_{t_2} and e_{t_3} form a basis for a three-dimensional representation which is *not* equivalent to the one which arose from the diagram

An easy demonstration of this fact is to calculate the character of $\sigma = (34)$. In this representation

$$\sigma e_{t_1} = e_{\sigma t_1} = e_{t_2}$$
$$\sigma e_{t_2} = e_{\sigma t_2} = e_{t_1}$$
$$\sigma e_{t_3} = e_{\sigma t_3} = -e_{t_3}$$

so the character of σ is -1. In the representation from

we had

$$\sigma e_{t_1} = e_{\sigma t_1} = e_{t_1}$$
$$\sigma e_{t_2} = e_{\sigma t_2} = e_{t_3}$$
$$\sigma e_{t_3} = e_{\sigma t_3} = e_{t_2}$$

so the character of σ was $+1$.

Of course, the other representations which we constructed from earlier Young diagrams also occur on the 12-dimensional space which we are now considering: the one- and two-dimensional ones once, and the three-dimensional one twice.

The final Young diagram is

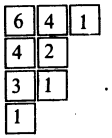
(1, 1, 1, 1) or  .

Corresponding to this diagram are 24 tabloids, and the representation on the space $\mathscr{F}_{(1,1,1,1)}$ is the regular representation. The new representation is the one-dimensional sign representation; each of the other representations also occurs, with a multiplicity equal to its dimension.

This example shows that it is useful to know the dimension of the new irreducible representation of S_n associated with a given Young diagram, so that we will know when we have found enough independent elements e_t to span the representation space. It would also be useful to be able to calculate characters without having to choose a basis explicitly. We now state without proof two useful computational recipes which achieve these goals. Proofs will be given in Appendix C.

The hook formula for dimensions

At any position in a Young diagram, we define its *hook length* to be the sum of the number of positions to its right plus the number of positions below it plus one. For example, we have placed the hook length in the corresponding position below:

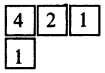


Then,

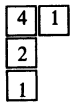
$$\dim F^{\lambda} = \frac{n!}{\prod(\text{all hook lengths in } \lambda)}.$$

For example, we may compute the dimensions of the representations of S_4 as follows (the number in each box is the hook length):

 $\dim F_{(4)} = \frac{4!}{4!} = 1$

 $\dim F_{(3,1)} = \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1} = 3$

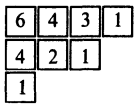
 $\dim F_{(2,2)} = \frac{4!}{3.2.2} = 2$



$$\dim F_{(2,1,1)} = \frac{4!}{4\cdot 2\cdot 1\cdot 1} = 3$$

$$\dim F_{(1,1,1,1)} = \frac{4!}{4!} = 1.$$

For S_8 the hook lengths of the entries of (4,3,1) and $\dim F_{(4,3,1)}$ are

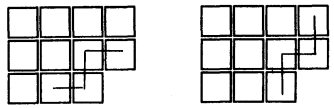


$$\dim F_{(4,3,1)} = \frac{8!}{6\cdot 4\cdot 4\cdot 3\cdot 2} = 70.$$

The Murnaghan–Nakayama rule

This is a rule for calculating a single entry in the character table of S_n; that is, for evaluating χ^λ on the conjugacy class having cycle structure corresponding to the partition $\mu = (\mu_1, \mu_2, \ldots)$, where $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_p$ and $\mu_1 + \cdots + \mu_p = n$.

A *skew hook* is a connected part of the rim of a diagram which can be removed so that the remaining boxes form a (smaller) diagram. Thus



show the only two skew 4-hooks in (4,4,3). If a skew hook starts on the ith row and ends on the jth row, then $i - j = l$ is called the leg length of the skew hook. Thus, the leg length of the skew hooks are 1 and 2, respectively.

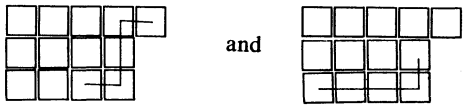
To find the value of χ^λ on $\mu = (\mu_1, \ldots, \mu_p)$, proceed as follows: draw all possible ways of removing a skew μ_1 hook from λ so as to obtain a diagram λ'. Then

$$\chi^\lambda(\mu) = \sum(-1)^l \chi^{\lambda'}(\bar{\mu})$$

where

$$\bar{\mu} = (\mu_2, \mu_3, \ldots, \mu_p),$$

and l is the leg length of the skew hook. For example, if $\lambda = (5,4,4)$ and $\mu = (5,4,3,1)$, then

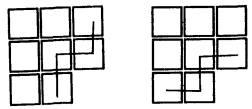


and

are the only skew 5-hooks so

$$\chi^{(5,4,4)}(5,4,3,1) = \chi^{3,3,2}(4,3,1) - \chi^{5,3}(4,3,1).$$

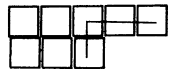
Applying the rule again we see that



are the only skew 4-hooks in (3,3,2) so

$$\chi^{3,3,2}(4,3,1) = \chi^{2,1,1}(3,1) - \chi^{3,1}(3,1)$$

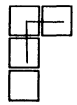
and



is the only skew 4-hook in (5,3) so

$$\chi^{5,3}(4,3,1) = \chi^{2,2}(3,1).$$

Now there is no skew 3-hook in (2,1,1) (since



does not yield an admissible diagram as it has 0 boxes in the first two rows) and similarly there is no skew 3-hook in (3,1). Therefore

$$\chi^{2,1,1}(3,1) = 0 = \chi^{3,1}(3,1).$$

As

is the only skew 3-hook in (2,2) we see that

$$\chi^{2,2}(3,1) = -\chi^1(1)$$

and

$$\chi^1(1) = 1.$$

So finally

$$\chi^{(5,4,4)}(5,4,3,1) = -1$$

by repeated application of the Murnaghan–Nakayama rule.

Since there is no skew 8-hook in (5,4,4), we conclude that $\chi^{(5,4,4)}$ vanishes on any conjugate class containing an 8-cycle.

Table 14. $n = 3$

	Conjugacy class		
	1	3	2
	(1, 1, 1)	(2, 1)	(3)
Partition			
(3)	1	1	1
(2, 1)	2	0	−1
(1, 1, 1)	1	−1	1

Table 15. $n = 4$

	Conjugacy class				
	1	6	8	3	6
	(1, 1, 1, 1)	(2, 1, 1)	(3, 1)	(2, 2)	(4)
Partition					
(4)	1	1	1	1	1
(3, 1)	3	1	0	−1	−1
(2, 2)	2	0	−1	2	0
(2, 1, 1)	3	−1	0	−1	1
(1, 1, 1, 1)	1	−1	1	1	−1

Table 16. $n = 5$

	Conjugacy class						
	1	10	20	15	30	20	24
	(1, 1, 1, 1, 1)	(2, 1, 1, 1)	(3, 1, 1)	(2, 2, 1)	(4, 1)	(3, 2)	(5)
Partition							
(5)	1	1	1	1	1	1	1
(4, 1)	4	2	1	0	0	−1	−1
(3, 2)	5	1	−1	1	−1	1	0
(3, 1, 1)	6	0	0	−2	0	0	1
(2, 2, 1)	5	−1	−1	1	1	−1	0
(2, 1, 1, 1)	4	−2	1	0	0	1	−1
(1, 1, 1, 1, 1)	1	−1	1	1	−1	−1	1

We list the character tables for S_n for low n in Tables 14–17. The reader can check any of the entries against the Murnaghan–Nakayama rule. Each irreducible representation F_λ is described by a partition λ. Each conjugacy class is described by its cycle structure. The number of elements is given above the cycle structure. The dimension of each F_λ is the character of the identity.

Table 17. $n = 6$

	Conjugacy class										
	1	15	40	45	90	120	144	15	90	40	120
	(1^6)	$(2, 1^4)$	$(3, 1^3)$	(2, 2, 1, 1)	(4, 1, 1)	(3, 2, 1)	(5, 1)	(2, 2, 2)	(4, 2)	(3, 3)	(6)
Partition											
(6)	1	1	1	1	1	1	1	1	1	1	1
(5, 1)	5	3	2	1	1	0	0	−1	−1	−1	−1
(4, 2)	9	3	0	1	−1	0	−1	3	1	0	0
(4, 1, 1)	10	2	1	−2	0	−1	0	−2	0	1	1
(3, 3)	5	1	−1	1	−1	1	0	−3	−1	2	0
(3, 2, 1)	16	0	−2	0	0	0	1	0	0	−2	0
(2, 2, 2)	5	−1	−1	1	1	−1	0	3	−1	2	0
(3, 1, 1, 1)	10	−2	1	−2	0	1	0	2	0	1	−1
(2, 2, 1, 1)	9	−3	0	1	1	0	−1	−3	1	0	0
$(2, 1^4)$	5	−3	2	1	−1	0	0	1	−1	−1	1
(1^6)	1	−1	1	1	−1	−1	1	−1	1	1	−1

3

MOLECULAR VIBRATIONS AND HOMOGENEOUS VECTOR BUNDLES

As a physical problem to guide us through the mathematics of this chapter, consider the following situation. Suppose we have a system of point masses held together by forces which keep the system fairly rigid. We may think of this system as a model of a molecule. (We say 'model' because, for the moment, we want to treat the problem as one in classical mechanics. Of course, we are interested in actual molecules, and hence should really use quantum mechanics. We shall discuss the quantum version later in this chapter.)

3.1 Small oscillations and group theory

We assume that we know the number of point masses but not the shape of the system at equilibrium. The system is in a 'black box' which we can shake at various frequencies and so determine the resonant frequencies of the system, which are the same as its frequencies of free oscillation about equilibrium. We want to show how to use this information, together with group theory, to determine the shape of the 'molecule' at equilibrium.

Let $\mathbf{q}$ be the vector which describes the deviation of the system from equilibrium by specifying the displacement of each 'atom'. Thus, if there are N 'atoms', $\mathbf{q}$ will be a vector in a $3N$-dimensional space, since each atom can move independently in three dimensions. (It is sometimes convenient to take out overall translational and rotational motion, i.e. remove six degrees of freedom so $\mathbf{q}$ varies in a $(3N-6)$-dimensional space.) From the theory of small oscillations we know that the behaviour of the system near equilibrium is described by a second-order differential equation of the form

$$\frac{d^2\mathbf{q}}{dt^2} + F\mathbf{q} = 0$$

where F is a self-adjoint operator. The eigenvalues of F determine the vibrational frequencies, and the eigenvectors determine the associated 'normal mode configuration'.

As an example, suppose that the 'molecule' consists of three identical 'atoms', each of mass m arranged in a line and joined by identical springs of force constant k (see

Fig. 3.1

Fig. 3.1). For simplicity, we assume that the atoms can be displaced only in one dimension, so that the vector

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

completely describes the deviation of the molecule from equilibrium. Then Newton's law says that

$$m\frac{d^2q_1}{dt^2} = k(q_2 - q_1)$$

$$m\frac{d^2q_2}{dt^2} = k(q_3 - q_2) - k(q_2 - q_1)$$

$$m\frac{d^2q_3}{dt^2} = -k(q_3 - q_2)$$

so that

$$\frac{d^2\mathbf{q}}{dt^2} + F\mathbf{q} = 0$$

where

$$F = \frac{k}{m}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The symmetry of the matrix F indicates that it represents a self-adjoint operator. The eigenvalues and eigenvectors of F correspond to the 'normal modes' of vibration of the system as in Table 18.

If the 'molecule' possesses a symmetry group G, then G acts on the space of displacements from equilibrium. (An exact description of this action will be presented in the next section.) As the group G acts as symmetries of the forces holding the 'molecule' together, the operator F must commute with the action of G. In the preceding example, G includes the interchange of molecules 1 and 3, and the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which interchanges the displacements q_1 and q_3, commutes with the matrix F. We are thus in the following situation. We are given a representation r of a group G on a vector space V and we are given a self-adjoint operator F on V which commutes with the

Table 18.

Eigenvalue	Eigenvector	Mode
0	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	uniform translation
$\frac{k}{m}$	$\begin{pmatrix}1\\0\\-1\end{pmatrix}$	symmetric vibration
$\frac{3k}{m}$	$\begin{pmatrix}-1\\2\\-1\end{pmatrix}$	anti-symmetric vibration

action of G. Let us first look at the extreme case where the action of G on V is irreducible. Then by Schur's lemma F must be a scalar multiple of the identity, i.e. F has just one eigenvalue. On the other hand, if we pick an operator F at 'random' on V, not requiring that it commute with G, then we would expect that F has n distinct eigenvalues, where $n = \dim V$. As usual in science, we argue backwards: if we observe only one eigenfrequency, where there 'should be' n, we conclude that there must be some reason; most likely that there is a group G which acts irreducibly on the space and commutes with F, forcing all the eigenvalues of F to coincide. More generally, let us drop the requirement that V be irreducible. Let $\mathbf{v}$ be an eigenvector of F with eigenvalue λ, so that

$$F\mathbf{v} = \lambda\mathbf{v}.$$

Then

$$Fr(a)\mathbf{v} = r(a)F\mathbf{v} = \lambda r(a)\mathbf{v}, \quad \text{for any } a \in G.$$

In other words, the space of all eigenvectors of F corresponding to the same eigenvalue λ is invariant under G. Since F is self-adjoint, we can decompose V into a direct sum of eigenspaces corresponding to the various distinct eigenvalues of F;

$$V = Z_1 \oplus \cdots \oplus Z_s.$$

Each of the eigenspaces Z_j is invariant under G. We can further decompose each Z_j into irreducibles under G. Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$$

is a decomposition of V into irreducibles, where each V_i lies in some Z_j and hence F has a constant eigenvalue on V_i. Thus F has at most p distinct eigenvalues, where p is the number of irreducible components in the decomposition. But the number of irreducible

components is independent of the particular choice of decomposition, and has nothing to do with F. It depends only on the representation.

Suppose, for the sake of illustration, that V has two possible groups acting on it, G_1 and G_2. Under the representation of G_1, V decomposes into

$$V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6,$$

six irreducible components, while under the representation of G_2

$$V = U_1 + U_2 + U_3 + U_4,$$

four irreducible components. Suppose that by observing the resonant frequencies, we see six distinct eigenvalues. Then we know that G_2 cannot be the correct group. If we observe just four frequencies, this is evidence (although not conclusive) that G_2 is the correct group. Thus molecular spectroscopy is an application of Schur's lemma!

The above description of a system in a 'box' which we shake to determine the resonant frequencies does not apply directly to actual molecules, since the laws of classical mechanics must be replaced by quantum mechanics. Instead of shaking the molecule, we shine some electromagnetic radiation on it and observe the frequencies at which the radiation is absorbed (or the displacement in the frequencies). We will explain the modification needed for the quantum theory (in particular the 'selection' rules which prevent some frequencies from being observed) in Section 3.6.

3.2 Molecular displacements and vector bundles

Consider the space of motions of (our molecular model of) carbon tetrachloride. At equilibrium the carbon atom lies at the center, and the four chlorine atoms at the vertices of a regular tetrahedron. We shall label the central carbon atom by C and the four chlorine atoms by 1, 2, 3, 4 (Fig. 3.2). In a small displacement from equilibrium, each of the atoms moves in its own three-dimensional vector space which describes the displacement of that atom from its equilibrium position. We label these spaces as

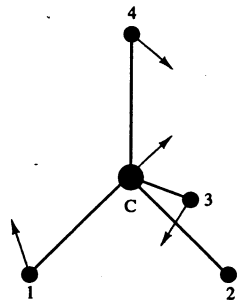


Fig. 3.2

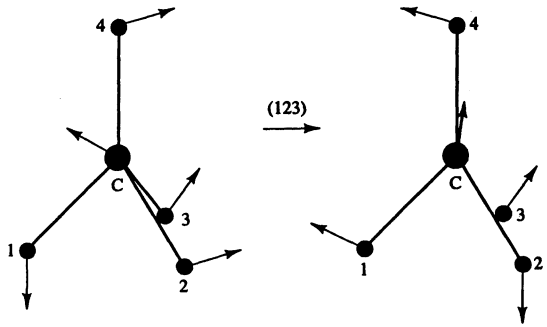


Fig. 3.3

E_C, E_1, E_2, E_3 and E_4. A displacement of the molecule as a whole moves each of the atoms, and so is a function f such that

$$f(\mathrm{C}) \in E_C \quad \text{and} \quad f(i) \in E_i, \quad i = 1, 2, 3, 4,$$

which tells how each atom has been displaced from its equilibrium.

Now we study how the group S_4 acts on the set of displacements. Consider, for example, the action of the element $(1,2,3) \in S_4$. On the molecule itself, at equilibrium, $(1,2,3)$ leaves C fixed, rotates the chlorine atoms 1, 2 and 3 and leaves 4 fixed (Fig. 3.3). It carries a small displacement of 1 (a vector in E_1) into a rotated displacement of 2 (a vector in E_2). Thus

$$\begin{aligned} &E_1 \to E_2 \\ &E_2 \to E_3 \\ (1,2,3): \quad &E_3 \to E_1 \\ &E_4 \to E_4 \\ &E_C \to E_C. \end{aligned}$$

Although the carbon atom is carried into itself, the transformation determined by $(1,2,3)$ on E_C is not the identity transformation, but a rotation. We can now begin to see how the group S_4, or the particular element $(1,2,3)$, acts on displacements of the molecule as a whole: the *new*, rotated, displacement at atom 2 depends on the old, original, displacement at atom 1. There are two things happening: the atoms are interchanged and the displacements themselves are rotated. In order to describe what is going on, it is simpler to pass to a more abstract setting, the notion of a vector bundle over a finite set.

Let M be a finite set. A *vector bundle* over M consists of a collection of vector spaces E_x, one vector space for each point x of M. We let $E = \bigcup_{x \in M} E_x$ (union, not direct sum) and sometimes talk of E as being the vector bundle. E is *not* a vector space. We define the map $\pi: E \to M$ by setting $\pi(\mathbf{v}) = x$ if $\mathbf{v} \in E_x$. That is, any $\mathbf{v} \in E$ belongs to one of the vector spaces E_x, and we let $\pi(\mathbf{v})$ be this particular x. The space $E_x = \pi^{-1}(x)$ is sometimes called the *fiber* over x, Fig. 3.4(*a*).

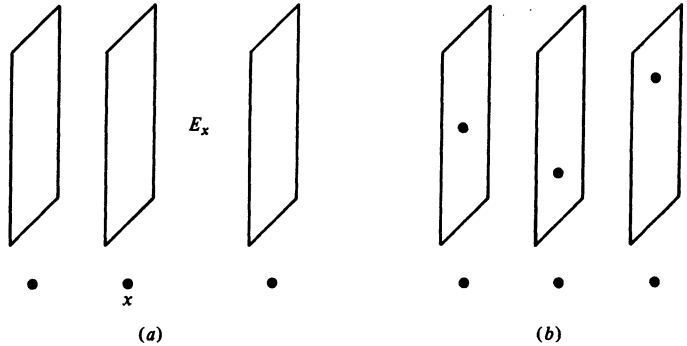


Fig. 3.4

Let E be a vector bundle over M. A *section* of E is a function f, which assigns a vector $f(x) \in E_x$ to each $x \in M$. Thus, f is a map from M to E with the property that

$$\pi \circ f = \text{identity}.$$

In the example of Section 3.1, we introduced a vector

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

to describe the complete displacement of the three-atom molecule. In keeping with the previous discussion, we should think of $\mathbf{q}$ as the *function* which assigns the displacement q_1 in space E_1 to atom 1, the displacement q_2 in the space E_2 to atom 2, and the displacement q_3 in the space E_3 to atom 3. When we draw a 'picture' of $\mathbf{q}$, e.g.

$$\begin{array}{ccc} \hookrightarrow & \hookleftarrow & \hookrightarrow \\ 1 & 2 & 3 \end{array}$$

we are representing the *function* $\mathbf{q}$ by its value on the set on which it acts.

More generally, let f_1 and f_2 be sections. Since $f_1(x)$ and $f_2(x)$ lie in the same vector space, E_x, it makes sense to add them. We can thus define the sum of two sections by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x).$$

Similarly, if c is any complex number and f is any section, we define cf by

$$(cf)(x) = cf(x).$$

In this way, *the space of all sections becomes a vector space.* We denote this vector space by $\Gamma(E)$. So $\Gamma(E)$ has the same dimension as the direct sum of the spaces E_x, but it is not to be thought of as the same space. The distinction will become crucial when we consider vector bundles over sets which are not discrete.

Suppose that the group G acts on M and that E is a vector bundle over M. We say

that G acts as a group of vector bundle morphisms on E, or that E is a *homogeneous vector bundle* under G, if

(1) G acts on E;
(2) the map $\pi: E \to M$ is a G morphism, i.e. $a\pi(v) = \pi(av)$ for all $a \in G$ and $v \in E$.

Condition (2) is the same as saying that $a: E_x \to E_{ax}$. We require also that

(3) the map $a: E_x \to E_{ax}$ is linear for each $a \in G$ and $x \in M$.

Thus, each a permutes the various vector spaces according to its action on M, and is a linear map from one vector space to another. Convince yourself that this action of a corresponds to the physical operation of performing a symmetry operation on a molecule whose atoms are displaced from equilibrium.

We can now let the group G act on $\Gamma(E)$ by setting

$$r(a)f(x) = a[f(a^{-1}x)]. \tag{2.1}$$

Notice that $f(a^{-1}x) \in E_{a^{-1}x}$ so that the right-hand side of (2.1) is an element of E_x, and thus $r(a)f$ is indeed again a section of E. It is clear that the map $f \rightsquigarrow r(a)f$ is a linear transformation, and so we obtain a linear representation of G on $\Gamma(E)$. In the case that M is the set of atoms of a molecule and E_x denotes the displacement of the atom x from its equilibrium position, $\Gamma(E)$ is the space of displacements of the molecule as a whole, and the action of G on $\Gamma(E)$ is our desired action of the symmetry group on the space of displacements.

As an example, let M be the atoms of a square planar 'molecule', and let $f(x)$, an element of $\Gamma(E)$, be the function which assigns to each atom the displacement shown in Fig. 3.5(*a*). If a is counter-clockwise rotation through 90°, then $r(a)f(x) = a[f(a^{-1}x)]$ tells us, for example, that $r(a)f(1) = af(4)$ and $r(a)f(2) = af(1)$. We obtain the displacement assigned to atom 1 by $r(a)f$ by performing a 90° counter-clockwise rotation on the displacement assigned to atom 4 by f, and so on. The net effect of this procedure is simply that Fig. 3.5(*b*) representing $r(a)f$, is obtained by rotating Fig. 3.5(*a*) representing f, counter-clockwise through 90°.

As a second example, consider the case of the 'trivial line bundle' where $E = M \times \mathbb{C}$. Over each point we have a copy of $\mathbb{C}$: the points of E_x are pairs (x, z), where z is a complex number, and $a \in G$ sends (x, z) into (ax, z). In other words, the group does not

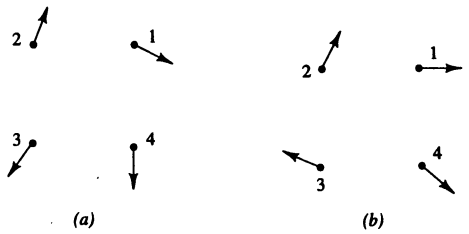


Fig. 3.5

act on the z component at all. In this case, a section f is a function of the form $f(x) = (x, g(x))$, where g is an ordinary complex-valued function on M. We may simply identify f with g, so that (2.1) becomes $r(a)f(x) = f(a^{-1}x)$. This is the representation of G on $F(M)$ that we studied in detail in Chapter 2. Our immediate task is to generalize many of the theorems we proved about the representation of G on $F(M)$ to the more general case of the representation of G on $\Gamma(E)$.

We shall denote the representation of G on $\Gamma(E)$ by r_E and its character by χ_E. In order to evaluate χ_E we introduce a convenient basis of $\Gamma(E)$, generalizing the δ function basis we used for $F(M)$. For each vector space, E_x, we introduce a basis $\mathbf{v}_{x1}, \ldots, \mathbf{v}_{xk_x}$, where $k_x = \dim E_x$. For any $\mathbf{v} \in E_x$ we define the section $f_{\mathbf{v}}$ by

$$f_{\mathbf{v}}(y) = \begin{cases} \mathbf{v} & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}.$$

It now follows, as for the case of $F(M)$, that

$$r_E(a)f_{\mathbf{v}} = f_{a\mathbf{v}}. \tag{2.2}$$

For example, if M is a triangular molecule and a is a 120° counter-clockwise rotation, typical diagrams representing $f_{\mathbf{v}}$ and $r(a)f_{\mathbf{v}}$ are as in Fig. 3.6.

As x ranges over all of M and the $\mathbf{v}_{xi}$ over a basis of each E_x, the sections $f_{\mathbf{v}_{xi}}$ form a basis of $\Gamma(E)$: they simply assign independent displacements to each 'atom' in turn. If we use this basis, it is clear from (2.2) that the only non-zero entries on the diagonal of the matrix representing $r_E(a)$ can come from those $\mathbf{v}_{xi}$ for which $ax = x$. For each x with $ax = x$, the element a maps E_x into itself, and

$$af_{\mathbf{v}_{xi}} = f_{a\mathbf{v}_{xi}}.$$

Now if (A_{xij}) is the matrix of the map $a: E_x \to E_x$ with respect to the basis $\mathbf{v}_{xi}$, then

$$a\mathbf{v}_{xj} = \sum_i A_{xij}\mathbf{v}_{xi}$$

and hence

$$af_{\mathbf{v}_{xj}} = \sum_i A_{xij} f_{\mathbf{v}_{xi}}.$$

Thus, for each fixed x the sum of the diagonal elements of the linear transformation $f \rightsquigarrow af$ coming from the $\mathbf{v}_{xi}$ is $\sum_i A_{xii}$, which is just the trace of the linear transformation $a: E_x \to E_x$. Summing over all the fixed points, x, of M, we obtain the celebrated

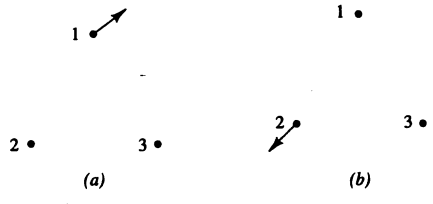


Fig. 3.6

Table 19. Character table of T_d

$24T_d$	$[e]$	$8[r_3]$	$3[r_2]$	$6[s_4]$	$6[\tau_d]$
χ_1	1	1	1	1	1
χ_2	2	−1	2	0	0
χ_3	1	1	1	−1	−1
χ_4	3	0	−1	1	−1
χ_5	3	0	−1	−1	1

Frobenius fixed point character formula:

$$\chi_E(a) = \sum_{ax=x} [\mathrm{tr}(a: E_x \to E_x)]. \tag{2.3}$$

For the case of the trivial line bundle, where $\Gamma(E) = \mathscr{F}(M)$, the map $a: E_x \to E_x$ is the identity, so the trace at each fixed point is one. Equation (2.3) then reduces to (5.1), Chapter 2.

For the case of molecular vibrations, all the summands on the right of (2.3) are the same, and (2.3) reduces to Wigner's formula:

$$\chi_E(a) = [\#\text{ of fixed points of } a] \cdot \chi^r(a). \tag{2.4}$$

By combining Wigner's formula with the use of character tables, as described in Section 7, Chapter 2, we can find the decomposition of the space of displacements of any molecule into irreducibles.

Let us illustrate this by considering the space of possible displacements of our molecule of CCl_4, which is assumed to have the symmetry of a tetrahedron, so that its symmetry group is T_d. The chlorine molecules are assumed to lie at the vertices of the tetrahedron, and the carbon atom at the center. All elements of T_d keep the carbon atom fixed and act on the chlorine atoms as T_d acts on the vertices of its tetrahedron. The character table of T_d is presented in Table 19. The character χ_5 corresponds to the action of T_d on $\mathbb{R}^3$ as the symmetries of the tetrahedron, and so this is the χ^r that we want to use in the right-hand side of Wigner's formula (2.4). The number of fixed points is 5 for the identity $[e]$, 2 for any 120° rotation $[r_3]$, 1 for any 180° rotation $[r_2]$, which corresponds to a permutation like (12) (34) of the tetrahedron but holds the central carbon atom fixed, 1 for the elements of class $[s_4]$, which correspond to permutations like (1234) which hold only the carbon atom fixed, and 3 for any reflection $[\tau_d]$, which interchanges two chlorine atoms while holding the other two fixed, and also holds the carbon atom fixed. Using (2.4), we find that the character χ_E is represented by the row

$$15 \quad 0 \quad -1 \quad -1 \quad 3.$$

(The last entry, for example, arises because a reflection through a plane containing an edge leaves two vertices of the tetrahedron fixed. Together with the central atom, this gives three fixed points. Multiplying by the value of χ_5 gives 3.) We now find how many

times each irreducible representation occurs in the decomposition of the representation of $\Gamma(E)$.

$$(\chi_E, \chi_1) = (1/24)(1\cdot 15 + 0 + 3\cdot(-1) + 6\cdot(-1) + 6\cdot 3) = 1$$
$$(\chi_E, \chi_2) = (1/24)(2\cdot 15 + 0 + 3\cdot 2\cdot(-1) + 0 + 0) = 1$$
$$(\chi_E, \chi_3) = (1/24)(15 - 3 + 6 - 18) = 0$$
$$(\chi_E, \chi_4) = (1/24)(45 + 3 - 6 - 18) = 1$$
$$(\chi_E, \chi_5) = (1/24)(45 + 3 + 6 + 18) = 3$$

so that

$$\chi_E = \chi_1 + \chi_2 + \chi_4 + 3\chi_5,$$

which can be checked by adding up the columns with the appropriate multiplicity.

It is informative to visualize the subspaces corresponding to this decomposition as actual subspaces of the space of displacements of the molecule. Since χ_1, χ_2 and χ_4 each occur with multiplicity one, each corresponds to a unique subspace. For χ_5, there is no unique way to pick out any three-dimensional subspace with character χ_5, although, as we know, the nine-dimensional space consisting of the direct sum of any three independent χ_5-type three-dimensional subspaces *is* invariantly defined. The representation with character χ_1 is the trivial representation. If we do not move the carbon atom at all, and move the chlorine atoms out along the axes joining them to the carbon atom, moving each chlorine atom the same amount, we get a configuration that is clearly invariant under T_d, and there is a one-dimensional space of such possible motions, cf. Fig. 3.7.

The space of displacements in which the carbon atom is fixed and the chlorine atoms move along the axes is an invariant four-dimensional space which contains the above space as a one-dimensional invariant subspace. It therefore contains as an invariant complement, a three-dimensional subspace.

The space of displacements in which the carbon atom is fixed and the chlorine atoms are displaced perpendicularly to the axes forms an eight-dimensional invariant subspace, which must therefore split into a two, a three and a three. One of these three-

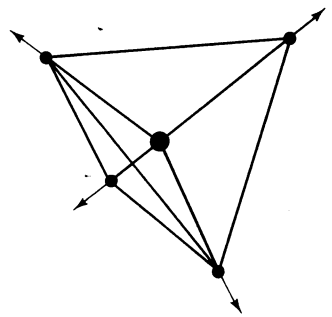
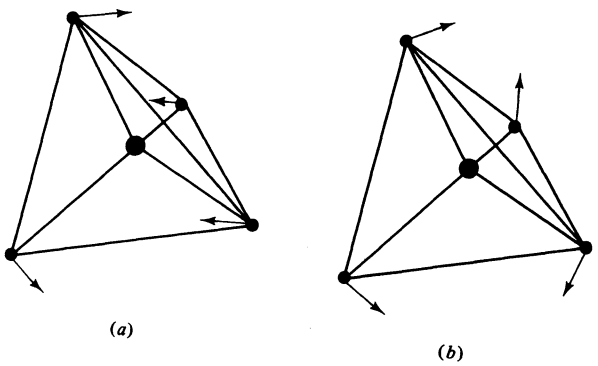


Fig. 3.7



(a) (b)

Fig. 3.8

dimensional subspaces can be identified as being given by an infinitesimal rotation of the molecule as a whole. It is an instructive exercise for the reader to verify that the group T_d acting on the space of infinitesimal rotations has character χ_4. (The space of infinitesimal rotations can be identified with the space of antisymmetric 3×3 matrices. These can be identified with vectors in three-space only if a choice of orientation is made. Reversing the orientation changes the identification by a factor of -1. That is why the character χ_4 differs from χ_5 by a factor of -1 precisely on the orientation reversing elements.) Thus the character of this eight-dimensional representation is $\chi_2 + \chi_4 + \chi_5$.

We draw two linearly independent displacements in the two-dimensional invariant subspace in Figs. 3.8(*a*) and (*b*).

The rigid translation of the molecule as a whole, where we draw the same vector in three-space at each molecule, clearly gives a three-dimensional invariant subspace with character χ_5. Actually, for physical reasons, we will want to separate the translations and the rotations of the molecule from the other displacements. This leaves us with a nine-dimensional space of vibrations, with character

$$\chi_1 + \chi_2 + 2\chi_5$$

There is no preferred way of picking the two three-dimensional vibrating subspaces.

3.3 Induced representations

Let $E \to M$ be a homogeneous vector bundle for the group G and let N be a subset of M. We can form $\pi^{-1}N = \bigcup_{x \in N} E_x$, which is now a vector bundle over N, which we denote by E_N. If N is mapped onto itself by all elements of G, i.e. is stable under G, then E_N is clearly a homogeneous vector bundle for G. We can identify $\Gamma(E_N)$ with a subspace of $\Gamma(E)$, namely the subspace consisting of all sections which vanish

outside of N, i.e those f for which $f(y) = 0$ for $y \notin N$. If $M = \cup M_i$ is a disjoint union, then we have the direct sum decomposition

$$\Gamma(E) = \Gamma(E_{M_1}) \oplus \cdots \oplus \Gamma(E_{M_k}).$$

If the M_i are invariant, in particular if they are orbits, we obtain the Mackey decomposition formula:

$$r_E = r_{E_{M_1}} \oplus \cdots \oplus r_{E_{M_k}} \tag{3.1}_r$$

and the corresponding character decomposition

$$\chi_E = \chi_{E_{M_1}} + \cdots + \chi_{E_{M_k}}. \tag{3.1}_\chi$$

So we might as well look at the case where G acts transitively on M. Let m be a point of M and $H = G_m$, the isotropy group of m. Each element of H maps E_m into E_m and acts linearly. In other words, we get a representation of H on E_m. Let us denote this representation by s. Let n be some other point of M. Since G acts transitively, we can write

$$n = am$$

for some $a \in G$, and if $n = bm$, then $b = ah$ for some $h \in H$. Any vector in E_n is the image of some vector in E_m under the action of a. In other words, we can write any vector in E_n as

$$\mathbf{w} = a\mathbf{u} \quad \mathbf{u} \in E_m.$$

If also

$$\mathbf{w} = b\mathbf{v}$$

then $b\mathbf{v} = a\mathbf{u}$ or $a^{-1}b\mathbf{v} = \mathbf{u}$ or

$$\mathbf{u} = s(h)\mathbf{v}.$$

This shows that the representation s determines the vector bundle E. We now show how to construct a homogeneous vector bundle from a representation of a subgroup.

So suppose we start with G and a subgroup H. We can reconstruct M as the coset space G/H: a point of $M = G/H$ is a coset aH. Suppose that s is a representation of the subgroup H on a vector space V. On the space $G \times V$ we introduce the equivalence relation

$$(gh, \mathbf{v}) \sim (g, s(h)\mathbf{v})$$

and let E denote the set of all equivalence classes. We denote the equivalence class of $(g, \mathbf{v}) \in G \times V$ by $[(g, \mathbf{v})]$. We write

$$E = G \times_H V.$$

The map sending $(g, \mathbf{v}) \rightsquigarrow gH \in M$ is clearly constant on equivalence classes, and hence defines a map from $E \to M$. Suppose that $x = gH$. Let E_x consist of all equivalence classes of elements $(g, \mathbf{v})$ as $\mathbf{v}$ ranges over V. We have an identification of the set E_x with the vector space V, which depends on the choice of g:

$$\phi_g : V \to E_x, \quad \phi_g(\mathbf{v}) = [(g, \mathbf{v})].$$

If we change our choice of g by replacing g by gh, we get

$$\phi_{gh}(\mathbf{v}) = [(gh, \mathbf{v})] = [(g, s(h)\mathbf{v})] = \phi_g(s(h)\mathbf{v})$$

so that

$$\phi_{gh} = \phi_g \circ s(h). \tag{3.2}$$

This equation shows that we may use the map ϕ_g to define a vector space structure on E_x which is independent of the particular choice of g. If $\mathbf{e}_i = \phi_g(\mathbf{v}_i)$, we define

$$\mathbf{e}_1 + \mathbf{e}_2 = \phi_g(\mathbf{v}_1 + \mathbf{v}_2).$$

In view of (3.2) and the fact that $s(h)$ acts linearly on V, the value of this sum does not depend on the choice of g, and similarly for the definition of multiplication of a vector by a scalar.

We have thus made E into a vector bundle over M. We define the action of the group G on this vector bundle by multiplication on the left:

$$a[(g, \mathbf{v})] = [(ag, \mathbf{v})].$$

It is obvious that this definition is independent of the choice of representative. It defines a linear map from $E_x \to E_{ax}$ by the very definition of the linear structure on these fibers. Thus E is a homogeneous vector bundle for G, and we obtain a representation of G on $\Gamma(E)$. This representation was obtained from the subgroup H and the representation s of H. It is called the representation of G *induced* from the representation s, and we will denote it by $(s \uparrow G)$.

Before proceeding with the general theory of induced representations, let us illustrate the concept with a specific example. Let $G = D_k$ be the dihedral group of order $2k$, and let $H = C_k$ be our choice of subgroup. Thus C_k is of index two in D_k, so $M = G/H = D_k/C_k$ consists of two points. If we choose some $g \notin H$, then the two points can be denoted by

$$x = H \quad \text{and} \quad y = gH.$$

The action of G on M is given by

$$\begin{aligned} hx &= x \quad & hy &= y \quad & h &\in H \\ (gh)x &= y \quad & (gh)y &= x \quad & h &\in H, \quad g \notin H. \end{aligned}$$

We can write $H = \{e, a, a^2, \ldots, a^{k-1}\}$, where a is the rotation through angle $2\pi/k$. Let us choose a one-dimensional representation, s, of H, so that

$$s(a) = \varepsilon,$$

where ε is a kth root of unity, $\varepsilon^k = 1$, and so

$$s(a^j) = \varepsilon^j.$$

Thus E consists of two complex lines, one sitting over each point of M, as shown in Fig. 3.9. Choosing a $\mathbf{v} \in \mathbb{C}$ and a $g \notin H$ then picks out a basis, $\mathbf{u} = [(e, \mathbf{v})]$ of E_x and $\mathbf{w} = [(g, \mathbf{v})]$ of E_y. We thus get a basis, $f_{\mathbf{u}}$, $f_{\mathbf{w}}$ of $\Gamma(E)$, where

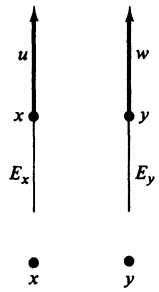


Fig. 3.9

$$f_{\mathbf{u}}(x) = \mathbf{u}, \quad f_{\mathbf{u}}(y) = 0$$

and

$$f_{\mathbf{w}}(x) = 0, \quad f_{\mathbf{w}}(y) = \mathbf{w}.$$

To compute further, we observe that the multiplication in G gives

$$ga^i g^{-1} = a^{-i} \quad g \notin H.$$

Now

$$\begin{aligned} a^i\mathbf{u} = a^i[(e, \mathbf{v})] &= [(a^i, \mathbf{v})] = [(e, s(a^i)\mathbf{v})] = [(e, \varepsilon^i\mathbf{v})] \\ &= \varepsilon^i\mathbf{u} \end{aligned}$$

so

$$a^i f_{\mathbf{u}} = \varepsilon^i f_{\mathbf{u}}$$

while

$$\begin{aligned} a^i\mathbf{w} = a^i[(g, \mathbf{v})] &= [(a^i g, \mathbf{v})] = [(ga^{-i}, \mathbf{v})] \\ &= [(g, s(a^{-i})\mathbf{v})] = [(g, \varepsilon^{-i}\mathbf{v})] \\ &= \varepsilon^{-i}\mathbf{w}. \end{aligned}$$

so

$$a^i f_{\mathbf{w}} = \varepsilon^{-i} f_{\mathbf{w}}.$$

Notice that a^i fixes both points x and y. Thus the matrix of a^i on $\Gamma(E)$ with respect to the basis $f_{\mathbf{u}}$, $f_{\mathbf{w}}$ is

$$\begin{pmatrix} \varepsilon^i & 0 \\ 0 & \varepsilon^{-i} \end{pmatrix}.$$

For the element ga^i, we have $(ga^i)x = y$ and $(ga^i)y = x$ and

$$\begin{aligned} ga^i\mathbf{u} = ga^i[(e, \mathbf{v})] &= [(ga^i, \mathbf{v})] = [(g, s(a^i)\mathbf{v})] = [(g, \varepsilon^i\mathbf{v})] \\ &= \varepsilon^i\mathbf{w} \end{aligned}$$

and

$$\begin{aligned} ga^i\mathbf{w} = ga^i[(g, \mathbf{v})] &= [(ga^i g, \mathbf{v})] = [(a^{-i}, \mathbf{v})] = [(e, s(a^{-i})\mathbf{v})] \\ &= [(e, \varepsilon^{-i}\mathbf{v})] = \varepsilon^{-i}\mathbf{u} \end{aligned}$$

and the matrix of ga^i is

$$\begin{pmatrix} 0 & \varepsilon^{-i} \\ \varepsilon^i & 0 \end{pmatrix}.$$

Since every element of G is either of the form a^i or ga^i, we have completely determined the matrices of the induced representation $s\uparrow G$:

$$a^i \rightsquigarrow \begin{pmatrix} \varepsilon^i & 0 \\ 0 & \varepsilon^{-i} \end{pmatrix} \quad ga^i \rightsquigarrow \begin{pmatrix} 0 & \varepsilon^{-i} \\ \varepsilon^i & 0 \end{pmatrix}.$$

The character of this representation is clearly given by

$$\chi(a^i) = \varepsilon^i + \varepsilon^{-i} \quad \text{and} \quad \chi(ga^i) = 0,$$

in agreement with (2.3). Notice that this representation is irreducible, unless $\varepsilon = \pm 1$, for if $\varepsilon \neq \pm 1$, the only subspaces invariant under a are the lines spanned by f_u or f_w and these are interchanged by g. For $\varepsilon = \pm 1$, $s\uparrow G$ is reducible.

We return to the general theory of induced representations. Let s be a representation of the subgroup H of the group G, and let σ be its character. We shall denote the character of the induced representation, $s\uparrow G$, by $\sigma\uparrow G$. The formula for $\sigma\uparrow G$ is given by (2.3). It is sometimes convenient to rewrite (2.3) in a slightly different form. The sum in (2.3) is over fixed points. To say that $x = gH$ is fixed under $a \in G$ means that $g^{-1}ag \in H$. Of course the element g is only determined by x up to right multiplication by an arbitrary element of H. The action of a on E_x can be described as follows: if $\mathbf{u} \in E_x$ is given by $\mathbf{u} = \phi_g \mathbf{v}$ for $\mathbf{v} \in V$, then

$$a\mathbf{u} = \phi_g(s_{g^{-1}ag})\phi_{g^{-1}}\mathbf{u}$$

so that

$$\operatorname{tr}[a: E_x \to E_x] = \operatorname{tr} s_{g^{-1}ag} = \sigma(g^{-1}ag).$$

The expression on the right makes sense because $g^{-1}ag \in H$, and does not depend on which g we pick with $x = gH$. For purposes of counting, it is sometimes easier to sum over *all* g with $g^{-1}ag \in H$, instead of summing over the fixed points, but then we will have counted each fixed point $\#H$ times. We divide by $\#H$ to compensate for the overcounting, and (2.3) becomes

$$\sigma\uparrow G(a) = (1/\#H) \sum_{\substack{g \in G \\ g^{-1}ag \in H}} \sigma(g^{-1}ag). \tag{3.3}$$

Let χ be a character of the group G, and let $\chi|_H$ denote the restriction of χ to H. Let us compute $(\sigma\uparrow G, \chi)_G$. We have

$$(\sigma\uparrow G, \chi)_G = \frac{1}{\#G}\sum (\sigma\uparrow G)(a)\overline{\chi(a)}$$

$$= \frac{1}{\#G}\frac{1}{\#H} \sum_{\substack{a \in G \\ h = b^{-1}ab \in H}} \sigma(b^{-1}ab)\overline{\chi(a)} \text{ by (3.3)}$$

$$= \frac{1}{\#G}\frac{1}{\#H} \sum_{\substack{h \in H \\ b \in G}} \sigma(h)\overline{\chi(bhb^{-1})}$$

$$= \frac{1}{\#H} \sum_{h \in H} \sigma(h)\overline{\chi(h)}$$

since $\chi(bhb^{-1}) = \chi(h)$

$$= (\sigma, \chi_H)_H.$$

Thus we have proved the *Frobenius reciprocity formula:*

$$(\sigma\uparrow G, \chi)_G = (\sigma, \chi|_H)_H. \tag{3.4}$$

If χ is the character of a representation of G on a vector space W, the left-hand side of (3.4) is just $\dim \operatorname{Hom}_G(W, \Gamma(E))$. If σ is the character of a representation of H on a vector space F (so the fiber of E over H is F), the right-hand side of (3.4) is just $\dim \operatorname{Hom}_H(W, F)$. Thus we can rewrite (3.4) as

$$\dim \operatorname{Hom}_G(W, \Gamma(E)) = \dim \operatorname{Hom}_H(W, F). \tag{3.5}$$

In fact, we can say more: that there is a natural identification of the two vector spaces $\operatorname{Hom}_G(W, \Gamma(E))$ and $\operatorname{Hom}_H(W, F)$. For this purpose it is convenient to have a different description of the space $\Gamma(E)$ and the action of G on it.

Let $x \in M = G/H$. Recall that for each $b \in G$ such that $x = bH$, we have identified E_x with the set of all $(b, \mathbf{v})$, where $\mathbf{v} \in F$. If $f \in \Gamma(E)$, then $f(x) \in E_x$, so we can write

$$f(x) = [(b, \hat{f}(b))],$$

where $\hat{f}(b) \in F$. If $c \in H$, then $f(x) = f(bH) = f(bcH)$, and

$$f(bcH) = [(bc, \hat{f}(bc))].$$

Thus

$$\hat{f}(bc) = s(c)^{-1}\hat{f}(b) \quad \text{for } c \in H. \tag{3.6}$$

Conversely, any function $\hat{f}: G \to F$ satisfying (3.6) defines a section of $\Gamma(E)$. Thus *we may identify* $\Gamma(E)$ *with the space of all functions from* G *to* F *satisfying* (3.6). Let us denote this space by $\hat{\Gamma}$. We now compare the action of G on both spaces. On $\Gamma(E)$, the representation r_E is given by

$$(r_E(a)f)(x) = af(a^{-1}x)$$

while on the space of functions the representation $\hat{r}$ is given by

$$\hat{r}(a)\hat{f}(g) = \hat{f}(a^{-1}g).$$

If $x = bH$, then

$$f(x) = [(b. \hat{f}(b))]$$

and

$$(r_E(a)f)(x) = a[(a^{-1}b, \hat{f}(a^{-1}b))]$$
$$= [(b, \hat{f}(a^{-1}b))]$$

so the function corresponding to $r_E(a)f$ is $\hat{r}(a)\hat{f}$ as required.

We can thus think of the induced representation either way. The section point of view is geometrical and convenient for certain computations. The function point of view is useful for certain proofs.

The Frobenius reciprocity theorem

Let u be an element of F. Define $c_u \in \hat{\Gamma}$ by

$$c_u(a) = \begin{matrix} 0 & \text{if} \quad a \notin H \\ s(a^{-1})u & \text{if} \quad a \in H \end{matrix}.$$

The map sending u to c_u is a map from F to $\hat{\Gamma}$ and, in fact, $c \in \mathrm{Hom}_H(F, \hat{\Gamma})$ since

$$c_{s(h)u}(a) = \begin{matrix} 0 & \text{if} \quad a \notin H \\ s(a^{-1}h)u & \text{if} \quad a \in H \end{matrix} = c_u(h^{-1}a) = (\hat{t}(h)c_u)(a).$$

Now let (t, W) be any representation of G. Then, by restriction, we can think of W as a representation space of H. Let $S: F \to W$ be an element of $\mathrm{Hom}_H(F, W)$. Define

$$T_S: \hat{\Gamma} \to W$$

by

$$T_S \hat{f} = \frac{1}{\#H} \sum_{a \in G} t(a) S \hat{f}(a)$$

Then

$$\begin{aligned} t(b) T_S \hat{f} &= \frac{1}{\#H} \sum_{a \in G} t(ba) S \hat{f}(a) \\ &= \frac{1}{\#H} \sum_{c \in G} t(c) S \hat{f}(b^{-1}c) \qquad a = b^{-1}c \\ &= T_S \hat{t}(b) \hat{f} \end{aligned}$$

so

$$T_S \in \mathrm{Hom}_G(\hat{\Gamma}, W).$$

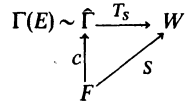
Also

$$\begin{aligned} T_S \cdot c_u &= \frac{1}{\#H} \sum_{a \in H} t(a) S s^{-1}(a) u \\ &= \frac{1}{\#H} \sum_{a \in H} S s(a) s^{-1}(a) u = Su. \end{aligned}$$

Given $S \in \mathrm{Hom}_H(F, W)$, we thus get a $T_S \in \mathrm{Hom}_G(\hat{\Gamma}, W)$ with

$$T_S \circ c = S.$$

This gives an injection of $\mathrm{Hom}_H(F, W)$ into $\mathrm{Hom}_G(\hat{\Gamma}, W)$ which is an isomorphism, since we already know that the dimensions are the same.



This is the Frobenius reciprocity theorem:

$$\mathrm{Hom}_H(F, W) \sim \mathrm{Hom}_G(\Gamma(E), W). \tag{3.7}$$

The Frobenius reciprocity theorem says that, in a sense, the operations of restriction and induction are reciprocal to one another. (They are adjoint, in the language of category theory.) We can formulate (3.7) as, say, that the induced representation can be viewed as the solution of a 'universal problem': given an H morphism, S, from F to the G representation space, W, there is a unique G morphism, T_S, from $\Gamma(E)$ to W such that $S = T_S \circ c$, where c is the map of $F \to \hat{\Gamma}(E)$ defined above.

For many purposes it is convenient to be able to replace the space $\Gamma(E)$ by a space of functions from M to a fixed vector space V, i.e. to replace the vector bundle E by a 'trivial bundle' $M \times V$. Let us describe one situation where this is always possible. Suppose, as before, that G acts transitively on M, that H is the isotropy group of some point $m \in M$ and that E is induced from some representation (ρ, V) of H. Thus, a typical point of E is an equivalence class

$$[(a, v)]$$

with

$$(ah^{-1}, \rho(h)v) \sim (a, v)$$

$a \in G$, $h \in H$, $v \in V$. Now suppose that

there is a representation, τ, of G on V such that $\tau(h) = \rho(h)$ for all $h \in H$.

Then let us define the map $\phi: E \to M \times V$ by

$$\phi[(a, v)] = (am, \tau(a)v).$$

This is well defined since

$$ah^{-1}m = am$$

and

$$\begin{aligned} \tau(ah^{-1})\rho(h)v &= \tau(a)\tau(h^{-1})\rho(h)v \\ &= \tau(a)\rho(h^{-1})\rho(h)v \\ &= \tau(a)v \end{aligned}$$

It is clear that ϕ is an isomorphism of vector bundles. Let us define the action of G on $M \times V$ by

$$b(m, w) = (bm, \tau(b)w).$$

Then

$$\phi(b[(a, v)]) = \phi([(ba, v)])$$

$$= (bam, \tau(ba)v)$$
$$= b(am, \tau(a)v).$$

Thus

$$b\phi = \phi b,$$

i.e. ϕ is a G morphism.

At the level of sections, our identification sends the function $\hat{f}$ in $\hat{\Gamma}$ into a V-valued function ψf on M by

$$(\psi \hat{f})(m) = \tau(a)\hat{f}(a) \quad \text{if} \quad m = aH.$$

Then

$$(\psi \hat{r}(b)\hat{f})(m) = \tau(a)\hat{f}(b^{-1}a) = \tau(b)\tau(b^{-1}a)\hat{f}(b^{-1}a) = \tau(b)(\psi \hat{f})(b^{-1}m).$$

If we think of $(\psi \hat{f})$ as defining a section of the bundle $M \times V$, this last expression on the right is exactly the action of b on $\psi \hat{f}$, evaluated at the point m. Thus ψ defines a G morphism between $\Gamma(E) \sim \hat{\Gamma}$ and $\Gamma(M \times V)$.

To summarize:

suppose that the representation (ρ, V) of H extends to a representation, τ, of G on V. Then there is a G isomorphism ψ between (3.8)

$$\Gamma(E) \quad \textit{and} \quad \mathscr{F}(M, V)$$

where $\mathscr{F}(M, V)$ denotes the space of functions from M to V. The morphism is defined as follows: if an $s \in \Gamma$ is given by

$$s(m) = [(a, u(a))], \quad \text{where } m = am_0$$

then

$$(\psi s)(m) = \tau(a)u(a). \tag{3.9}$$

Here the group G acts on $\mathscr{F}(M, V)$ by

$$(bh)(m) = \tau(b)h(b^{-1}m), \quad h \in \mathscr{F}(M, V)$$

3.4 Principal bundles

Let us go back and examine once more the construction of the vector bundle E from the representation of H on V. In the construction, we used the action of H on G by right multiplication, in order to define the action of H on $G \times V$ which gave our equivalence relation. But we did not use the full fact that G was a group until we wanted to define the action of G on E. It is best to separate these two phenomena, and discuss the construction of E in a more general setting. For this we postulate the necessary properties of the right action that we need. Assume that we are given a set P together with a right action of H on P, i.e. a map

$$P \times H \to P, \quad (p, h) \to ph$$

with

$$p(h_1 h_2) = (ph_1)h_2.$$

We further assume that the action is *free* in the sense that

$$ph_1 = ph_2 \text{ for some } p \in P \text{ implies } h_1 = h_2.$$

The set P is called a principal H bundle. It is a bundle over $M = P/H$. In other words, let M be the set of orbits of the right action of H on P. We let π denote the projection of $P \to M$ which assigns to each point its orbit. Given an $x \in M$ and two points p_1 and $p_2 \in \pi^{-1}(x)$, there is a unique $h \in H$ with $p_1 = p_2 h$. Conversely, given $p_1 \in \pi^{-1}(x)$ and $h \in H$, the element $p_2 = p_1 h$ again belongs to $\pi^{-1}(x)$. Thus each $\pi^{-1}(x)$ looks like a copy of H but with no preferred identity element. It is much like the relation between an affine space and a vector space. We say that *P is a principal bundle over M, with structure group H*. Now suppose that we are given some representation, σ, of H on a vector space, V. As before, we can define the (left) action of H on $P \times V$ by

$$h(p, v) = (ph^{-1}, \sigma(h)v).$$

As before, the quotient space

$$E = (P \times V)/H$$

is a vector bundle over M. It is called the vector bundle *associated* to the representation σ, or, more succinctly, an associated bundle.

We will now show that there is a natural identification between sections of E and certain kinds of functions from P to V. Let s be a section of E. Define the function f_s: $P \to V$ by

$$s(\pi(p)) = [(p, f_s(p))].$$

In other words, what we are saying is as follows. For each $x \in M$ the element $s(x)$ lies in E_x and can be represented as the equivalence class $s(x) = [(p, w)]$ for some w in V. The choice of w depends on which p we pick in $\pi^{-1}(x)$, but having picked a p the w is determined, and we define the function $f_s(p)$ by setting $f_s(p) = w$. If we chose a different element of $\pi^{-1}(x)$, say the element ph for some h in H, then we would have to replace w by $\sigma^{-1}(h)w$. In other words, the function f_s satisfies the identity

$$f_s(ph) = \sigma(h^{-1})f_s(p). \tag{4.1}$$

Conversely, suppose f is a function satisfying this identity. Then define the section s of E by setting

$$s(x) = [(p, f(p))] \text{ for any } p \text{ with } \pi(p) = x.$$

The identity satisfied by f guarantees that this is well defined.

In particular, taking the case $P = G$, we see that the space $\Gamma(E)$ can be identified with the subspace of the space of V valued functions, namely those functions satisfying (4.1), on G. This was the construction we used in the preceding section.

Before going on to the next topic, we mention some definitions connected to

principal bundles. Although we will not make any immediate use of these definitions, they play a key role in the present-day theory of elementary particles. Let $P \to M$ be a principal bundle with structure group H. A map $\phi: P \to P$ is called an *automorphism* of P if it is one-to-one and surjective and commutes with the action of H, i.e.

$$\phi(ph) = \phi(p)h \quad \text{for all} \quad h \in H \quad \text{and} \quad p \in P.$$

In other words, an automorphism of P is an H morphism which is bijective. If ϕ is such an automorphism, then $\pi(\phi(p))$ clearly depends only on $\pi(p)$. Thus ϕ determines a one-to-one transformation of M onto itself, which we denote by $\bar{\phi}$. The group of all automorphisms of P will be denoted by Aut P. We have shown that there is a homomorphism from Aut P to $S(M)$, the group of all one-to-one transformations of M, the homomorphism sends ϕ into $\bar{\phi}$. The kernel of this homomorphism is the group of those ϕ which induce the identity transformation on M, i.e. those $\phi \in$ Aut P which satisfy

$$\pi \circ \phi = \pi.$$

This subgroup of Aut P is called the *gauge group* of P and is denoted by Gau P.

It is interesting to observe that the elements of Gau P can be identified with sections of an associated bundle. Let H act on itself by conjugation, and consider the corresponding associated bundle, $H(P)$. Since conjugation preserves group multiplication, each fiber of $H(P)$ is a group – it makes sense to multiply two elements of $H(P)$ which lie over the same point of M. For each x, the group sitting over x is isomorphic to H, but there is no preferred isomorphism. The space of sections $\Gamma(H(P))$ becomes a group under pointwise multiplication (just as the space of sections of a vector bundle became a vector space). If we think of a section as a function from P to H, the product of two sections is given as

$$(f_1 f_2)(p) = f_1(p) f_2(p).$$

We claim that each section $f \in \Gamma(H(P))$ defines an element $\phi = \phi_f$ of Gau P by the formula

$$\phi(p) = pf(p).$$

This certainly defines ϕ as a transformation of P with $\pi \circ \phi = \pi$. To check that ϕ commutes with the action of H, observe that

$$\begin{aligned} \phi(ph) &= (ph) f(ph) \\ &= (ph)\sigma(h^{-1}) f(p) \\ &= (ph) h^{-1} f(p) h = pf(p)h = \phi(p)h. \end{aligned}$$

It is clear that $\phi_{f_1 f_2} = \phi_{f_1} \circ \phi_{f_2}$ and that $\phi_{f^{-1}} = (\phi_f)^{-1}$ so that we have a homomorphism from the group $\Gamma(H(P))$ to Gau P. Conversely, given $\phi \in$ Gau P and $p \in P$, we know that $\phi(p)$ projects onto the same point as p, and hence we can write $\phi(p) = ph$ for some $h \in H$. This h depends on p, so we write it as $f(p)$. The preceding equations can now be read backwards to conclude that $f(ph) = h^{-1} f(p) h$, i.e. that $f \in \Gamma(H(P))$. Thus the mapping between $\Gamma(H(P))$ and Gau P is an isomorphism.

3.5 Tensor products

Let (ρ_1, V_1) and (ρ_2, V_2) be representations of the same group G. We can construct a representation $\rho_1 \otimes \rho_2$ on the space $V_1 \otimes V_2$ by defining*

$$(\rho_1 \otimes \rho_2)(a) = \rho_1(a) \otimes \rho_2(a).$$

It is easy to check that this is indeed a representation of G, which is called the tensor product of the two original representations. Let χ_1 and χ_2 be the characters of ρ_1 and ρ_2. Since

$$\operatorname{tr}(A \otimes B) = \operatorname{tr} A \cdot \operatorname{tr} B$$

for any linear transformations A and B on V_1 and V_2, we conclude that the character of $\rho_1 \otimes \rho_2$ is simply the product function $\chi_1 \chi_2$. In many applications, it is necessary to decompose $V_1 \otimes V_2$ into irreducibles under G. Using the character tables, we have the tools at our disposal to solve this problem. For example, suppose we consider the group $S_4 = T_d$, whose character table is given in Table 9. Suppose we take $V_1 = V_2 = \mathbb{R}^3$ with the representation given by the standard action of T_d, whose character is χ_5, corresponding to the last row of the table. The character of the representation on $V_1 \otimes V_2 = \mathbb{R}^3 \otimes \mathbb{R}^3$ is then χ_5^2, which take on the values $(9, 0, 1, 1, 1)$ (in the order of conjugacy classes listed in Table 9). A direct computation then shows that

$$(\chi_5^2, \chi_1) = (1/24)(1 \cdot 9 \cdot 1 + 8 \cdot 0 \cdot 1 + 3 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot 1) = 1$$

and similarly,

$$\begin{aligned} (\chi_5^2, \chi_2) &= 1 \\ (\chi_5^2, \chi_3) &= 0 \\ (\chi_5^2, \chi_4) &= 1 \end{aligned}$$

and

$$(\chi_5^2, \chi_5) = 1.$$

So

$$\chi_5^2 = \chi_1 + \chi_2 + \chi_4 + \chi_5,$$

and this yields the corresponding decomposition of the space $\mathbb{R}^3 \otimes \mathbb{R}^3$.

Suppose that we are given a bilinear map $f: V_1 \times V_2 \to W$ which commutes with the action of G in that

$$f(\rho_1(a)v_1, \rho_2(a)v_2) = \rho(a) f(v_1, v_2)$$

where ρ is a given representation of G on W. Any such bilinear map corresponds to a

* The relation with the definition of tensor product given in Section 2.6 is the following. There we defined the 'external' tensor product of two representations of two groups. Applying that definition here would give us a representation of $G \times G$. Restricting to the 'diagonal' subgroup $\{(a, a)\}$ which is isomorphic to G gives the tensor product here.

linear map $F: V_1 \otimes V_2 \to W$ with

$$f(v_1, v_2) = F(v_1 \otimes v_2).$$

The map F is a G morphism. Suppose that f, and hence F, is surjective. This means that any *irreducible component of W must occur as an irreducible component of $V_1 \otimes V_2$* and its multiplicity in W is at most equal to its multiplicity in $V_1 \otimes V_2$. We shall have occasion to apply this remark in the next section where we discuss some quantum mechanical applications of group theory. Roughly speaking, the situation that we will encounter there will be where V_1 is some G invariant subspace of $\mathrm{Hom}(V, V)$ with $V = V_2$. The bilinear map will be the map which sends (A, v) into Av and W is the linear span of such Av.

3.6 Representative operators and quantum mechanical selection rules

In this section we present, in somewhat sketchy form, one of the principal tools used in the application of group theory to quantum mechanics. We begin with a description of the structure of quantum mechanics. A quantum mechanical system consists of a separable complex Hilbert space, $\mathscr{H}$. A *pure state* of the system consists of a one-dimensional subspace. Usually a unit vector, ϕ, is chosen in this subspace. We let P_ϕ denote the projection onto the subspace spanned by ϕ. A *physical observable* is represented by a self-adjoint operator, A, on the Hilbert space, $\mathscr{H}$. Each pure state assigns a probability distribution to the observed values of any observable. The expected value of the observable A in the state corresponding to ϕ is

$$\langle \phi, A\phi \rangle = \mathrm{tr}\, AP_\phi$$

(provided that $\phi \in \mathrm{dom}(A)$ so that the expression makes sense. From now on, we will not make explicit the hypotheses concerning domains, in order not to clutter up the discussion). Here $\langle\ ,\ \rangle$ denotes the scalar product on $\mathscr{H}$. The variance of the probability distribution of the observed values of A in the state ϕ is given by

$$\langle \phi, A^2\phi \rangle - \langle \phi, A\phi \rangle^2.$$

In particular, the observable A will take on a definite value in the pure state, ϕ, if and only if ϕ is an eigenvector of A, i.e. $A\phi = \lambda\phi$, in which case λ is the observed value of A in the state, ϕ.

As a typical sort of example of the kind of Hilbert space that might arise, consider the space $L^2(\mathbb{R}^3)$ of all square integrable functions on Euclidean three space, with respect to the usual measure. A pure state is then a square integrable function of norm one. If B is any subset of $\mathbb{R}^3$, let 1_B be the indicator function of B, i.e. $1_B(x) = 1$ for $x \in B$ and $1_B(x) = 0$ for $x \notin B$. Let A_B denote the self-adjoint operator consisting of multiplication by 1_B. Then

$$\langle A_B\phi, \phi \rangle = \int_{\mathbb{R}^3} 1_B(x)|\phi(x)|^2\, dx = \int_B |\phi(x)|^2\, dx$$

is interpreted as the probability of being in the set B when in the state ϕ. Notice that if G is any group of Euclidean motions, then G acts as a group of unitary operators on $L^2(\mathbb{R}^3)$, where, as usual, $(a\phi)(x) = \phi(a^{-1}x)$.

More generally, let $\mathscr{H}$ be the Hilbert space of a quantum system. We say that G acts as a group of automorphisms if G acts on the pure states, and, for any two unit vectors, ϕ and ψ, we have

$$|\langle g\phi, g\psi \rangle|^2 = |\langle \phi, \psi \rangle|^2.$$

The meaning of this equation is the following. Each state ϕ defines an observable, P_ϕ. The self-adjoint operator, P_ϕ, can be thought of as the observable asserting that the system is in the state ϕ. Then $\langle P_\phi\psi, \psi \rangle = |\langle \phi, \psi \rangle|^2 = \mathrm{tr}\, P_\phi P_\psi$ is thought of as the probability of observing the system in the state ϕ when it is known to be in the state ψ. The equation $|\langle g\phi, g\psi \rangle|^2 = |\langle \phi, \psi \rangle|^2$ asserts that these probabilities are unchanged. It is a theorem due to Wigner (see Appendix D) that any such transformation of the states actually arises from either a unitary or an anti-unitary transformation of $\mathscr{H}$. That is, there exists either a unitary operator, U_g, of $\mathscr{H}$ such that $g\phi = U_g\phi$, or there exists an anti-unitary operator, V_g such that $g\phi = V_g\phi$. (To say that V_g is anti-unitary means that V_g is an anti-linear map, i.e. that

$$V_g(\alpha\phi + \beta\psi) = \bar{\alpha}V_g\phi + \bar{\beta}V_g\psi$$

for any complex numbers α and β and any vectors ϕ and ψ, that V_g is invertible, and that

$$\langle V_g\phi, V_g\psi \rangle = \langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}.$$

The operator U_g (or V_g) is not completely determined, since the vector ϕ itself is not quite determined by the state it defines. It turns out that U_g (or V_g) is determined up to scalar factor (the scalar being of absolute value one). The product of two anti-unitary transformations is a unitary transformation, and hence the subgroup G_U consisting of those U which correspond to unitary transformations is of index two in G. Let us restrict attention to this subgroup. The unitary operator U_g which is associated to $g \in G$ is only determined up to multiplication by a complex number of absolute value one. So all we can say, in general, is that, if we take any two elements g_1 and g_2 in G,

$$U_{g_1g_2} = c(g_1, g_2)U_{g_1}U_{g_2}$$

where $c(g_1, g_2)$ is some complex number of absolute value one depending on g_1 and g_2. If we could choose the U_g (by multiplying by an appropriate factor of absolute value one depending on g) so that $c(g_1, g_2) = 1$ for all g_1 and g_2, then we would get a unitary representation of G on $\mathscr{H}$. For many groups it is possible to prove that one can always make such a choice, and in many situations one is actually given a unitary representation. (In general the map $g \rightsquigarrow U_g$ is called a projective representation of G. It is always possible to enlarge the group G to a group G' so that a projective representation of G corresponds to an ordinary representation of G'. We do not want to go into these points here, although both anti-unitary operators (which arise in the study of time reversal) and projective representations (which arise in the study of the Galilean group) have important physical applications.)

With the above remarks in mind, we shall assume, for the rest of the discussion, that

we are given a unitary representation of the finite group, G, on the Hilbert space $\mathscr{H}$.

Let ξ be any vector in $\mathscr{H}$. Then the set of all linear combinations of the vectors $a\xi$, as a ranges over G, clearly forms a finite-dimensional subspace of $\mathscr{H}$, which is invariant under the action of the group, G. Thus every vector lies in some finite-dimensional invariant subspace of $\mathscr{H}$. Of course, each such subspace can be decomposed into an orthogonal direct sum of irreducible finite-dimensional subspaces. Let W_i and W_j be irreducible (finite-dimensional) subspaces corresponding to two inequivalent representations of G. Let P denote the map from W_i to W_j given by orthogonal projection in $\mathscr{H}$: if $\xi \in W_i$, then $\xi' = P\xi \in W_j$ is the unique vector in W_j such that

$$\langle \xi, \eta \rangle = \langle \xi', \eta \rangle \quad \text{for all} \quad \eta \in W_j.$$

It follows that $gP = Pg$ for all $g \in G$ and hence, from Schur's lemma, that $P = 0$. In other words, *irreducible subspaces corresponding to inequivalent representations are orthogonal.*

Let F_i be the subset consisting of those η which belong to some irreducible subspace, W_j, with representation, r_j, inequivalent to the ith irreducible representation, r_i. Then $\mathscr{H}_i = (F_i)^{\perp}$ is a closed subspace, and every vector $\xi \in \mathscr{H}_i$ belongs to an irreducible representation isomorphic to r_i. We clearly have

$$\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_p$$

giving a decomposition of $\mathscr{H}$.

Let $L(\mathscr{H})$ denote the space of linear operators on $\mathscr{H}$. (In order to avoid complications about domains, the reader may prefer to replace $L(\mathscr{H})$ by the space of bounded linear operators or by some space or operators with a common dense domain.) The group G acts on $L = L(\mathscr{H})$ by sending $T \in L$ into gTg^{-1}. This gives a representation of G on L.

Let r be some representation of G on some auxiliary vector space, V (usually finite dimensional). By a *representation operator* belonging to r, or, more simply, an r-operator, we shall mean some element, R, of $\mathrm{Hom}_G(V, L)$. Thus $R: V \to L$ is a linear map such that

$$R(gv) = gR(v)g^{-1} \quad \text{for} \quad g \in G \quad \text{and} \quad v \in V.$$

(In the physics literature, a representation operator is frequently called a *tensor operator.*) The value, $R(v)$, of the r-operator at some $v \in V$ we shall call a *representative operator.* Thus a representative operator is an actual operator on $\mathscr{H}$, i.e. an element of L, whereas the r-operator, R, is not an operator but a map from V to the space of operators.

Let W_1 and W_2 be irreducible subspaces of $\mathscr{H}$, and let $R: V \to L$ be a representation operator. The set of all vectors of the form

$$R(v)\eta, \quad v \in V, \quad \eta \in W_2$$

clearly forms an invariant subspace of $\mathscr{H}$ equivalent to the image of $V \times W_2$ under the bilinear map of $V \times W_2 \to \mathscr{H}$ sending (v, η) to $R(v)\eta$. We know that W_1 will be orthogonal to this subspace, unless $V \otimes W_2$ contains a subrepresentation equivalent to W_1. We thus obtain the fundamental quantum mechanical selection rule

> if W_1 and W_2 are irreducible subspaces and $R: V \to L$ is a representation operator, then
>
> $$\langle \eta_1, R(v)\eta_2 \rangle = 0 \quad \text{for all} \quad \eta_i \in W_i, v \in V \tag{6.1}$$
>
> unless $V \otimes W_2$ contains a subrepresentation equivalent to W_1.

The simplest kinds of representative operators correspond to the trivial representation of G. These are just the operators which commute with the action of G. They are called 'scalar' operators. Suppose that H is a self-adjoint scalar operator, and let

$$U(t) = \exp(-itH/\hbar)$$

(here $\hbar = h/2\pi$, where h is Planck's constant) be the associated one parameter group of unitary operators. It then follows that $U(t)$ is again a scalar operator, i.e. commutes with the action of G. In quantum mechanics, one is frequently given a self-adjoint operator, H, called the Hamiltonian, whose associated one parameter group describes the time evolution of the system. To say that the Hamiltonian is a scalar operator means that the time evolution of the system commutes with the action of G.

Suppose that H is a scalar operator and that ξ is an eigenvector of H with eigenvalue λ:

$$H\xi = \lambda\xi.$$

Then

$$Ha\xi = aH\xi = a\lambda\xi = \lambda(a\xi)$$

so that the space of eigenvectors with eigenvalue λ is invariant under the action of G. Thus, eigenvalues of scalar operators will tend to be degenerate, with multiplicity equal (at least) to the dimensions of the various irreducible representations of G.

One of the most important ways of applying (6.1) is in conjunction with time-dependent perturbation theory which we now describe. Suppose that the Hamiltonian of the system has the form

$$H = H_0 + H_1,$$

where $H_1 = H_1(t)$ should be thought of as representing the effect of some external disturbance. For example, we might want to consider the behavior of a vibrating molecule under the influence of some electromagnetic radiation. Then H_0 would be the Hamiltonian describing the time evolution of the unperturbed vibrations of the molecule, while H_1 describes the effect of the radiation.

We suppose that we know the time evolution of the unperturbed Hamiltonian, i.e. that we know the one parameter group

$$U(t) = \exp(-itH_0/\hbar),$$

and wish to find the one parameter family of unitary transformations corresponding to H. That is, we are looking for a one parameter family, $V(t)$, of unitary trans-

formations which satisfy the differential equation

$$i\hbar V'(t) = HV(t).$$

Let us proceed formally to try to solve this equation by the method of variation of constants. That is, we set

$$V(t) = U(t)B(t)$$

so that

$$V'(t) = U'(t)B(t) + U(t)B'(t)$$

and hence

$$i\hbar V'(t) = H_0 V(t) + i\hbar U(t)B'(t).$$

Thus B must satisfy the differential equation

$$i\hbar B'(t) = (U^{-1}H_1 U)B(t).$$

Suppose that H_0 has a complete set of eigenstates ϕ_n with eigenvalues E_n, so that

$$H_0\phi_n = E_n\phi_n$$

and therefore

$$U(t)\phi_n = \exp(-itE_n/\hbar)\phi_n.$$

For any $\xi \in \mathscr{H}$, we can write $B(t)\xi = \sum_n c_n(t)\phi_n$ and the above differential equation becomes

$$i\hbar \sum c_n'(t)\phi_n = (U^{-1}H_1 U)(\sum c_n(t)\phi_n).$$

Take the scalar product of both sides with ϕ_m. On the left we get $i\hbar c_m'(t)$, when the ϕ_n are chosen to the orthonormal. On the right we get $\sum \langle U^{-1}H_1 U\phi_n, \phi_m\rangle = \sum \langle H_1 U(t)\phi_n, U(t)\phi_m\rangle = \sum \exp[i(E_m - E_n)t/\hbar]\langle H_1\phi_n, \phi_m\rangle$. So we obtain the differential equations

$$i\hbar c_m'(t) = \sum \exp[i(E_m - E_n)t/\hbar]\langle H_1\phi_n, \phi_m\rangle c_n(t). \tag{6.2}$$

Let us assume that we start with the state ϕ_k at time 0, so that we have the initial conditions $c_k(0) = 1$, $c_j(0) = 0$ for $j \neq k$. We are assuming that H_1 is 'small' so that a first approximation to the solution can be obtained by regarding the c's on the right as taking on their constant initial values. We write $c_m(t) = \delta_{km} + c_m^{(1)}(t) + \cdots$, where the approximation $c_m^{(1)}(t)$ is given by

$$i\hbar c_m^{(1)}(t) = \int_0^t \exp[i(E_m - E_k)t/\hbar]\langle H_1\phi_m, \phi_k\rangle. \tag{6.3}$$

There are now several types of $H_1(t)$'s that arise in applications of (6.3). In the case of radiation, for example, it is reasonable to assume that H_1 is a periodic function of the time as a sum or integral of operators of the form:

$$H_1(t) = Ae^{-i\omega t} + Be^{i\omega t}, \quad \omega = 2\pi\nu.$$

The fact that H_1 is to be Hermitian requires $B = A^*$. Let us set

$$\omega_{mk} = (E_m - E_k)/\hbar.$$

Then the integral in (6.3) becomes

$$i\hbar c_m^{(1)}(t) = \langle A\phi_k, \phi_m\rangle \frac{\exp[i(\omega_{mk} - \omega)t] - 1}{\omega_{mk} - \omega} + \langle A^*\phi_k, \phi_m\rangle \frac{\exp[i(\omega_{mk} + \omega)t] - 1}{\omega_{mk} + \omega} \tag{6.4}$$

for those values of ω and t for which the left-hand side is sufficiently small, so that it still may be considered a good approximation to the solution of (6.2). This will be the case if ω is not too close to $\pm\omega_{mk}$ or if t is not too large. Suppose, to fix the ideas, that we consider the situation where $E_m > E_k$. Then $|c_m(t)|^2$ can be regarded as the 'transition probability at time t' for unperturbed state, ϕ_k, whose energy was E_k, to 'absorb the quantum of energy $h\nu$' and pass into the state ϕ_m whose energy is E_m in the unperturbed system. According to (6.4), for moderate values of t, this transition probability will be non-negligible only for ω close to ω_{mk}. Plugging into (6.4) and ignoring the second term, we obtain the approximation

$$\hbar^2|c_m^{(1)}(t)|^2 = t^2|\langle A\phi_m, \phi_n\rangle|^2 \left|\frac{\sin^2\zeta(\omega, t)}{\zeta^2(\omega, t)}\right|, \quad \text{where } \zeta = (\omega_{mk} - \omega)t/2. \tag{6.5}$$

This is the formula for a fixed frequency, ω. If we are given a whole spectrum of perturbations of the form $\int A(\omega)e^{i\omega t}\,d\omega$ + (adjoint), we must integrate (6.5) with respect to ω. Again, the major contribution comes from the values of ω near ω_{mk}, so that if A is approximately constant near ω_{mk}, we can pull it out of the integral sign to obtain (with A now denoting $A(\omega_{mk})$)

$$t^2|\langle A\phi_m, \phi_k\rangle|^2 \int \frac{\sin^2\zeta}{\zeta^2}\,d\omega.$$

By a change of variables, the integral becomes

$$4t^{-1}\int (\sin^2 x/x^2)\,dx = 4\pi t^{-1}$$

so that

$$|c_m(t)|^2 = 4\pi t|\langle A\phi_m, \phi_k\rangle|^2/\hbar^2. \tag{6.6}$$

Thus, up to constant factors, and to the extent that the approximations are valid, the expression $|\langle A\phi_m, \phi_k\rangle|^2$ gives the transition probabilities per unit time.

That transition probabilities are non-negligible only for ω close to ω_{mk} can be formulated as saying that the system described by H_0 can only 'absorb' or 'emit' energy to the vibrating system in discrete units corresponding to differences in the energy levels on H_0. It took about a century of investigation of spectra to realize that the observed spectral lines of atoms or molecules could best be described as differences between various values. The Planck–Einstein relation between energy and frequency, $E = h\nu$, led to the understanding that one was really dealing with differences of energy. Not *all* differences $E_k - E_m$ make their appearance in the spectrum, so this raised the question of why certain differences do occur and others do not; i.e. what are the 'selection rules'? In terms of (6.4) or (6.5), why should $\langle Aq_k, q_n\rangle$ vanish? The answer of course is group theoretical:

It will usually happen in our applications that A is a representative operator for some group, so that we can apply (6.1) to conclude that certain of the $\langle A\phi_m, \phi_k\rangle$ must vanish. We speak of the corresponding transition as being *forbidden* by symmetry considerations. Frequently the argument is used in reverse: we notice that certain transitions do not occur, and attempt to explain their absence on the grounds of symmetry. In other words, we use the observed transitions to deduce the structure of the underlying symmetry group.

Another application of (6.5) is to the situation where $\omega = 0$, but where one averages over the final states, instead of over ω. This is the situation that arises in a crude approximation to the analysis of collision processes. Here we assume that some perturbing 'particle' passes near our system. We approximate the situation by assuming that there is a constant perturbation applied for a short period of time. Thus $\omega = 0$, and (6.5) becomes

$$|c_m(t)|^2 = 4|\langle A\phi_m, \phi_k\rangle|^2 \frac{\sin^2 \omega_{mk} t/2}{\omega_{mk}^2}.$$

We now assume that there is a whole family of states whose energies are all close to one another and close to the energy of the state ϕ_k, and for which the values of $\langle A\phi_m, \phi_k\rangle$ are approximately constant. Then the transition probability from ϕ_k to any one of these states is obtained by summing the above formula over m to obtain

$$4|\langle A\phi_m, \phi_k\rangle|^2 \sum_m \frac{\sin^2 \omega_{mk} t/2}{\omega_{mk}^2}.$$

We assume that the energy levels are sufficiently closely spaced to justify approximating the sum by an integral. We assume that the number of states whose energy is in an interval about E_m is given by the integral of the density $\rho(E_m)$, and, since $dE_m = \hbar\, d\omega_{mk}$, the above sum is replaced by the integral

$$\hbar \int \frac{\sin^2 \omega_{mk} t/2}{\omega_{mk}^2} \rho(E_m)\, d\omega_{mk}.$$

For moderately large values of t (but not too large to render the approximations invalid), the major contribution of the integral will come from the neighborhood of $\omega_{mk} = 0$ and we obtain the formula

$$\text{rate of transition} = (2\pi/\hbar)|\langle A\psi, \phi_k\rangle|^2 \rho(E_k) \tag{6.7}$$

where ψ is one state among a family of states having energies close to E_k, and for which the $\langle A\psi, \phi_k\rangle$ is approximately constant, and ρ gives the density of states per energy level for this family. Equation (6.7) is known as Fermi's 'golden rule no. 2' and plays a key role in the study of collision processes. Again we see that (6.1) implies that certain transitions are forbidden if A is a representative operator for a suitable group.

In any specific application, we must determine the representation associated to the operator A and the representation associated to the unperturbed eigenstates ϕ_j. For example, we shall show in the next section that the semiclassical theory of radiation suggests that the operator A to be taken for induced emission or absorption of

radiation is a 'vector operator', i.e. A transforms like an element of $\mathbb{R}^3$ under a given action of G on $\mathbb{R}^3$. (The operator A is associated with the electric dipole moment of the system, and the dipole moment is a vector in $\mathbb{R}^3$.)

In a given situation, we must also determine the eigenvalues, $E_0, E_1, \ldots$ of H_0. In principal, any allowed transition from E_n to E_k can occur. But the system must be in state E_m and it must be given enough energy (or be able to give up enough energy to the oscillating perturbation) to get to E_k. In practice, of course, we observe many copies of the same system, not one; we shine light on a sample of a given substance, not on a single molecule. At certain temperatures, most of the copies may be in a given energy state. For instance, at low temperatures most of the molecules might be expected to be in the 'ground state', E_0. Thus we would be primarily interested in the $E_0 \to E_k$ transitions.

In the case of a (harmonic) oscillator, the eigenvalues of H_0 were worked out at the very beginning of quantum mechanics. For a one-dimensional oscillator whose classical frequency is ω_0, the quantum energy levels are

$$\tfrac{1}{2}\hbar, (\omega_0 + \tfrac{1}{2})\hbar, \quad (2\omega_0 + \tfrac{1}{2})\hbar, \ldots.$$

More generally, if we are given a finite-dimensional space V for the classical oscillator, then the Hilbert space for the quantum system can be written as the (Hilbert space) direct sum

$$\underset{\substack{\text{ground}\\ \text{state}}}{\mathbb{C}} \oplus \underset{\substack{\text{fundamental}\\ \text{or one-phonon}\\ \text{states}}}{V} \oplus \underset{\substack{\text{first harmonic}\\ \text{or two-phonon}\\ \text{states}}}{S^2(V)} \oplus \underset{\substack{\text{three-phonon}\\ \text{states}}}{S^3(V)} \oplus \cdots$$

where $S^k(V)$ denotes the kth symmetric power (the space of symmetric k-tensors) of V. Suppose that

$$V = V_1 \oplus \cdots \oplus V_k$$

where the V_i are eigenspaces of the classical oscillator with eigenvalues $\omega_1, \ldots, \omega_k$. Then

$$S^2(V) = S^2(V_1) \oplus \cdots \oplus S^2(V_k) \oplus V_1 \otimes V_2 + \cdots$$

and the eigenvalues of the quantum oscillator are $2\omega_1, 2\omega_2, \ldots\ 2\omega_k, \omega_1 + \omega_2$, etc. Similarly the eigenvalues of the quantum oscillator on $S^n(V)$ are $\omega_{i_1} + \cdots + \omega_{i_n}$.

For the case of a vibrating molecule at laboratory temperatures, the most important transitions are those from the ground state (of lowest energy) to a 'fundamental state', where the change in energy is (proportional to) an eigenfrequency of the classical vibrating system. Now the ground state is invariant under the group, i.e. transforms according to the trivial representation, while each of the fundamental states transforms according to the representation associated to the given eigenvalue of the vibrating system. So, if ϕ_0 denotes the ground state, then $A\phi_0$ transforms according to $\mathbb{R}^3$, while a fundamental state ϕ_f transforms according to an appropriate irreducible representation W_f occurring in the decomposition of the classical vibrating system. The fundamental transition $\phi_0 \to \phi_f$ will be *allowed* if W_f occurs in the decomposition of $\mathbb{R}^3$ (or rather, its complexification, $\mathbb{C}^3$) under the group G, and will be *forbidden* otherwise.

We can thus determine which of the irreducible components of a vibrating molecule makes its appearance as a fundamental transition.

Thus, for example, in a molecule like methane, whose symmetry group is T_d, we have computed the decomposition of the character of the nine-dimensional space of vibrations as

$$\chi_1 + \chi_2 + 2\chi_5.$$

Here χ_5 is the character of the $\mathbb{R}^3$ actions. So, although there will generically be four frequencies, only two of them will be observed as absorption lines corresponding to transitions from the ground state to a fundamental state.

The transitions we have described so far have to do with absorption. Experimentally, a red-hot body emits infra-red radiation which is collimated and passed through a sample of the substance to be studied. It is then dispersed by a prism or grating and the amount of radiation in each small portion of the spectrum is measured. This is compared to the amount emitted. In this way, one detects the absorption spectrum. There is another procedure discovered by Raman in 1928 which, fortunately, has different selection rules. In Raman spectroscopy, an intense monochromatic beam of light, of (any) frequency ν, is brought to bear on the sample and observations are made at right angles to the beam. Most of the scattered light has the same frequency, ν, as the incident beam. But there will be a small amount of light at frequency $\nu - \nu_R$ (and less at $\nu + \nu_R$), where ν_R is known as the Raman line. The displacement ν_R is independent of the incident frequency and is associated with a 'tensor' operator, i.e. a representative operator, transforming according to the representation on the space of symmetric tensors $S^2(\mathbb{R}^3)$. Again, at room temperatures, the only important Raman transitions will be from the ground state to the fundamental state. So we must compare the fundamental state with $S^2(\mathbb{R}^3)$ to check which Raman lines will be observed. For example, we have seen that the representation of T_d on $\mathbb{R}^3 \otimes \mathbb{R}^3$ has character

$$\chi_5^2 = \chi_1 + \chi_2 + \chi_4 + \chi_5,$$

and we know that χ_4 is the character of T_d on the three-dimensional space of anti-symmetric tensors. So the representation on $S^2(\mathbb{R}^3)$ has character

$$\chi_1 + \chi_2 + \chi_5.$$

So all ground state to fundamental transitions can occur in Raman spectroscopy.

The combination of infra-red and Raman spectroscopy can be very effective. For example, suppose that we have a molecule of the form AB_4 and we consider three possible shapes (Fig. 3.10). We can evaluate the number of fundamental eigenvalues and which are infra-red or Raman admissible. We let the reader check that we obtain Table 20 (we have already worked out the result for T_d).

In principle, we should be able to distinguish experimentally between these (or other) alternatives. (In practice, the experimental results may not be clear cut.)

In the next section we shall sketch the proof that the infra-red selection rules are associated to vector operators. We refer the reader to specialized treatises for the proof that the Raman selection rules are associated to $S^2(\mathbb{R}^3)$ second order perturbation theory.

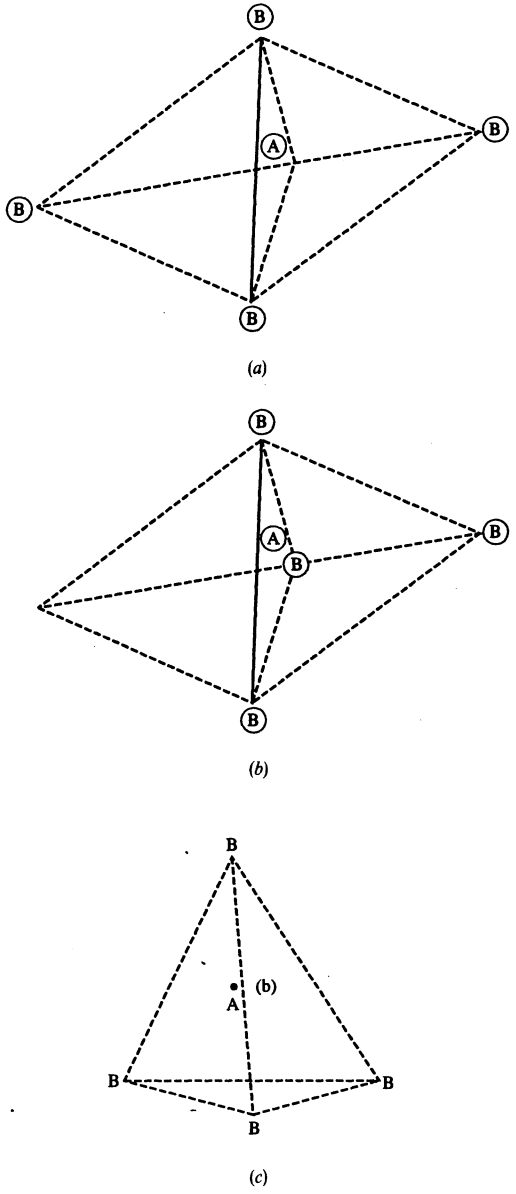


Fig. 3.10

Table 20.

Group	Number of eigenvalues	Infra-red	Raman
C_{2v}	9	8	9
C_{3v}	6	6	6
T_d	4	2	4

Let us apply the results of the results we have derived in this chapter to describe the infra-red and Raman spectra of the buckyball. We know that the space of vibrational states has dimension $174 = 180 - 6$. We wish to prove the following.

(1) That there are at most 46 distinct vibrational modes. In other words, the space of classical vibrational states decomposes into 46 irreducible representations.
(2) Of these (at most) four are visible in the infra-red. In other words, exactly four of the 46 irreducibles are equivalent to the representation of the group I_h on the (complexification of) ordinary three-dimensional space, $\mathbb{R}^3$.
(3) (At most) ten lines are visible in the Raman spectrum. More precisely, the complexification of the space $S^2(\mathbb{R}^3)$, which is six-dimensional, decomposes into a direct sum of the trivial representation (given by the scalar matrices) and a five-dimensional representation (corresponding to the traceless tensors). This five-dimensional representation is irreducible and occurs eight times in the space of vibrational states, whereas the trivial representation occurs twice.

The above facts concerning the spectrum of the buckyball have all been verified experimentally. The two different kinds of Raman spectra, corresponding to the trivial and to the five-dimensional representation, can be distinguished through the use of polarized light.

To begin, we must list all the irreducibles of I_h. Since I_h is the direct product of I with $\mathbb{Z}_2$, it is enough to find the irreducibles of I and then label each with a + or − sign according to the trivial or sign representation of the $\mathbb{Z}_2$ component.

Now $I \sim A_5$, which is a subgroup of S_5. The elements of A_5 consist of the identity, 20 three-cycles, 15 elements of the form $(ab)(cd)$, and 24 five-cycles. Any two three-cycles are conjugate by an even permutation, since the odd permutation, (de), commutes with (abc). So the 20 three-cycles form a single conjugacy class in A_5. Similarly, the odd permutation (ab) commutes with $(ab)(cd)$ and hence the 15 elements of the form $(ab)(cd)$ form a single conjugacy class. On the other hand, a five-cycle such as $\sigma = (12345)$ does not commute with any odd element, since it cannot carry the elements of any two-cycle or any four-cycle into itself. Now $\sigma^2 = (13524)$ is conjugate to σ in S_5 by the odd element (2354) and hence by no even element. Thus the 24 five-cycles split up in A_5 into two conjugacy classes, those conjugate to σ and

those conjugate to σ^2. Thus there are five conjugacy classes in all, and so five inequivalent irreducible representations.

We now go back to Table 16 of Chapter 2 and see what happens when we restrict the irreducible representations of S_5 to A_5. Direct computation shows that the first three lines restrict to irreducibles of A_5. The fourth line, the six-dimensional representation of S_5, does not remain irreducible; the character satisfies $(\chi, \chi)_{A_5} = 2$. So it splits into two irreducibles of dimensions, say, d_1 and d_2. Since

$$1 + 16 + 25 + d_1^2 + d_2^2 = 60,$$

we must have

$$d_1 = d_2 = 3.$$

One of these three-dimensional representations we already know – the complexification of the representation identifying A_5 with I acting as rotational symmetries of the icosahedron. Since all elements are rotations, having trace $1 + 2\cos\theta$, we see that, for this representation,

$$\chi(abc) = 0$$

and

$$\chi((ab)(cd)) = -1.$$

If σ corresponds to rotation through angle $2\pi/5$, then

$$\chi(\sigma) = 1 + 2\cos 2\pi/5, \quad \chi(\sigma^2) = 1 + 2\cos 4\pi/5.$$

A direct check shows that this is an irreducible character, and interchanging the roles of σ and σ^2 in this last equation gives the character of the other three-dimensional representation, as can be checked by adding the two together to get the character of the reducible six-dimensional representation coming from S_5.

We may label the irreducibles of I as **1**, **4**, **5**, **3**, and **3′**. Thus the regular representation of I decomposes into

$$1 + 4 + 5 + 3 + 3 = 16$$

irreducible representations.

Let $E \to B$ denote the vector bundle describing the displacements of the carbon atoms in the buckyball from their equilibrium positions. It is a homogeneous vector bundle with respect to the groups I and I_h. The group I has a trivial isotropy group at any point of B. Hence the space $\Gamma(E)$ transforms as the direct sum of three copies of the regular representation of I. It thus decomposes into 48 irreducibles. We remove two copies of **3** when we subtract off overall translations and overall rotations. Hence there are (at most) 46 distinct vibrational modes.

Let X be a vertex of the buckyball and let H be the isotropy group of X in I. So H consists of two elements, the identity and the reflection, r_X, in the plane passing through X and bisecting the buckyball. The representation of I_h given by (the complexification of) its action on three-dimensional space is labeled as $\mathbf{3}^-$, since the inversion operator (the generator of the $\mathbb{Z}_2$ component) acts as $-Id$. The vector bundle E is induced from the restriction of this representation to H. The group H has only two irreducible representations, both one-dimensional: the trivial

representation, $\mathbf{1}_+$, and the 'sign' representation, $\mathbf{1}_-$, which assigns the value -1 to r_X. Thus the three-dimensional representation $\mathbf{3}^-\downarrow$ of H must decompose into a sum of these one-dimensional representations. To see what this decomposition is, choose coordinates so that r_X is reflection in the y, z plane. Then the matrix representing r_X is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which shows that we have the block decomposition

$$\mathbf{3}^-\downarrow = \mathbf{1}_- \oplus \mathbf{1}_+ \oplus \mathbf{1}_+. \tag{6.8}$$

But now Frobenius reciprocity says that $\mathbf{3}^-$ occurs once in the representation induced from $\mathbf{1}_-$ and twice in the representation induced from $\mathbf{1}_+$. Hence it follows from (6.8) that $\dim \mathrm{Hom}(\mathbf{3}^-, \Gamma(E)) = 5$. Subtracting one copy of $\mathbf{3}^-$ corresponding to overall translations shows that there are exactly four vibrational lines in the infra-red. This proves statement (2) above.

We can use Frobenius reciprocity together with Table 16 to compute the multiplicities of all the irreducibles in the induced representations. Indeed, the reflection, r_X, is the product of the inversion $-Id$ with a rotation ρ_X through 180° lying in I:

$$r_X = -\rho_X.$$

Let $K = \{e, \rho_X\}$ be the corresponding two element subgroup of I, and $\mathbf{1}_{K\pm}$ its one-dimensional representations. Then for any $+$ representation of I_h we have

$$\dim \mathrm{Hom}_H(\mathbf{1}_\pm, \mathbf{k}^+\downarrow) = \dim \mathrm{Hom}_K(\mathbf{1}_{K\pm}, \mathbf{k}^+\downarrow),$$

whereas for the $-$ representations the signs are reversed:

$$\dim \mathrm{Hom}_H(\mathbf{1}_\pm, \mathbf{k}^-\downarrow) = \dim \mathrm{Hom}_K(\mathbf{1}_{K\mp}, \mathbf{k}\downarrow).$$

So, for example,

$$\dim \mathrm{Hom}_{I_h}(\mathbf{5}^+, \mathbf{1}_+\uparrow) = \dim \mathrm{Hom}_H(\mathbf{1}_+, \mathbf{5}^+\downarrow) = \dim \mathrm{Hom}_K(\mathbf{1}_{K+}, \mathbf{k}^+\downarrow) = \tfrac{1}{2}(5+1) = 3.$$

Proceeding in this way we compute the decompositions

$$\mathbf{1}_+\uparrow = \mathbf{1}^+ \oplus 2\times\mathbf{3}^- \oplus \mathbf{3}^+ \oplus 2\times\mathbf{3}'^- \oplus \mathbf{3}'^+ \oplus 2\times\mathbf{4}^+ \oplus 2\times\mathbf{4}^- \oplus 3\times\mathbf{5}^+ \oplus 2\times\mathbf{5}^- \tag{6.9}$$

and

$$\mathbf{1}_-\uparrow = \mathbf{1}^- \oplus 2\times\mathbf{3}^+ \oplus \mathbf{3}^- \oplus 2\times\mathbf{3}'^+ \oplus \mathbf{3}'^- \oplus 2\times\mathbf{4}^+ \oplus 2\times\mathbf{4}^- \oplus 3\times\mathbf{5}^- \oplus 2\times\mathbf{5}^+. \tag{6.10}$$

We can use (6.9) and (6.10) to determine the number of lines in the Raman spectrum. In the Raman experiment the operator, A, transforms like an element of $S^2(\mathbb{R}^3)^+$. Using the characters of I, a direct computation shows that

$$\mathbf{3}\otimes\mathbf{3} = \mathbf{3}\oplus\mathbf{1}\oplus\mathbf{5}.$$

The first summand on the left is just the space of anti-symmetric tensors. So we see

that the complexification of $S^2(\mathbb{R}^3)$ transforms, under I_h, like $\mathbf{1}^+ \oplus \mathbf{5}^+$. (The $\mathbf{5}^+$ corresponds to traceless tensors and the $\mathbf{1}^+$ corresponds to multiples of the identity. The reason for the $+$ is that the parity operator, P, has no effect on tensors of even degree.) Let us examine these two components separately. Notice that $\mathbf{1}^+$ does not occur at all on the right-hand side of (6.10), and occurs once on the right-hand side of (6.9). Hence we conclude from (6.8) and Frobenius reciprocity that

$$\dim \mathrm{Hom}_{I_h}(\mathbf{1}^+, Vib) = 2. \tag{6.11}$$

In other words there should be two Raman lines corresponding to the $\mathbf{1}^+$ representation of I_h. The $\mathbf{5}^+$ occurs twice on the right-hand side of (6.10), and three times on the right-hand side of (6.9). Hence we conclude from (6.8) and Frobenius reciprocity that

$$\dim \mathrm{Hom}_{I_h}(\mathbf{5}^+, Vib) = 2+3+3 = 8. \tag{6.12}$$

So there should be eight lines corresponding to $\mathbf{5}^+$. All ten lines have been observed. In fact that $\mathbf{1}^+$ lines can be distinguished, experimentally, from the $\mathbf{5}^+$ lines through the use of polarized light.

We should point out that decompositions such as (6.9) and (6.10) together with Schur's lemma offer a powerful tool for the location of the eigenvalues of invariant operators and not merely for counting the number of distinct eigenvalues, which is the use we have made so far. For example, the 60-dimensional space, $\mathbf{1}^+$, plays a central role in the electronic structure of the buckyball, and, hopefully, in understanding the high temperature superconductivity of doped buckyball crystals. So it will be important to analyze invariant operators, R, on this space, that is matrices, R, which lie in $\mathrm{Hom}_{I_h}(\mathbf{1}^+, \mathbf{1}^+)$. Such a matrix has size 60×60. But Schur's lemma implies that there cannot be any interaction between inequivalent components. In other words, R must have a block decomposition corresponding to the right-hand side of (6.9), where the size of the blocks is given by the multiplicities of the various irreducible representations, i.e. by the various coefficients occurring on the right-hand side of (6.9). The largest such coefficient is 3. So, although R is a 60×60 matrix, in searching for the eigenvalues of R, the worst computation that we have to do is diagonalizing a 3×3 matrix, and this can be done analytically.

3.7 The semiclassical theory of radiation

In this section we regard the ambient radiation as giving a periodic perturbation of the quantum mechanical system describing the atom or molecule, as described in the preceding section. In particular, we will be concerned with describing the 'first order transition operator', A, which arises from first order perturbation theory. The main result of this section is that, for absorption or emission spectra, the operator A transforms as an element of $\mathbb{R}^3$ (i.e. is a 'vector operator'). The reader may want to accept this assertion on faith and immediately proceed to the next section.

According to Lorentz, the force $\mathbf{F}$ acting on a (spinless) particle of charge q in an

electromagnetic field is given by

$$\mathbf{F} = q\left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})\right]$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\times$ denotes vector product. (Later on we shall discuss how this equation is to be modified for particles which possess an intrinsic angular momentum, i.e. a spin.) We can treat $\mathbf{E}$ as a differential form of degree one: $\mathbf{E} = E_x\,dx + E_y\,dy + E_z\,dz$ on (x, y, z, t) space and $\mathbf{B}$ as a differential form of degree two: $\mathbf{B} = B_x\,dy \wedge dz - B_y\,dx \wedge dz + B_z\,dx \wedge dy$. We can then write the electromagnetic field as a two-form

$$\begin{aligned}\Omega &= \mathbf{E} \wedge c\,dt + \mathbf{B}\\ &= c(E_x\,dx \wedge dt + E_y\,dy \wedge dt + E_z\,dz \wedge dt)\\ &\quad + B_z\,dy \wedge dz - B_y\,dx \wedge dz + b_z\,dx \wedge dy.\end{aligned}$$

Then the Lorentz force, $\mathbf{F}$, is the space component of

$$\frac{-q}{c}\,i(\mathbf{w})\Omega$$

where $\mathbf{w}$ is the four-vector

$$\begin{aligned}\mathbf{w} &= \mathbf{v} + (\partial/\partial t)\\ &= (v_x, v_y, v_z, 1).\end{aligned}$$

The Maxwell equations for the electromagnetic field without sources are

$$d\Omega = 0, \quad d{\star}\Omega = 0,$$

where $\star$ denotes the star operator relative to the Lorentz metric in which

$$(dx, dx) = (dy, dy) = (dz, dz) = -1,$$
$$(dx, dy) = 0, \text{ etc.} \quad (dx, dt) = 0, \text{ etc.} \quad \text{and} \quad (c\,dt, c\,dt) = 1$$

so that, explicitly,

$$\begin{aligned}{\star}\,dx &= -dy \wedge dz \wedge c\,dt\\ {\star}\,dy &= dx \wedge dz \wedge c\,dt\\ {\star}\,dz &= -dx \wedge dy \wedge c\,dt\\ {\star}\,c\,dt &= dx \wedge dy \wedge dz\end{aligned}$$

and

$$(dx \wedge dy) = -dz \wedge c\,dt, \text{ etc.}$$
$$(dx \wedge c\,dt) = dy \wedge dz, \text{ etc.}$$

The first of the Maxwell equations, $d\Omega = 0$, implies that we can write

$$\Omega = d\Xi$$

for some form

$$\Xi = A_x\,dx + A_y\,dy + A_z\,dz - c\phi\,dt,$$

where $\mathbf{A} = (A_x, A_y, A_z)$ is called the vector potential and ϕ is called the scalar potential. (Of course Ξ is only determined up to the addition of $d\psi$ for some function ψ – replacing Ξ by $\Xi + d\psi$ is known as a guage transformation.) Explicitly,

$$E_x = -(1/c)(\partial A_x/\partial t) - \partial\phi/\partial x,$$

etc. and

$$B_x = \partial A_z/\partial y - \partial A_y/\partial z,$$

etc. The equations of motion become (for a particle of mass m)

$$m\frac{d^2x}{dt^2} = -q\frac{\partial\phi}{\partial x} - \frac{q}{c}\frac{\partial A_x}{\partial t} + \frac{q}{c}\left[\frac{dy}{dt}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \frac{dz}{dt}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right]$$

etc., which can be written as

$$\frac{dx}{dt} = \frac{1}{m}\left(p_x - \frac{q}{c}A_x\right)$$

$$\frac{dp_x}{dt} = \frac{q}{mc}\left(\frac{dx}{dt}\frac{\partial A_x}{\partial x} + \frac{dy}{dt}\frac{\partial A_y}{\partial x} + \frac{dz}{dt}\frac{\partial A_z}{\partial x}\right) - q\frac{\partial\phi}{\partial x}$$

etc., or

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x}, \quad \frac{dp_x}{dt} = -\frac{\partial H}{\partial x}$$

etc., where

$$H = \frac{1}{2m}\left[\left(p_x - \frac{q}{c}A_x\right)^2 + \left(p_y - \frac{q}{c}A_y\right)^2 + \left(p_z - \frac{q}{c}A_z\right)^2\right] + q\phi.$$

If we compare this with the free Hamiltonian

$$H_0 = \frac{1}{2m}[p_x^2 + p_y^2 + p_z^2],$$

we see that H may be obtained from H_0 by replacing the energy momentum vector (p_x, p_y, p_z, E) by $(p_x - (q/c)A_x, \ldots, E + q\phi)$ in the Hamiltonian. This scheme is the one we shall use in the quantum mechanical formulation of the problem as well. In quantum mechanics the momentum, p_x, becomes the operator $-i\hbar(\partial/\partial x)$, so that the expression $(p_x - (q/c)A_x)^2$ becomes the operator

$$\left[-i\hbar(\partial/\partial x) - \frac{q}{c}A_x\right]^2 = -\hbar^2\frac{\partial^2}{\partial x^2} + i\hbar\frac{q}{c}A_x\frac{\partial}{\partial x} + i\hbar\frac{q}{c}\frac{\partial A_x}{\partial x} + \frac{q^2}{c^2}A_x^2.$$

Now the second Maxwell equation in free space, $d\Omega = 0$, implies (as is easily seen) that we can, by an appropriate gauge transformation, arrange that $\phi = 0$, and $\partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z = 0$. Thus, substituting this choice of gauge into the quantum mechanical expression for H, we obtain

$$\frac{1}{2m}\left[-\hbar^2\Delta + i\hbar\frac{q}{c}\left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right) + \frac{q^2}{c^2}\|A\|^2\right]$$

as the Hamiltonian of a particle of mass m and charge q in the presence of an electromagnetic field in free space. For the case of an electromagnetic field of weak to

moderate strength, the last term in this expression is negligible in comparison with the first two. If we have a system of such charged particles, with an internal energy V in the presence of an electromagnetic field, the Hamiltonian can be written as $H = H_0 + H_1$, where now

$$H_0 = -\sum \frac{\hbar^2}{2m_j}\Delta_j + V$$

and

$$H_1 = -\sum \frac{q_j}{m_j c} A_j \cdot p_j$$

with $A_j \cdot p_j$ denoting the operator $-i\hbar(A_x\partial/\partial x + A_y\partial/\partial y + A_z\partial/\partial z)$ for the jth particle. Let us first assume that we are dealing with an electromagnetic field given by a plane wave,

$$\mathbf{A}(v,t) = \tfrac{1}{2}\| A \| u[\exp i(\omega t - k\cdot v) + \exp i(k\cdot v - \omega t)],$$

where $\| A \|$ is a constant, u is a (constant) vector giving the polarization of the radiation, and k is the wave vector determining the direction of propagation and the wavelength of the radiation. Now the wavelengths of visible and ultra-violet radiation are of the order of magnitude 10^{-5} cm at the least, while a rough estimate of molecular dimensions gives a diameter of about 10^{-8} cm. Thus $|k\cdot v| < 10^{-3}$ as v varies over a region of size 10^{-8}, i.e. in the expansion

$$e^{ik\cdot v} = 1 + ik\cdot v + \cdots$$

we can ignore all but the constant term. Thus, in the formula for H_1, we may assume that the A_j's take on the same constant value. In equation (6.6), we must evaluate a sum of terms of the form

$$\langle (u\cdot p_j)\phi_m, \phi_m\rangle = u\cdot\langle p_j\phi_m, \phi_n\rangle$$

where, on the right, p_j denotes the vector operator corresponding to the momentum of the jth particle. Now ϕ_m and ϕ_n are eigenvectors of H_0 with eigenvalues E_m and E_n, while we have the commutation relations

$$r_j H_0 - H_0 r_j = \frac{i\hbar}{m_j} p_j,$$

where r_j denotes the position operator for the jth particle. Therefore,

$$\left\langle \left(\sum \frac{q_j}{m_j c} p_j\right)\phi_m, \phi_n \right\rangle = (i\hbar c)^{-1}\langle(\textstyle\sum q_j(r_j H_0 - H_0 r_j))\phi_m, \phi_n\rangle$$

$$= (i\hbar c)^{-1}(E_m - E_n)\langle(\textstyle\sum q_j r_j)\phi_m, \phi_n\rangle.$$

Now the expression $\beta = \sum q_j r_j$ is the total dipole moment of the system, and is clearly a vector in $\mathbb{R}^3$, classically, and a vector operator quantum mechanically. Substituting into (6.6) of the preceding section gives, for the transition probability per unit time,

$$\frac{\pi^2 \nu_{mn}^2}{c^2\hbar^2}\| A \|^2 |u\cdot\langle\beta\phi_m, \phi_n\rangle|^2, \tag{7.1}$$

where, as usual, $E_m - E_n = h\nu_{mn}$. Averaging over possible directions of polarization, u (and arbitrary initial phases), gives, for the induced transition probability per unit time, under the assumption of homogeneous radiation,

$$\frac{\pi^2 \nu_{mn}^2}{c^2\hbar^2}\| A \|^2 \| \langle\beta\phi_m, \phi_n\rangle \|^2 \tag{7.2}$$

where $\langle\beta\phi_m, \phi_n\rangle \in \mathbb{C}^3$ is the expectation value of the vector operator β between the states ϕ_m and ϕ_n and $\| \langle\beta\phi_m, \phi_n\rangle \|$ is the length of this expectation. Formula (7.1) shows that the transition probabilities for induced emission or absorption is given by a vector operator, as indicated at the beginning of this section. We can give an intuitive interpretation of (7.2) as follows: it follows from the equation $E = -(1/c)\partial A/\partial t - d\phi$, with $\phi = 0$ and $A = \| A^0 \| u \cos 2\pi\nu_{mn} t$, that

$$E = \frac{2\pi\nu_{mn}}{c}\| A^0 \| u \sin 2\pi\nu_{mn} t,$$

for radiation of frequency ν_{mn}. We can then rewrite (7.1) as

$$\frac{1}{2\hbar^2}\overline{|\langle(E\cdot\beta)\phi_m, \phi_n\rangle|^2}$$

where $\overline{}$ denotes the time average, since the average value of $\sin^2 2\pi\nu_{mn} t$ is $\frac{1}{2}$. Now, classically, $E\cdot\beta$ is the energy of the dipole β in the electric field, E. Thus, we can say that within the approximations that we have adopted it is the 'electric dipole energy' which is responsible for the transitions from one stationary state to another with corresponding absorption or emission of light.

Let us return to the formulas for induced emission and absorption, (7.1) and (7.2). The time average of the density of radiation associated with a plane wave is, according to classical electromagnetic theory, given by $(1/8\pi)\| E^0 \|^2$, where $E = E^0 \sin 2\pi\nu t$. Taking the time average of (7.2) then gives us a formula relating the transition probabilities, P_{mn}, from the state ϕ_m to the state ϕ_n and the density, $\rho(\nu_{mn})$, of radiation at frequency ν_{mn}. Namely,

$$P_{mn} = B_{mn}\rho(\nu_{mn}) \quad \text{where} \quad B_{mn} = \frac{2\pi}{\hbar^2}\| \langle\beta\phi_m, \phi_n\rangle \|^2.$$

The coefficients, $B_{mn} = B_{nm}$, are known as the Einstein transition probability coefficients. They were introduced by Einstein in 1917 as part of an argument relating induced absorption and emission with spontaneous emission, using thermodynamic considerations and the Planck radiation law. We sketch his argument. According to the Boltzmann distribution law, the probability for a system to be in a state of energy E_m is proportional to $\exp - E_m/kT$, where k is Boltzmann's constant and T is the absolute temperature. If N_m denotes the number of systems in the state with energy E_m, then we expect that

$$N_m/N_n = \exp -(E_m - E_n)/kT = \exp - h\nu_{mn}/kT.$$

Now a system in an excited state can be expected to emit radiation even in the absence

of an ambient electromagnetic field. Suppose that we let Q_{mn} denote the probability of spontaneous transition from the state ϕ_m to the state ϕ_n, where we assume that $E_m > E_n$. Then the expected number of systems making the transition from the state ϕ_m to the state ϕ_n as a result of both spontaneous and induced emission will be given by

$$N_m\{Q_{mn} + B_{mn}\rho(\nu_{mn})\},$$

while the expected number of systems making the reverse transition as a result of absorption of radiation is

$$N_n B_{nm}\rho(\nu_{mn}) = N_n B_{mn}\rho(\nu_{mn}).$$

At equilibrium, the number of expected transitions in both directions should be equal, yielding the equation

$$\exp - h\nu_{mn}/kT = \frac{B_{mn}\rho(\nu_{mn})}{Q_{mn} + B_{mn}\rho(\nu_{mn})},$$

or

$$Q_{mn}/B_{mn} = [\exp(h\nu_{mn}/kT) - 1]\rho(\nu_{mn}).$$

In the above expression, we can substitute the Planck law of radiation,

$$\rho(\nu) = (8\pi h\nu^3/c^3)[\exp(h\nu/kT) - 1]^{-1}$$

to obtain

$$Q_{mn} = (8\pi h\nu^3/c^3)B_{mn} = (16\pi^2\nu_{mn}^3/c^3 h)\|\langle\beta\phi_m, \phi_n\rangle\|^2. \tag{7.3}$$

Thus, to within the accuracy of the approximations that we have been using, the probability for spontaneous emission depends on only the matrix element of the dipole moment between the stationary states. It is not difficult to check that a rough estimate of the next terms in the expansion of the electromagnetic field shows that they can be expected to be of much smaller magnitude than $(\varepsilon a)^2$, where ε is the electronic charge and a is of the size of a typical radius. If we use this expression for our estimate of the order of magnitude of the dipole transitions, we see that the next order terms are considerably smaller. Of course, some of the matrix elements will vanish on account of selection rules. However, the contribution of the 'dipole transitions' will predominate.

We will not present a detailed discussion of the theory of Raman scattering. Roughly speaking, the transitions are due to the interaction with the ambient electromagnetic field of the dipole moments induced by the field. This depends on the so-called polarizability tensor of the molecule, which measures its response to the electromagnetic field. This is a symmetric tensor, which is why the representation space associated with the Raman effect is $S^2(\mathbb{R}^3)$.

The actual analysis of real spectra is, of course, much more complicated than the simple scheme described above and requires much experience and intuition. As some allowed fundamental lines may in fact be relatively weak, there will be confusion with overtone transitions. The practice of spectroscopy thus becomes a profession which cannot be adequately described within the confines of a book like this.

3.8 Semidirect products and their representations

Let G be a group and N a subgroup. Recall that N is called a *normal* subgroup if $aNa^{-1} = N$ for all $a \in G$. Then we can define the quotient group G/N. We multiply aN and bN by the rule

$$aN \cdot bN = (ab)N.$$

This is well defined since

$$anbn' = abb^{-1}nbn' = (ab)n''n'$$

with $n'' = b^{-1}nb \in N$.

For example, consider the group of Euclidean motions in $\mathbb{R}^3$. It consists of all 4×4 matrices of the form

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \quad A \in SO(3), \; v \in \mathbb{R}^3.$$

Such a matrix sends the point x in $\mathbb{R}^3$ into $Ax + v$. This can be written as a matrix multiplication:

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + v \\ 1 \end{pmatrix}.$$

The inverse of

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \text{ is } \begin{pmatrix} A^{-1} & -A^{-1}v \\ 0 & 1 \end{pmatrix}$$

as can be checked by matrix multiplication. Let N denote the subgroup consisting of all translations. So N consists of all matrices of the form

$$\begin{pmatrix} I & w \\ 0 & 1 \end{pmatrix}$$

where I is the (3×3) identity matrix. Then

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A^{-1} & -A^{-1}v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & Aw \\ 0 & 1 \end{pmatrix}. \tag{8.1}$$

So N is a normal subgroup. Notice that there are two further special properties that hold for this example:

(i) N is Abelian.

(ii) There is a subgroup $H\left(\text{consisting of all } \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right)$ such that every element of G can be written uniquely as hn with $h \in H$ and $n \in N$.

Suppose that G is any group with a normal Abelian subgroup, N, and some other

subgroup H such that every element of G can be written in a unique way as hn, where $h \in H$ and $n \in N$. If G is finite, this implies that $\#G = \#H \cdot \#N$ and $H \cap N = \{e\}$. Conversely, if these last two conditions hold, then every element of G can be written uniquely as hn; indeed, we cannot have $hn = e$ unless $h = e$ and $n = e$, since $hn = e$ implies $h = n^{-1} \in H \cap N$. Therefore, $hn = h'n'$ implies $h = h'$ and $n = n'$ and so there are $\#H \cdot \#N$ distinct elements hn. Since $\#H \cdot \#N = \#G$, this implies that every element of G can be uniquely written in the desired form.

Examples

$G = D_n, N = C_n, H = \{e, r\}$, where $r \notin C_n$.

$G = S_4, N = \{e, a, b, c\}$, where $a = (12)(34)$,
$b = (13)(24)$, $c = (14)(23)$ and
$H \sim S_3 = \{e, (12), (13), (23), (123), (132)\}$.

We say that G is the *semidirect product* of H and N, and write $G = H \circledS N$.

The group H acts on N by conjugation:

$$h \text{ sends } n \text{ into } hnh^{-1} \in N.$$

For the case of the Euclidean group, (8.1) gives an explicit formula for this action. We can generalize the Euclidean example as follows: let H be any group and let us be given a representation τ of H on a vector space N. We let G consist of all pairs (h, n) and define multiplication by

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 + \tau(h_1)n_2). \tag{8.2}$$

(Of course, in writing (8.2) we are really writing the multiplication law for 'matrices'

$$\begin{pmatrix} h & n \\ 0 & 1 \end{pmatrix}$$

with h_1 acting on n_2 via τ.) Then

$$(h, n)^{-1} = (h^{-1}, -\tau(h^{-1})n)$$

and

$$(h, m)(I, n)(h, m)^{-1} = (I, \tau(h)n). \tag{8.3}$$

We can then identify N with the subgroup of G consisting of all (I, n), $n \in N$. It becomes a normal subgroup. We can identify H with the subgroup of G consisting of all $(h, 0)$, $h \in H$. Then it is immediate that G is the semidirect product of H and N. In this case, taking $m = 0$ in (8.3) shows that the conjugation action of H on N is just the representation τ.

A very important case for us is where $H = Sl(2, \mathbb{R})$, $N = \mathbb{R}^{1,3}$ and τ is the homomorphism from $Sl(2, \mathbb{R})$ to the Lorentz group defined at the very beginning of this book. The resulting semidirect product is then the *Poincaré group*.

Let us return to the finite case. Since N is Abelian, all of its irreducible representations are one dimensional. Let N^* denote the space of all one-dimensional characters on N. Now N^* is a set of functions on N. Since H acts on N, it also acts on the space of all functions, $\mathcal{F}(N)$ by the rule

$$hf(n) = f(n')$$

where n' is the image of n under h^{-1},
i.e.

$$hf(n) = f(h^{-1}nh).$$

In particular, if $\chi \in N^*$ is a character, so is $h\chi$, where

$$h\chi(n) = \chi(h^{-1}nh). \tag{8.4}$$

We will now show how to find all the irreducible representations of a semidirect product.

Suppose that G is the semidirect product of H and N. Let (ρ, V) be an irreducible representation of G. Decompose V under N:

$$V = V_{\chi_1} + \cdots + V_{\chi_k}$$

where

$$V_{\chi_j} = m_j W_j$$

and (σ_j, W_j) is the irreducible (one-dimensional) representation of N with character χ_j. Thus V_{χ_j} consists of those vectors $v \in V$ which satisfy

$$\rho(n)v = \chi_j(n)v.$$

For any $a \in G$ and $v \in V_{\chi_j}$

$$\begin{aligned} \rho(n)\rho(a)v &= \rho(a)\rho(a)^{-1}\rho(n)\rho(a)v \\ &= \rho(a)\rho(a^{-1}na)v \\ &= \rho(a)\chi_j(a^{-1}na)v \\ &= (a\chi_j)(n)\rho(a)v \end{aligned}$$

so

$$\rho(a)V_{\chi_j} = V_{a\chi_j}.$$

Thus, if $V_{\chi_j} \neq \{0\}$ and if N_j^* denotes the orbit of G acting on N^* containing χ_j then the subspace

$$V_{\chi_{i_1}} + \cdots + V_{\chi_{i_p}} \qquad \chi_{i_l} \in N_j^*$$

is an invariant subspace of V. Since V is irreducible, we conclude that

$$V = V_{\chi_{i_1}} + \cdots + V_{\chi_{i_p}} \qquad \chi_{i_l} \in N_j^*.$$

In other words $m_i = 0$ unless $\chi_i \in N_j^*$. If we think of V_{χ_i} as a vector space situated over

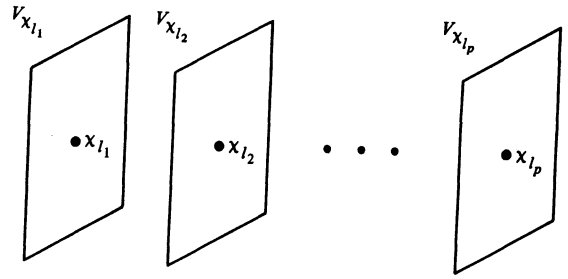


Fig. 3.11

$\chi_i \in N_j^*$, we have constructed a vector bundle E over N_j^* on which G acts (see Fig. 3.11). So we have

$$\begin{array}{c} E \\ \downarrow \\ N_j^* \end{array}$$

and (ρ, V) is equivalent to the action of G on $\Gamma(E)$.

Let G_j be the isotropy group of $\chi_j \in N_j^*$. Then G_j is the semidirect product of L_j and N, where $L_j \subset H$ is the isotropy group of H acting on N^* at χ_j. G_j is represented on V_{χ_j} and each element $n \in N$ acts on V_{χ_j} by $\chi_j(n)I$. In order for G acting on $\Gamma(E)$ to be irreducible, the group G_j must act irreducibly on V_{χ_j}. This means that L_j must act irreducibly on V_{χ_j}. We have thus shown that every irreducible representation of G is constructed as follows:

(1) Decompose N^* into orbits under G:

$$N^* = N_1^* \cup \cdots \cup N_r^*$$

and pick a $\chi_j \in N_j^*$ for each orbit.

(2) Let $L_j \subset H$ be the subgroup fixing χ_j and let $G_j = L_j \textcircled{s} N \subset G$. Choose an irreducible representation of L_j, say (ρ_j, V_j) and consider it as a representation of G_j by

$$\rho_j(hn)v = \chi_j(n)\rho(h)v.$$

(3) Construct the corresponding vector bundle E and let G act on $\Gamma(E)$.

(4) The space V_j can be identified with the δ-sections concentrated at χ_j and can be characterized as those sections s which satisfy $\rho(n)s = \chi_j(n)s$; in other words

$$V_j = (\Gamma(E))_{\chi_j}.$$

If we start with an irreducible representation (ρ_j, V_j) of L_j and construct $\Gamma(E)$, then $\Gamma(E)$ must be irreducible. In fact, if $\Gamma(E) = \Gamma(E)' + \Gamma(E)''$, then we must have

$$\Gamma(E)_{\chi_j} = \Gamma(E)'_{\chi_j} + \Gamma(E)''_{\chi_j}$$

as a decomposition into invariant subspaces under L_j. But $\Gamma(E)_{\chi_j} = V_j$ is irreducible.

Table 21.

	e	R_1	R_2	R_3
χ_0	1	1	1	1
χ_1	1	i	-1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	-1	i

Hence $\Gamma(E)''_{\chi_j} = \{0\}$ and hence $\Gamma(E)''_{\chi} = 0$ for all $\chi \in N_j^*$ so $\Gamma(E)'' = 0$. Thus $\Gamma(E)$ *is* irreducible.

Example The irreducible representations of D_4

Step 1. $D_4 = H \textcircled{s} N$, where $H = \{e, \alpha_1\}$ and $N = \{e, R_1, R_2, R_3\}$.

Step 2 Form N^*, the set of characters of N ($\cong C_4$) (see Table 21).

Step 3 Let H act on N^* by $h\chi_i(n) = \chi_i(h^{-1}nh)$ (see Table 22). The action breaks up N^* into three orbits as in Fig. 3.12.

Step 4 Pick an orbit and a point in the orbit. Find L_i, the isotropy subgroup, which fixed the point. For the single-element orbits $L_i = H$. For the double-element orbit $L_i = \{e\}$.

Step 5 Consider the two-element orbit and choose χ_1 to be identified with the coset N. Then χ_3 is identified with the coset $\alpha_1 N$. Construct a vector bundle over the two points by taking as basis elements $e_1 = [(e, v_0)]$ and $e_2 = [(\alpha_1, v_0)]$.

Step 6 Calculate representation matrices by letting $G = D_4$ act on basis elements. Use the rule $[(b, v_0)] \sim [(b, \sigma(l)v_0)]$, where $\sigma(l) = \chi_1(n)\rho(h)$ and $l = hn$. Since $H = \{e\}$, $\rho(h) \equiv 1$. So

$$\begin{aligned} R_1[(e, v_0)] &= [(R_1, v_0)] = [(eR_1, v_0)] = [(e, \sigma(R_1)v_0)] \\ &= [(e, \chi_1(R_1)v_0)] = [(e, iv_0)] = i[(e, v_0)] = ie_1 \end{aligned}$$

Fig. 3.12

Table 22.

	χ_0	χ_1	χ_2	χ_3
e	χ_0	χ_1	χ_2	χ_3
α_1	χ_0	χ_3	χ_2	χ_1

and

$$R_1[(\alpha_1, v_0)] = [(R_1\alpha_1, v_0)] = [(\alpha_1 R_3, v_0)] = [(\alpha_1, \sigma(R_3)v_0)]$$
$$= [(\alpha_1, \chi_1(R_3)v_0)] = [(\alpha_1, -iv_0)] = -ie_2.$$

Thus R_1 is represented by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

For α_1 we have

$$\alpha_1[(e, v_0)] = [(\alpha_1, v_0)] = e_2$$

and

$$\alpha_1[(\alpha_1, v_0)] = [(e, v_0)] = e_1.$$

So α_1 is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Since $R_1\alpha_1 = \beta_1$ we can, with these two matrices, recover all of the matrix representations of the group. So this orbit gave the two-dimensional representation of D_4.

Step 6' We apply Step 6 to each of the two remaining orbits. Suppose we consider the orbit with χ_2. Now $L_2 = H$ is not trivial and there are two representations of the little group as shown in Table 23.

The choice of using either ρ_1 or ρ_2 in $\sigma(l) = \chi_2(n)\rho_i(h)$ will give *two* distinct one-dimensional representations.

There is only one basis element now, $[(e, v_0)]$.

$$R_1[(e, v_0)] = [(eR_1, v_0)] = [(e, \chi_2(R_1)\rho_i(e)v_0)]$$
$$= -[(e, v_0)]$$

for both representations, since $\rho_i(e) = 1$ for $i = 1, 2$. However,

$$\alpha_1[(e, v_0)] = [(e\alpha_1, v_0)] = [(e, \chi_2(e)\rho_i(\alpha_1)v_0)]$$
$$= [(e, v_0)]$$

since $\rho_1(\alpha_1) = 1$ but $\rho_2(\alpha_1) = -1$. Hence our two representations are as shown in Table 24.

Again, these will generate the whole group, so we are done.

Table 23.

	e	α_1
ρ_1	1	1
ρ_2	1	-1

Table 24.

	R_1	α_1
3	-1	1
4	-1	-1

It remains to do the third orbit, which consists of χ_0. The little group is H, and R_1 is obviously represented by $+1$ since $\chi_0 \equiv 1$. For α, we have again

$$\alpha_1[(e, v_0)] = [(e, \rho_i(\alpha_1)v_0)] = \pm[(e, v_0)].$$

So the other two one-dimensional representations are as in Table 25.

This is summarized in Table 26, which of course checks with the known character table for D_4.

Let M be an orbit of H in N^*. Let $H_0 \in H$ be the isotropy group of $\chi_0 \in M$. Let (ρ, V) be a representation of H_0. For future use, let us write once again the action of various elements of G on $\Gamma(E)$, where E is the associated vector bundle: if $s \in \Gamma(E)$ and $n \in N$, then

$$(ns)(\chi) = \chi(n)s(\chi). \tag{8.5}$$

To evaluate the action of $h \in H$ on $\Gamma(E)$ in a convenient way, we first make some preliminary remarks. We know that the most general element of E can be written as an equivalence class $[(a, v)]$, where $a \in H$ and $v \in V$, cf. Section 3.3. Let us choose some 'cross section'

$$r: M \to H \text{ such that } r(\chi)\chi_0 = \chi.$$

We can write any section s of E as

$$s(\chi) = [(r(\chi), \psi(\chi))]$$

where $\psi: M \to V$. Then if $a \in H$, we have

$$(as)(\chi) = as(a^{-1}\chi)$$
$$= [(ar(a^{-1}\chi), \psi(a^{-1}\chi))]$$
$$= [(r(\chi)r(\chi)^{-1}ar(a^{-1}\chi), \psi(a^{-1}\chi))]$$
$$= [(r(\chi), \rho(r(\chi)^{-1}ar(a^{-1}\chi))\psi(a^{-1}\chi))].$$

Table 25.

	R_1	α_1
1	1	1
2	1	−1

Table 26.

	e	R_1, R_3	R_2	$\alpha_1, \alpha_2 = R_2\alpha_1$	$\beta_1 = R_1\alpha_1, \beta_2 = R_3\alpha_1$
1	1	1	1	1	1
2	1	1	1	−1	−1
3	1	−1	1	1	−1
4	1	−1	1	−1	1
5	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}, \begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}$

Thus, *as* corresponds to the function ψ_a, where

$$\psi_a(\chi) = \rho(r^{-1}(\chi)ar(a^{-1}\chi))\psi(a^{-1}\chi). \tag{8.6}$$

This rather formidable expression simplifies for elements of H_0 if r has nice properties. Suppose that

$$r(a\chi) = ar(\chi)a^{-1}, \quad a \in H_0 \tag{8.7}$$

i.e. that the section r is equivariant for the conjugation action of a on H. Then

$$r^{-1}(\chi)ar(a^{-1}\chi) = r^{-1}(\chi)aa^{-1}r(\chi)a = a$$

and (8.6) simplifies to

$$\psi_a(\chi) = \rho(a)\psi(a^{-1}\chi), \quad a \in H_0 \tag{8.8}$$

We recall from the end of Section 3.3 that if the representation, ρ, of H_0 extends to a representation τ of H, then we can identify $\Gamma(M)$ with $\mathscr{F}(M, V)$ and (the analogue of) (8.8) is true for all $a \in H$:

$$a\psi(\chi) = \tau(a)\psi(a^{-1}\chi). \tag{8.9}$$

3.9 Wigner's classification of the irreducible representation of the Poincaré group

Let us now see how to formulate the results of the preceding section for the case of groups like the Euclidean group or the Poincaré group. We first must decide how to define N^* and then describe the action of H on N^*. For the finite case, a character χ *must* take on values of absolute value one. For the infinite case, we must add this as an additional condition in the definition. We also require that χ be a continuous function. Thus N^* consists of all continuous functions $\chi: N \to \mathbb{C}$ such that $|\chi| \equiv 1$ and

$$\chi(n_1n_2) = \chi(n_1)\chi(n_2). \tag{9.1}$$

Now

$$n_1 = \begin{pmatrix}I & w_1\\0 & 1\end{pmatrix} \quad n_2 = \begin{pmatrix}I & w_2\\0 & 1\end{pmatrix}$$

$$n_1n_2 = \begin{pmatrix}I & w_1 + w_2\\0 & 1\end{pmatrix}, \quad w_L \in V,$$

where V is a finite-dimensional vector space on which H acts. So writing

$$\chi(w) \text{ instead of } \chi\begin{pmatrix}I & w\\0 & 1\end{pmatrix}$$

(9.1) becomes

$$\chi(w_1 + w_2) = \chi(w_1)\chi(w_2). \tag{9.2}$$

This implies that χ must be of the form χ_l where, for $l \in V^*$,

$$\chi_l(w) = e^{il(w)}.$$

It follows from (8.3) (with $m = 0$) that under the conjugation action of H on N, the element A^{-1} sends

$$n = \begin{pmatrix}I & w\\0 & 1\end{pmatrix} \text{ into } \begin{pmatrix}I & \tau(A^{-1})w\\0 & 1\end{pmatrix}.$$

So the action of H on N^* is given by

$$A\chi_l = \chi_{\tau(A)^{*-1}l}. \tag{9.3}$$

Suppose that the $\tau(A)$ preserve some non-degenerate scalar product on N. Then we can use this scalar product to identify N with N^*. Thus $l(w)$ becomes $l \cdot w$

$$\chi_l(w) = e^{il \cdot w}$$

and $\tau(A)^{*-1} = \tau(A)$ so (9.3) simplifies to

$$A\chi_l = \chi_{\tau(A)l}.$$

Up to now in this book we have only discussed representations of finite groups, not groups such as the Poincaré group. We will give some of the basic definitions

and theorems in the next chapter. Suffice it to say that we will require all our representations to be unitary and to be continuous (in the appropriate sense). For groups such as the Poincaré group or the Euclidean group, it turns out that the typical irreducible representation will be infinite dimensional. Nevertheless, it is true that the method of the preceding section applies to give all irreducible representations of a semidirect product. This was proved by Wigner for the case of the Poincaré group and generalized to the case of a general semidirect product by Mackey.

Of course, in order to describe the representations involved, we must replace the finite and discrete by the continuous. We give some indication of how to do this in the next chapter. For the moment, let us follow the prescription of the preceding section if we want to label the representations. We must

(i) describe the orbits of $Sl(2,\mathbb{C})$ acting on $\mathbb{R}^{1,3}$;

(ii) pick a point on each orbit and find its isotropy group;

(iii) list all the representations of the groups in (ii).

Recall from Chapter 1 that an $A \in Sl(2,\mathbb{C})$ acts on

$$P = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}$$

by

$$\tau(A)P = APA^*,$$

and that

$$m^2 = \det P = p_0^2 - p_1^2 - p_2^2 - p_3^2$$

is preserved. Thus each orbit must be contained in a level set of m^2. For $m^2 > 0$, the hyperboloid $\det P = m^2$ has two sheets corresponding to $p_0 > 0$ and $p_0 < 0$ (see Fig. 3.13). For $m^2 = 0$, we have the forward 'light cone' with $p_0 > 0$, the backward light cone with $p_0 < 0$ and the isolated point orbit consisting of the origin alone. For $m^2 < 0$, the hyperboloids $\det P = m^2$ are connected and are hence orbits. We therefore get the table of orbits, Table 27, where we have indicated a representative point on each orbit.

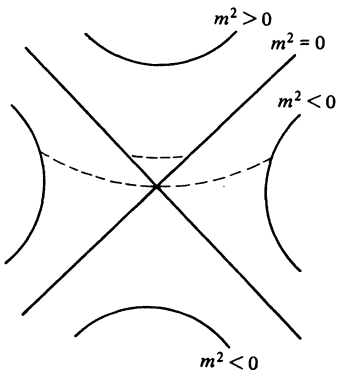


Fig. 3.13

Table 27.

Orbit	Representative point
$m^2 > 0, p_0 > 0$	$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$
$m^2 > 0, p_0 < 0$	$\begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}$
$m^2 = 0, p_0 > 0$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$
$m^2 = 0, p_0 < 0$	$\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$
$\{0\}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$m^2 < 0$	$\begin{pmatrix} 0 & \lvert m\rvert i \\ -\lvert m\rvert i & 0 \end{pmatrix}$

For each of these types, let us compute the isotropy group of the representative point.

(i) $$A\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}A^* = AA^*\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

so the condition that

$$A\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}A^* = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

is the same as $AA^* = I$ so the isotropy group for the first two cases is $SU(2)$. Let us now turn to case (ii).

(ii) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 2|a|^2 & 2a\bar{c} \\ 2c\bar{a} & 2|c|^2 \end{pmatrix}$$

so

$$A\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}A^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

holds if and only if

$$|a|^2 = 1 \text{ and } c = 0.$$

We are thus dealing with the group of all matrices of the form

$$\begin{pmatrix} e^{i\theta} & b \\ 0 & e^{-i\theta} \end{pmatrix}.$$

The product of two such matrices is given by

$$\begin{pmatrix} e^{i\theta} & b \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} e^{i\theta'} & b' \\ 0 & e^{-i\theta'} \end{pmatrix}=\begin{pmatrix} e^{i(\theta+\theta')} & e^{i\theta}b'+be^{-i\theta'} \\ 0 & e^{-i(\theta+\theta')} \end{pmatrix}$$

and we have the conjugation

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}=\begin{pmatrix} 1 & be^{2i\theta} \\ 0 & 1 \end{pmatrix}.$$

If we write $b = x + iy$, we see that the isotropy group in question is just the semidirect product of the circle $SO(2)$ and the plane, with θ acting as rotation through angle 2θ. Thus the group in question is the double cover (on account of the factor 2) of the group of Euclidean motions. Let us denote this group by $\tilde{E}(2)$. The same computation gives $\tilde{E}(2)$ as the isotropy group of the point $\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$. The isotropy group of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is clearly the whole of $Sl(2, \mathbb{C})$. So the remaining case (iii) is $\begin{pmatrix} 0 & |m|i \\ -|m|i & 0 \end{pmatrix}$.

(iii) The set of A satisfying

$$A\begin{pmatrix} 0 & |m|i \\ -|m|i & 0 \end{pmatrix}A^*=\begin{pmatrix} 0 & |m|i \\ -|m|i & 0 \end{pmatrix}$$

is clearly the same as the set of A satisfying

$$A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A^*=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $A^t=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ denote the transpose of $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A^t=\begin{pmatrix} 0 & ad-bc \\ -(ad-bc) & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus we must have

$$A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A^*=A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A^t$$

which can happen if and only if

$$A^*=A^t,$$

i.e. all the entries of A are real. Thus the isotropy group in this case is $Sl(2.\mathbb{R})$.

In the physics literature, the isotropy group of a representative point on an orbit is called the 'little group' (of the point or of the orbit). We can thus complete Table 27 to obtain Table 28.

For the next step, we must classify the irreducible representations of the little groups. We will compute all the irreducible representations of $SU(2)$ in the next chapter. We will find that they are parametrized by a number s which can take on

Table 28.

Orbit	Representative point	Little group
$m^2>0, p_0>0$	$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$	$SU(2)$
$m^2>0, p_0<0$	$\begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}$	$SU(2)$
$m^2=0, p_0>0$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\tilde{E}(2)$
$m^2=0, p_0<0$	$\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$	$\tilde{E}(2)$
$\{0\}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$Sl(2, \mathbb{C})$
$m^2<0$	$\begin{pmatrix} 0 & \|m\|i \\ -\|m\|i & 0 \end{pmatrix}$	$Sl(2, \mathbb{R})$

non-negative half integer values:

$$s=0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots.$$

Thus, corresponding to our first family of orbits, the representations are parametrized by a continuous parameter $m^2>0$ and the discrete parameter s. In physical terms, m is the 'rest mass'

$$m^2=p_0^2-p_1^2-p_2^2-p_3^2$$

and s can be identified with the 'intrinsic spin' of a point particle. We shall discuss its meaning and implication in the next chapter.

The group $E(2)$ is itself a semidirect product, so we can compute its representations by the same method. The orbits of $SO(2)$ acting on $\mathbb{R}^2$ are circles and the one-point orbit consisting of the origin. It turns out that the representations corresponding to the circles do not arise in physics. The little group of $\{0\}$ is $SO(2)$. Its irreducible unitary representations are all one dimensional and are given by

$$\theta \to e^{in\theta}. \quad n=0, \pm 1, \pm 2, \ldots.$$

Since θ corresponds to rotation by 2θ it is usual to write $\theta=\psi/2$, $0 \leqslant \psi \leqslant 4\pi$ and $n=2s$ so the irreducibles corresponding to the orbit $\{0\}$ are labeled by

$$s=0, \pm\tfrac{1}{2}, \pm 1, \ldots.$$

For $m^2<0$, we must list all the irreducible unitary representations of $Sl(2, \mathbb{R})$.

Outside of the trivial representation these are all infinite dimensional and their description lies outside the scope of this book. Fortunately, no particles are known to exist with $m^2 < 0$. (They would be 'tachyons' travelling faster than the speed of light, although some recent theories demand their existence.)

Thus, the physically relevant representations of $Sl(2,\mathbb{C}) \circledS \cdot \mathbb{R}^{1,3}$ with $p_0 > 0$ are parametrized by

$$m^2 > 0 \quad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

and

$$m^2 = 0 \quad s = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots$$

It is time to pause and consider the significance of this result, which was proved by Wigner some 50 years ago.

Speculation as to the ultimate constituents of matter occurs in the earliest scientific thought. The concept of an element as a 'substance from which all other bodies are made or derived' was held at the very beginning of occidental philosophy. Thales of Miletus (about 640–547 Before the Common Era) regarded 'water' as the basis of all things. (His ideas may have been influenced by still earlier doctrines having their origin in ancient Egypt.) His followers accepted his idea of a primordial substance as the basis of all bodies, but they endeavored to determine some other general element such as 'fire' or 'spirit' or 'love' or 'hatred'. Around 500 BCE Heraclitus suggested that the fundamental elements were earth, air, fire and water, and that none could be obtained from one another by physical means. A little later Democritus (460–370 BCE) claimed that matter was composed of many different kinds of minute hard particles or 'atoms', and similar ideas were put forward by Epicurus (341–276 BCE). Aristotle (384–322 BCE) in his *Physics*, adopted the point of view of Heraclitus. Furthermore, he attributed the motions of matter to the tendency of the constituent elements to return to their 'natural' place.

In any attempt to classify the elementary constituents of matter we must come to grips with two problems: what is the meaning of the word 'elementary' and when should two objects be regarded as the 'same'? The second of these questions immediately brings into consideration the theory of groups, and, in particular, those groups which are intimately related to our conception of space. For example, we may wish to regard a particle as unchanged if we pick it up and place it down at some other location in a horizontal plane. Thus, the location of a particle in a plane is not an 'intrinsic' property of a particle; the group of symmetries of the Euclidean plane enters as a 'group of symmetries' of the classification problem. It would not make sense to talk of the 'natural' horizontal location of a particle. In a geocentric conception of the universe, the 'natural' vertical location of a particle (or, more precisely, its 'natural' distance from the center of the earth) makes perfectly good sense. Hence, in Aristotle's theory, one could conceive that each element had its own (vertical) 'natural' place. With the overthrow of the geocentric conception, and with the advent of Newtonian physics, we tend to believe, in some form or other, in the isotropy of space and can no longer single out a vertical or radial direction as being preferred. Similarly, our conceptions about space time will change the kinds of properties that can be allowed for the

description of particles. We regard two freely moving bodies which differ only in their kinetic energy as being the 'same'. This reflects the fact that we do not believe in the existence of an 'absolute rest frame'. By a geometrically admissible transformation of space time (either in Galilean relativity or in Einstein's special relativity) we can arrange that a freely moving massive body be at rest. It is thus clear that in any theory of classification of elementary particles there must be some group, G, in the background of our thoughts, and that this group should contain the group associated to our geometric conceptions of space and time. Of course, the group might be larger, reflecting symmetries that go beyond geometry. Needless to say, our conceptions about the geometry of space and time are strongly influenced by our theory of the nature of matter.

The first problem, that of defining the meaning of the word 'elementary' is much more difficult to discuss at this level of generality, without specifying a more detailed choice of theory. The way in which objects are 'put together' will tend to vary from one theory to another. Nevertheless, a brief glance at some examples might prove instructive even at this stage. If we believe in the undulatory theory of light, then, in an intuitive sense that can be made quite precise, the most 'elementary' kind of wave is a plane wave. (Here the elementary waves are 'put together' by Fourier analysis.) Now a plane wave cannot be localized in space. If we try by physical means (say a screen with an aperture) to localize (a physical approximation to) a plane wave, we find that the localized wave is much more complicated than the plane wave we started with, due to diffraction effects. We learn two lessons: our 'particles' may tend to be somewhat ideal, and we may have to give up on the notion of localizability in space. A similar situation obtains in currently popular theories which are based on 'relativistic quantum mechanics'. In these theories it requires a great deal of energy to confine a particle in a small region of space, and when these energies become sufficiently great, there is spontaneous production of new particles. Thus we may have to be more Aristotelian than Democritean in our outlook.

All of the recent theories of elementary particles have been shaped by the paper by Wigner, containing the classification of the irreducible representations of $Sl(2,\mathbb{R}) \circledS \mathbb{R}^{1,3}$ described above. It is difficult to overestimate the importance of this paper, which will certainly stand as one of the great intellectual achievements of our century. It has not only provided a framework for the physical search for elementary particles, but has also had a profound influence on the development of modern mathematics, in particular the theory of group representations. From our point of view, we can summarize Wigner's main points as follows. The logic of physics is quantum mechanics. Hence, a symmetry group of the system manifests itself as a unitary (or possibly anti-unitary) representation. Ignoring the anti-unitaries (for example, by considering connected groups), one posits that

(i) an elementary particle 'is' an irreducible unitary representation of the group, G of physics, where these representations are required to satisfy certain physically reasonable restrictions, and where

(ii) the group G of physics is the group $Sl(2,\mathbb{C}) \circledS \mathbb{R}^{1,3}$, the semidirect product of $Sl(2,\mathbb{C})$ and the translation group in Minkowski space, i.e. the group G is the double

(universal) covering of the Poincaré group, P, where P denotes the group of those transformations of special relativity which can be continuously deformed to the identity.

Wigner computes all the irreducible representations of G and finds, as we have seen, that

(iii) the representations satisfying the physical conditions can be parametrized by two parameters m and s, where m is a non-negative real number and where s is constrained by

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \text{ etc. if } m \text{ is positive}$$

and

$$s = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \text{ etc. if } m = 0.$$

Here the parameter m can be identified with the rest mass of the particle, and the parameter s can be identified with its spin (or spin and helicity if $m = 0$). The spin is a notion related to the 'intrinsic angular momentum' of the particle.

Wigner also considers the minor modification in this scheme that has to be made if we replace the connected group of symmetries of special relativity by disconnected groups. (The full groups have four components, corresponding to the possibilities of space inversion, time inversion, and their product. The current belief is that the correct group has two components corresponding to simultaneous space and time inversion, but that this transformation must be accompanied by reversal of all electric charges.)

In the past 50 years or so since Wigner's work various modifications have been introduced. First of all, even at the time of publication of Wigner's article, the scheme was not completely satisfactory in so far as, although it explained the observed spins, it failed to explain why only certain mass values were observed and not others. Also, it did not deal with electric charge, a clearly conserved property of observed particles on the same footing as mass and spin. In the intervening years various other invariant 'quantum numbers' of particles have been registered, such as isotopic spin, strangeness, etc. We will discuss these in Chapters 4 and 5. The main thrust of recent theories (or at least most of them) has been to modify (ii) by enlarging the group G. The most successful of such theories, involving quarks and electroweak unification, indicates that the group $SU(3)$ should be contained, in some way, in the group G. Recent speculation concerning the so-called 'supersymmetries' suggests the notion of a group might have to be slightly enlarged. However, point (i) in the dogma remains unchanged. This is because in quantum mechanics the 'state' of a system is a unit vector in Hilbert space and any symmetry manifests itself as a unitary (or anti-unitary) transformation of this Hilbert space. To the extent that we accept quantum mechanics as providing the underlying foundational theory of physics, we cannot make major changes in (i). We shall enlarge on these points in the next two chapters.

3.10 Parity

One issue that is convenient to discuss here is that of parity: do the basic laws of physics distinguish between right and left handedness? Put another way, let $\mathscr{P}$ denote

the operator of spatial inversion:

$$\mathscr{P} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}.$$

In terms of 2×2 matrices, the operation $\mathscr{P}$ has a very simple expression. For any matrix

$$M = \begin{pmatrix} c & d \\ f & g \end{pmatrix}$$

let M^a denote its 'cofactor matrix'

$$M^a = \begin{pmatrix} g & -d \\ -f & c \end{pmatrix}$$

so

$$MM^a = (\det M)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$(M_1 M_2)^a = M_2^a M_1^a$$

for any two matrices M_1 and M_2. Now if

$$X = \begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix}$$

then

$$X^a = \begin{pmatrix} x_0 + x_3 & -x_1 - ix_2 \\ -x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

So we see that when we represent points of Minkowski space as self-adjoint 2×2 matrices, we have

$$\mathscr{P}(X) = X^a.$$

Notice that $A^a = A^{-1}$ if $\det A = 1$, so

$$\begin{aligned} \mathscr{P}(AXA^*) &= (A^*)^a X^a A^a \\ &= A^{*-1} X^a A^{-1} \end{aligned}$$

for $A \in Sl(2, \mathbb{C})$.

Notice that if the group H in our semidirect $H \circledS N$ product is enlarged from $H = Sl(2, \mathbb{C})$ so as to include $\mathscr{P}$, then we see from the preceding equation that

$$\mathscr{P}A = A^{*-1}\mathscr{P}$$

or

$$(\mathscr{P}A\mathscr{P}^{-1})X = A^{*-1}X \tag{10.1}$$

for $A \in Sl(2, \mathbb{C})$ and X self-adjoint.

Similarly, are the laws of physics invariant under the time reversal transformation $\mathscr{T}$ given by

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} ?$$

If the laws of nature are invariant under both parity and time reversal, then they are also invariant under their product $\mathscr{P}\mathscr{T}$. This would mean that we would be replacing the connected component of the Lorentz group by all four components. So we are taking up the question raised in the preceding section as to the number of connected components of the Lorentz group that occur as symmetries of the laws of physics.

If we include more components in the Lorentz group, this means that the group H in our semidirect product, $H \circledS N$, must be enlarged from $Sl(2,\mathbb{C})$ to a group whose action on space time includes various components of the Lorentz group. We would then be interested in the representations of $H \circledS N$. So our first question should be – what are the possibilities for H? This is a rather subtle question. We shall return to it at the end of this section. For the moment we will investigate some consequences of an H whose action on space time includes $\mathscr{P}$ and explain how this led to the remarkable discovery in 1956 that parity is *not* a symmetry of nature.

Since at the moment we are interested only in $\mathscr{P}$, we may restrict to a subgroup if necessary and assume that the action of H on Minkowski space has exactly two components. That is, if we let π denote the homomorphism

$$\pi: H \to Lor$$

of H into the Lorentz group, we are assuming that $\pi(H)$ has two components, the non-identity component containing $\mathscr{P}$. We also assume that π is two-to-one, so the identity component of H is $Sl(2,\mathbb{C})$. Let $\mathscr{Q}$ be an element of H such that

$$\pi(\mathscr{Q}) = \mathscr{P}.$$

It follows from (10.1) that

$$\pi(\mathscr{Q}A\mathscr{Q}^{-1}) = \pi(A^{*-1})$$

so either

$$\mathscr{Q}A\mathscr{Q}^{-1} = A^{*-1}$$

or

$$\mathscr{Q}A\mathscr{Q}^{-1} = -A^{*-1}$$

for all $A \in Sl(2,\mathbb{C})$. Since $Sl(2,\mathbb{C})$ is connected and conjugation in any group leaves the identity fixed, we conclude that

$$\mathscr{Q}A\mathscr{Q}^{-1} = A^{*-1}, \text{ for all } A \in Sl(2,\mathbb{C}). \tag{10.2}$$

Since $\pi(\mathscr{Q}^2) = \mathscr{P}^2 = \mathrm{id}$ we conclude that either

$$\mathscr{Q}^2 = I \tag{10.3a}$$

or

$$\mathscr{Q}^2 = -I. \tag{10.3b}$$

Both of these possibilities exist for very natural choices of the group H, as we shall see at the end of this section, so we have to take both of them into account.

Let us now look at the irreducible representations of $H \circledS N$ by the orbit method for semidirect products. Each of the orbits that we computed in the preceding section for the group $Sl(2,\mathbb{C})$ is mapped into itself by $\mathscr{P}$ as can be checked directly. Hence the orbits for H are the same as for $Sl(2,\mathbb{C})$; what changes is the isotropy group. (This would not be the case if we included all four components, since time reversal interchanges the two sheets of the double sheeted hyperboloid, for example.)

So let us look at the orbits. The cases $m^2 > 0$ and $m^2 = 0$ behave differently. For $m^2 > 0$ the transformation $\mathscr{P}$ preserves the representative point

$$ml = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}.$$

Furthermore, $A^{*-1} = A$ for $A \in SU(2)$. Thus the isotropy group contains $\mathscr{Q}$ and $SU(2)$ and $\mathscr{Q}$ commutes with all elements of $SU(2)$. There are thus two possibilities. In *case (a)* the isotropy group is the direct product

$$\mathbb{Z}_2 \times SU(2) \text{ generated by } \mathscr{Q} \text{ and } SU(2) \text{ with } \mathscr{Q}A = A\mathscr{Q}, A \in SU(2)\ \mathscr{Q}^2 = I. \tag{10.4a}$$

In *case (b)* it is not the direct product but the

$$\text{group generated by } \mathscr{Q} \text{ and } SU(2) \text{ with } \mathscr{Q}A = A\mathscr{Q}, A \in SU(2)\ \mathscr{Q}^2 = -I. \tag{10.4b}$$

We must now investigate the irreducible representations of these isotropy groups. Since $\mathscr{Q}$ lies in the center of the isotropy group in both cases, it must be represented by a scalar matrix. In case (a) this scalar value must be ± 1 and either possibility can occur. In other words, the labels of the irreducible representations are now $(s, \pm)$. The symbol $\pm$ is called the *intrinsic parity* of the representation. Thus $(1, -)$ labels a representation of spin one and negative intrinsic parity. Let us now examine case (b). If s is an integer, then $-I$ is represented by the identity matrix and it follows from (10.4*b*) that the scalar representing $\mathscr{Q}$ is ± 1. On the other hand, if s is not integral, then $-I$ is represented by $-\mathrm{id}$ and hence $\mathscr{Q}$ is represented by $\pm \mathrm{i}$. So in case (b) the representations are also labeled by $(s, \pm)$ but $\pm$ means ± 1 for integer s and $\pm \mathrm{i}$ for half-integer s. If the group of nature includes the parity transformation, then every positive-mass particle would have its own intrinsic parity, and total parity would be conserved in all interactions. But the meaning of this total parity conservation would be different for fermions, according to whether case (a) or (b) holds. Since $(\pm 1)^3 = \pm 1$, whereas $(\pm \mathrm{i})^3 = -(\pm \mathrm{i})$, cases (a) and (b) are really different. Assuming that there is a fermion with no other conserved quantum numbers, then a decay of one particle into three would be permitted in case (a) but not in case (b).

Let us now turn to the mass-zero orbits. The transformation $\mathscr{P}$ does not preserve the point $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$; indeed

$$\mathscr{P}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Let $U=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then the operator $\pi(U)\mathscr{P}$ does preserve $\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$ since

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}.$$

A direct computation shows that $\pi(U)\mathscr{P}$ acts as reflection about the line through $\mathbf{e}_1$ in the $\mathbf{e}_1$, $\mathbf{e}_2$ plane. So the isotropy group is now generated by $\tilde{E}(2)$ and the element $U(2)$. Let us define $\mathscr{R}=U\mathscr{Q}$. A direct check using (10.2) shows that conjugation by $U(2)$ sends every element of $\tilde{E}(2)$ into its complex conjugate. Furthermore, U commutes with $\mathscr{Q}$ since U is unitary. Once again we must consider two cases. In case (a)

$$\mathscr{R}^2=U^2\mathscr{Q}^2=-I \tag{10.5a}$$

so the isotropy group is a twisted form of double cover of the group of Euclidean motions. In case (b) we have

$$\mathscr{R}^2=U^2\mathscr{Q}^2=I. \tag{10.5b}$$

Notice that the roles of cases (a) and (b) are reversed here. Case (b) is now the direct product $\mathbb{Z}_2$ with $\tilde{E}(2)$. In both cases the isotropy groups are semidirect products of the group of translations in the plane with a group acting on the plane linearly. As in the preceding section, we will, for 'physical grounds', only be interested in those representations of the 'little group' which correspond to zero orbit in the plane. The 'little little group', i.e. the subgroup of the isotropy group which fixes the origin, is either (case b)

$$D_\infty=\mathbb{Z}_2\circledS U(1)$$

$$\text{generated by } \mathscr{R} \text{ and } e^{i\theta}\in U(1) \text{ with } \mathscr{R}e^{i\theta}\mathscr{R}^{-1}=e^{-i\theta} \text{ and } \mathscr{R}^2=1 \tag{10.6b}$$

or (case a)

$$\text{generated by } \mathscr{R} \text{ and } e^{i\theta}\in U(1) \text{ with } \mathscr{R}e^{i\theta}\mathscr{R}^{-1}=e^{-i\theta} \text{ and } \mathscr{R}^2=-1. \tag{10.6a}$$

The group D_∞ is itself a semidirect product so we can use the Wigner–Mackey technique a third time. In fact D_∞ is a continuous analogue of the dihedral groups; it is the group of symmetries of a two sided disk. The characters of $U(1)$ are the integers (which we label as $2s$, where $s=0,\frac{1}{2},-\frac{1}{2},1,-1$, etc.) and the element $\mathscr{R}$ sends s into $-s$. So, just as in the case of the odd dihedral groups, there is a family of two-dimensional representations parametrized by $j=\frac{1}{2},1,\frac{3}{2}$, etc. And there are two one-dimensional representations corresponding to $s=0$. The two-dimensional representations split into the sum of the one-dimensional $s=j$ and $s=-j$ representations. The story for case (a) is much the same, except that there will be four one-dimensional representations instead of two. In any event we see that if the group of nature does *not* contain the parity transformation then the $m=0, s\neq 0$ particles come in 'right handed' ($s>0$) and 'left handed' ($s<0$) versions. If parity is a symmetry then it interchanges this handedness and we must combine the $\pm j$ representations to get an irreducible of the full group. Let us summarize (in a form common to both cases (a) and (b)) the effects of the presence or absence of parity as a symmetry of nature in Table 29.

Table 29.

	For $m^2>0$	For $m^2=0$
If the laws of nature are invariant under $\mathscr{P}$	There is a property called 'intrinsic parity' conserved under interactions	Right handedness and left handedness makes no sense: the $s\neq 0$ irreducible representations of the little group are two dimensional
If the laws of nature are not invariant under $\mathscr{P}$	There is no intrinsic parity	The $m=0$, $s\neq 0$ particles come in right- and left-handed versions. The irreducible representations of the little group are all one dimensional

Until 1956 it was a commonly held dogma that the laws of nature should be invariant under $\mathscr{P}$. Thus, values of intrinsic parity were assigned to various particles, and corresponding selection rules were observed for many interactions. But it was then observed that while *some* interactions preserved parity, others did not. Whether we want to include $\mathscr{P}$ or not depends on the type of interaction.

In 1956, Madame Wu (after the suggestion of Lee and Yang) performed a crucial experiment proving that radioactive decay does *not* conserve parity. One such experiment is the following. Some cobalt 60 is placed inside a coil of wire. The Co^{60} nucleus is unstable – one of its neutrons decays into a proton emitting an electron and an anti-neutrino; this is known as β-decay. When an electric current is passed through the coil, more electrons are emitted in one direction perpendicular to the area element of the coil than in the other (Fig. 3.14). Reversing the direction of the current changes the preferred direction. Now an area element such as described by current can be related to a direction (i.e. a vector) only through a choice of orientation of space. (In more mathematical language, an area element is an element of $\Lambda^2(\mathbb{R}^3)$ while a vector is an element of $\mathbb{R}^3$. We have seen that these spaces are equivalent under the action of $SO(3)$ but *not* of $O(3)$.) Thus parity is not conserved in β-decay.

This experiment could, in principle, have been performed any time after the discovery of β-decay. In fact, it was not done because everyone took for granted on

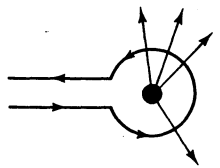


Fig. 3.14

some *a priori* grounds that parity was conserved. The conservation of parity was called into question by a bold suggestion of Lee and Yang to resolve the so-called 'θ-τ puzzle' as follows. In the early 1950s, new particles were discovered which decay, with a lifetime of about 10^{-10} s, into pions. Some of these particles decayed into two pions and were called θ-particles. Others decayed into three pions and were called τ-particles. Now, previous work on the pions indicated that they were particles of spin 0 and negative intrinsic parity. Hence, two pions (the tensor product of two pions) would have positive intrinsic parity, while three would have negative intrinsic parity again. Thus, assuming conservation of parity, the θ-particle would have positive intrinsic parity while the τ-particle would have negative intrinsic parity. However, in all other respects, the θ and τ seemed exactly alike, in particular they had exactly the same mass. The bold suggestion of Lee and Yang was that perhaps they were the same particle, but that parity is not conserved in this type of decay. In fact, it is now generally believed that parity is not conserved in the 'weak interactions'.

Although parity is not conserved in β-decay, there *is* a discrete symmetry that is conserved. Classical electromagnetism is invariant under a change of sign of all electric charges. The fields reverse their signs, but there is a compensating change of sign in the Lorenz law so the forces are unchanged. Suppose there is an operation C which maps each type of particle into its anti-particle and which has the effect of changing the sign of all 'internal quantum numbers' such as electric charge. (The notion of an anti-particle was one of the more startling predictions of Dirac's theory of the electron in the late 1920s.) Then symmetry is restored in the Wu experiment if we apply CP instead of P.

Experiments done in 1964 by Fitch and Cronin seem to indicate that CP is not conserved. I do not fully understand the issues involved in the correct interpretation of this experiment, which clearly shows that CP and CPT are not *both* conserved. It follows from the locality axioms, that quantum field theory implies that CPT is a symmetry of nature, and hence that CP is violated in the Fitch–Cronin experiment. But other, group theoretical, hypotheses might favor CP. I have my own views on the subject, which I will not expand on in this book.

Perhaps we should pause here to introduce some terminology. There are four fundamental types of forces or 'interactions'.

(i) The 'nuclear force' or *strong interaction* is the force that binds the atomic nucleus together. It is obviously much stronger than the electromagnetic force because it overcomes the mutual electrical repulsion of the protons. In fact, it is about 100 times as strong as the electromagnetic force. It is also of short range, acting only over distances of order 10^{-13} cm. This accounts for the instability of larger nuclei, since the nuclear force acts only on nearest neighbors while the electrical repulsion acts between all pairs of protons. The time scale of a strong interaction is, roughly, the time it takes to traverse 10^{-13} cm with the speed of light, i.e. about 10^{-23} s.

(ii) Electromagnetic forces.

(iii) The *weak interaction* responsible for radioactive β-decay. They are 10^{-12} times as strong as the electromagnetic interaction. They are also short range and their typical time scale is about 10^{-10} s.

(iv) Gravitational forces. These are long range but only 10^{-35} times as strong as the electromagnetic interactions. Due to the cancellation of positive and negative charges, and the absence of such cancellation in gravity, it is the cumulative effect of gravity that we experience in everyday life. But, on the scale of elementary particles, it can be ignored.

Not all particles participate in all forces. Particles which participate in the strong interactions are called *hadrons*. Those hadrons which have half integral spin, $s = \frac{1}{2}, \frac{3}{2}, \ldots$ are called *baryons*. Those with integer spin $s = 0, 1, 2, \ldots$ are called *mesons*. Thus the nucleons p and n are baryons while the pions π^+, π^0, π^- are mesons. The hadrons also engage in the electromagnetic and weak interactions. Particles of spin $\frac{1}{2}$ which have weak but not strong interactions are called *leptons*. Examples are the electron and the neutrino. The photon is a mass 0, $s = \pm 1$ particle.

Let us now return to the discussion of some of the issues involved in considering the discrete symmetry groups of space time. Wigner's theorem, as we explained in Section 3.6, asserts that every symmetry of a quantum system can be realized by a unitary or an anti-unitary operator, and that this operator is determined only up to multiplication by a scalar phase factor. The product of two anti-unitaries is a unitary. So as long as we considered connected groups, where every element is the square of some other, no anti-unitaries can arise. In fact, Wigner chose to represent the time reversal symmetry by an anti-unitary transformation, and this choice has been accepted in the physics literature for the past 60 years. The mathematical reasons for this choice are quite persuasive: $\mathscr{T}$ carries the 'forward mass shell', the orbit $m^2 > 0$, $p_0 > 0$, onto the 'backward mass shell' $m^2 > 0$, $p_0 < 0$, and vice versa. Hence, if $\mathscr{T}$ were represented by a unitary operator, we would have to include representations corresponding to both mass shells if we want to get an irreducible representation of the enlarged group. Although this is not an insuperable difficulty, it can be avoided if we represent $\mathscr{T}$ by an anti-unitary transformation. This is because the representation of spin s corresponding to the backward mass shell is anti-unitarily equivalent to the forward one.

A second problem relates to the issue of ordinary versus projective transformations. Even if we only had to deal with unitary transformations, Wigner's theorem only guarantees that we have a projective representation of the group. That is, we can assign to each group element a an operator $\rho(a)$ which is only determined up to phase, and

$$\rho(ab) = c(a, b)\rho(a)\rho(b).$$

The function c is called a *multiplier* of the representation. It depends, to some extent, on how we choose ρ. If we replace $\rho(a)$ by $f(a)\rho(a)$, where f is some complex-valued function on the group with $|f| = 1$, then clearly c will be replaced by d, where

$$d(a, b) = c(a, b)f(a)f(b)/f(ab).$$

Now it is a theorem that for the group $Sl(2, \mathbb{C}) \circledS \mathbb{R}^{1,3}$ we can choose the operators in any projective representation so that the c is identically one, i.e. every projective representation is given by an ordinary representation. Hence our classification of the ordinary irreducible unitary representations of $Sl(2, \mathbb{C}) \circledS \mathbb{R}^{1,3}$ is, in fact, a classification of its projective irreducible representations as well. In fact, in Wigner's paper on the

classification of the representations of the Poincaré group he classifies the irreducible projective representations of $Lor \circledS \mathbb{R}^{1,3}$ and allows for disconnected components.

For projective representations, however, a knowledge of the irreducible components (as projective representations) does not determine the representation. The reason for this is quite easy to see by example. Since all one-dimensional unitary transformations are just multiplications by a phase factor, and since we are allowed, in a projective representation, to change any operator by multiplication by a phase factor, it follows that all one-dimensional projective representations of a group G are the same. But now suppose that we are given a projective representation on a space which is a direct sum of a number (say even two) of one-dimensional invariant subspaces. We are now allowed only one overall phase factor for each group element, not a separate phase factor for each one-dimensional subspace. Hence two such representations on spaces of the same dimension need not be equivalent. In terms of our physical problem, a knowledge of the irreducible projective representations of the Poincaré group is insufficient once we want to consider interactions between particles, which will involve a direct sum of the irreducible representations.

The theory of projective representations (for finite groups and complex linear representations) was completely worked out by Schur in two astounding papers in 1904 and 1907, both in *J. für Math.* Each projective representation of a group G comes equipped with its own multiplier, and two projectively equivalent representations have multipliers differing in (multiplication by) a 'trivial multiplier'. The equivalence classes of multipliers, mod 'trivial multipliers', is what we call today $H^2(G, \mathbb{C}^*)$, the second cohomology of G with values in the non-zero complex numbers. Schur called this group $M(G)$ (the group of Multiplikatoren); we shall stick to Schur's notation to avoid confusion with topological cohomology groups. Thus, for example, a computation shows that

$$M(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2.$$

This computation will be important for us because we will want to think of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as the group consisting of $I, \mathscr{P}, \mathscr{T}$ and $\mathscr{P}\mathscr{T}$. Schur shows that for any group G we can find a central extension, that is a group G^* and a homomorphism of $G^* \to G$ with kernel A lying in the center of G; we write this as

$$1 \to A \to G^* \to G \to 1,$$

such that any projective representation of G is projectively equivalent to one that can be lifted to an ordinary representation of G^* with the property that *the elements of A are represented by scalar operators.* Conversely any representation of G^* which satisfies the italicized condition clearly descends to give a projective representation of G. If G^* is as small as possible (essentially this means that A has the same size as $M(G)$) then Schur calls G^* a 'representation group' of G. The reason is that the study of projective representations of G can be reduced to the study of certain ordinary representations of G^*, namely those for which the italicized condition holds. (It holds automatically for irreducible representations by Schur's lemma.) But the representation group is not uniquely determined. This is again a homological question completely treated by Schur. For the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ we have the two possibilities for the group

G^* which must be of order eight:

$$1 \to \mathbb{Z}_2 \to G_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

and

$$1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

where G_2 is the quaternionic group $G_2 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ of unit quaternions and D_4 is the dihedral group of order eight. In a sense, the group G_2 is nicer, because $M(G_2) = \{0\}$ whereas $M(D_4) = \mathbb{Z}_2$.

Now, projective representations are very unpleasant objects. You cannot take direct sums of two projective representations unless the multipliers agree; when you take tensor products you have to multiply the multipliers, and so on. There is a lot of messy bookkeeping involved. The natural thing to do, and one that is done automatically and instinctively, is to pass from the group G to the group G^*, and to *forget about the above italicized condition.* That is, we use G^* instead of G as the group of physics. We then want to consider all honest unitary representations of G^*. For the case of no discrete symmetries, this involved passing from the proper Lorentz group to $Sl(2, \mathbb{C})$ – there was no ambiguity in the choice of G^*. However, for the full Lorentz group there are several choices (in fact eight in all). Let us describe two very natural choices arising from the study of Clifford algebras. (For details about Clifford algebras see the treatment in Greub, *Multilinear algebra*, or in Bamberg and Sternberg, vol. II.)

Let $V = \mathbb{R}^{p,q}$ be real Cartesian $p+q$ space with a non-degenerate quadratic form of signature p pluses and q minuses. We denote its orthogonal group by $O(p, q)$. The corresponding Clifford algebra, denoted by $C(p, q)$, is the associative algebra generated by the elements of V subject to the relations

$$uv + vu = 2(u, v)\mathbf{1}$$

where $(,)$ denotes the scalar product associated to the bilinear form and $\mathbf{1}$ is the unit element in the algebra. The group $Pin(p, q)$ is the group generated multiplicatively in $C(p, q)$ by 'unit' vectors in V, that is by vectors e satisfying $(e, e) = \pm 1$. There is a two-to-one homomorphism, $\phi: Pin(p, q) \to O(p, q)$ determined by

$$\phi(e)v = eve^{-1} \text{ for all 'unit' vectors } e \text{ and all } v \in V.$$

Notice that if we take $v = e$ we get $\phi(e)e = eee^{-1} = e$, whereas if we take v orthogonal to e then the Clifford identities say that $ev = -ve$, so that $\phi(e)v = -vee^{-1} = -v$. Thus $\phi(e)$ is the negative of the reflection through the hyperplane perpendicular to e. In particular, in ordinary space time, if we take e to be a time-like vector, then $\phi(e)$ is just the parity operator $\mathscr{P}$. But this shows that the double covers

$$Pin(1, 3) \to O(1, 3) \quad \text{and} \quad Pin(3, 1) \to O(3, 1)$$

are different. In the first case $e^2 = \mathbf{1}$, whereas in the second case $e^2 = -\mathbf{1}$. Put another way, in each of these groups there will be two elements $\pm \mathscr{Q}$ covering the parity transformation, $\mathscr{P}$. But in $Pin(1, 3)$ $\mathscr{Q}^2 = id$, whereas in $Pin(3, 1)$ $\mathscr{Q}$ is of order four, with $\mathscr{Q}^2 = -I \in Sl(2, \mathbb{C})$. These are precisely the alternatives (a) and (b) discussed at the beginning of this section.

If we look at the inverse images of the four-element group $\{I, \mathscr{P}, \mathscr{T}, \mathscr{P}\mathscr{T}\}$ in the *Pin* groups, for $Pin(3,1)$ we get G_2 and for $Pin(1,3)$ we get D_2. We should point out (see the references cited above) that

$C(1,3) \sim H(2)$, the algebra of 2×2 matrices over the quaternions

and

$C(3,1) \sim R(4)$, the algebra of 4×4 matrices over the reals.

Thus, for example, the smallest dimension for a real representation of $C(1,3)$ is eight, and the spin representation for $C(3,1)$ has four real dimensions (the Majorana spinors).

The existence of the two alternatives (a) and (b) mentioned at the beginning of this section has an interesting history. The fact that there were two types of spinors seems to have been known to Racah in the 1930s. It was rediscovered by Fermi and Yang in the 1940s; they raised the question as to whether this could have an observable physical consequence. This, in turn, led to a celebrated paper by Wick, Wigner and Wightman in which the notion of a 'superselection rule' was introduced. This declared that this question might (or should) not have relevant physical consequences. My own feeling is that in the light of recent work on supersymmetry this question should be reexamined.

As we indicated above, from a mathematical point of view the groups $Pin(1,3)$ and $Pin(3,1)$ seem most natural as candidates for the cover of the four-component Lorentz group. However, for the sake of completeness we list all eight possibilities in the following.

The group generated by $\mathscr{P}$ and $\mathscr{T}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The overall possibilities for its inverse image in the double cover must be extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, so $1 \to G \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$. Each of $\mathscr{P}$, $\mathscr{T}$, $\mathscr{P}\mathscr{T}$ have two covering elements. Call them (temporarily) $\pm p, \pm t, \pm pt$. Each of these is of order either two or four, so there are eight possible choices for these orders (two for each of p, t, and pt). Elementary arguments show that each of these possibilities occurs for exactly one central extension. The possibilities are:

- the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (all elements of order two);
- the group D_4, where the kernel $\mathbb{Z}_2$ is identified as rotation through 180°. In D_4 four elements have square *id* (the four reflections) and two (rotations through $\pm 90°$) have square -1 (the non-trivial element in the central $\mathbb{Z}_2$). So the 90° rotation element must cover one of the three elements, $\mathscr{P}$, $\mathscr{T}$, or $\mathscr{P}\mathscr{T}$. So there are three choices in all, corresponding to which of the $\pm p, \pm t, \pm pt$ is of order four;
- the group $\mathbb{Z}_2 \times \mathbb{Z}_4$, where the kernel $\mathbb{Z}_2$ is taken to be the $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$. In this group there are four elements of order four and four of order two. So two of the $\pm p, \pm t, \pm pt$ are of order four and one of order two. Again three possibilities;
- the quaternionic group, $\mathscr{Q} = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$, with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, etc. All the $\pm p, \pm t, \pm pt$ are of order four.

So there are eight possibilities in all. It turns out that all these can occur for the Lorentz group exactly once. Thus there are eight double covers for the Poincaré group.

Summary of the relation between the *Pin* groups and the discrete symmetries

Facts common to Pin(3,1) and Pin(1,3) e_0 covers the parity operator, $\mathscr{P}$; $\mu = e_0 e_1 e_2 e_3$ covers $-id = \mathscr{P}\mathscr{T}$ and $\mu^2 = -1$; $e_1 e_2 e_3$ covers $\mathscr{T}$.

Differences In $Pin(1,3)$ $e_0^2 = 1$, $(e_1 e_2 e_3)^2 = 1$. The group is D_4 with μ being rotation through 90°, e_0 a bisector reflection, and $e_1 e_2 e_3$ a diagonal reflection.

In $Pin(3,1)$ $e_0^2 = -1$, $(e_1 e_2 e_3)^2 = -1$. The group is G_2, the quaternionic group.

3.11 The Mackey theorems on induced representations, with applications to the symmetric group

In this section we return to the general study of the representation $s \uparrow G$ induced up to G from a representation s of a subgroup H on a vector space V. Thus, from the representation s we construct the vector bundle $E \to M$, where $M = G/H$, and $s \uparrow G$ is the corresponding action of elements of G on $\Gamma(E)$. All of this is described in Section 3.3. We let σ denote the character of s and $\sigma \uparrow G$ the character of $s \uparrow G$. We recall the Frobenius fixed point character formula, equation (3.3)

$$(\sigma \uparrow G)(a) = (1/\#H) \sum_{\substack{g \in G \\ g^{-1}ag \in H}} \sigma(g^{-1}ag). \tag{11.1}$$

We now want to consider the following situation. Let K be some other subgroup of G. We can consider the restriction of the representation $s \uparrow G$ to K. Let us denote it by $(s \uparrow G) \downarrow K$. The subgroup K need not act transitively on M.

If we let M_i denote the orbits of M under K, and $E_i = \pi^{-1}(M_i)$, then we can apply the decomposition (3.1) to the representation $(s \uparrow G) \downarrow K$, where we are now considering E as a homogeneous vector bundle for the subgroup, K. To see what (3.1) actually says in this case, we describe the orbits M_i more explicitly, and describe the vector bundles E_i as being induced from certain representations s_i of subgroups K_i of K. Here are the details. Let us pick a point x_i in each orbit, M_i. We can write $x_i = g_i H$ for some $g_i \in H$. Then we can write

$$M_i = K g_i H, \tag{11.2}$$

i.e. the orbits, M_i, of M under K are precisely the distinct double cosets Kg_iH. To identify E_i as an induced vector bundle, we consider the isotropy group, K_{x_i} of the point, x_i. For notational convenience, let us denote this subgroup by K_i, so that

$$K_i = (g_i H g_i^{-1}) \cap K. \tag{11.3}$$

Let us set

$$V_i = E_{x_i}.$$

Recall from the beginning of Section 3.3 that the element g_i defines an isomorphism φ_{g_i}: $V \to V_i = E_{x_i}$ given by $\varphi_{g_i}(v) = [(g_i, v)]$.

Then $k_i \in K_i$ acts on V_i according to the representation defined by

$$k_i \varphi_{gi} v = \varphi_{g_i} s(g_i^{-1} k_i g_i) v. \tag{11.4}$$

The right-hand side makes sense, since the element $g_i^{-1} k_i g_i$ belongs to H. Let us identify V with V_i via φ_{g_i}, and let us define the representation, s^i, of K_i by

$$s^i(k_i) = s(g_i^{-1} k_i g_i). \tag{11.5}$$

It follows from the above considerations that the homogeneous vector bundle, E_i, is isomorphic to the bundle induced over $M_i = K/K_i$ by the representation, s^i of K_i. Then (3.1) becomes the Mackey subgroup decomposition formula

$$(s \uparrow G) \downarrow K \sim (s^1 \uparrow K) \oplus \cdots \oplus (s^n \uparrow K) \tag{11.6}$$

where there is one summand for each double coset, Kg_iH, and where the representations s^i are given by (11.5) on the subgroups K_i of K given by (11.3) and the $s^i \uparrow K$ are the corresponding induced representations of K.

We wish to apply (11.6) to the following very important special case. Let H' and H'' be two subgroups of a group, G. Then their direct product, $H' \times H''$ is a subgroup of the direct product, $G \times G$ of G with itself. We can consider G as the 'diagonal' subgroup of $G \times G$ consisting of all pairs of the form (g, g). Thus, in (11.6) G will play the role of K, $G \times G$ will play the role of G and $H = H' \times H''$ will be the H. If we are given a representation, s', of H' on V' and a representation, s'', of H'' on V'', we obtain a representation of $H' \times H''$ on the space $\mathrm{Hom}(V', V'')$ by setting

$$(h', h'')T = h'' T h'^{-1}.$$

Let s be this representation, so that

$$S(h)T = h'' T h'^{-1} \quad \text{if} \quad h = (h', h'') \in H = H' \times H''.$$

We wish to apply (11.6), but before doing so we make some identifications. We can write

$$M = (G \times G)/H = (G \times G)/H' \times H'' = (G/H') \times (G/H'')$$
$$= M' \times M''$$

where we have set $M' = G/H'$ and $M'' = G/H''$. Let E be the vector bundle induced by s over M, and E', E'' the vector bundles corresponding to s', s''. Let (x', x'') be a point of M. There is a natural identification of

$$E_{(x', x')} \text{ with } \mathrm{Hom}(E'_{x'}, E''_{x''}).$$

Indeed, if $x' = g'H''$ and $x'' = g''H''$, then the left-hand side consists of all equivalence classes

$$[((g', g''), T)] \qquad T \in \mathrm{Hom}(V', V'')$$

while the right-hand side becomes identified with $\mathrm{Hom}(V', V'')$ if we use $\varphi_{g'}$ to identify V' with $E'_{x'}$ and $\varphi_{g''}$ to identify V'' with $E''_{x''}$. It is easy to check that the identification is independent of the particular choice of g' and g''.

Let $F \in \Gamma(E)$ be a section of E over M. We can think of F as assigning an element, $F(x', x'') \in \mathrm{Hom}(E_{x'}, E_{x''})$ to the point (x', x'') in view of the above identification. Furthermore, we can regard such an F as giving a 'kernel' which defines a map from $\Gamma(E') \to \Gamma(E'')$. Indeed for any $u \in \Gamma(E')$ define $Fu \in \Gamma(E'')$ by the formula

$$(Fu)(x'') = \sum_{x' \in M'} F(x', x'') u(x').$$

In the sum on the right, $u(x') \in E'_{x'}$ and $F(x', x'')$ maps $E'_{x'} \to E''_{x''}$, and so we have indeed defined a section of E''. Thus each F defines a map of $\Gamma(E') \to \Gamma(E'')$ and the map so obtained depends linearly on F. It is easy to see that we have defined an isomorphism

$$\Gamma(E) \sim \mathrm{Hom}(\Gamma(E'), \Gamma(E'')).$$

The group G acts on both sides, and this is an equivalence of the corresponding representations. Thus we have identified the representation $(s \uparrow (G \times G)) \downarrow G$ with the representation of G on $\mathrm{Hom}(\Gamma(E'), \Gamma(E''))$.

In the case that the representations s' and s'' are both the trivial one-dimensional representations of H' and H'', the space $\Gamma(E')$ and $\Gamma(E'')$ reduce to the ordinary function spaces $F(M')$ and $F(M'')$. The identification that we have just described then becomes the identification of $F(M' \times M'')$ with $\mathrm{Hom}(F(M'), F(M''))$ that we used in Section 5, Chapter 2. We will soon see that the Mackey formula (11.6) will give us a generalization to vector bundles of (5.2) which was so important in the proof of the basic facts about characters in Chapter 2.

So much for the left-hand side of (11.6) in the case we are considering. To deal with the right-hand side we must examine the G, H double cosets in $G \times G$ and choose a convenient representative for each of them. Given any $(g_1, g_2) \in G \times G$, its double coset consists of all products of the form

$$(g, g)(g_1, g_2)(h', h'') = (g g_1 h', g g_2 h'')$$

where $h' \in H'$ and $h'' \in H''$. If we choose $g = (g_1 h')^{-1}$ we get all elements of the form $(e, H' g_3 H'')$, where $g_3 = g_1^{-1} g_2$. Thus the G, H double coset in $G \times G$ gives rise to an H', H'' double coset in G. It is easy to see that this relation is one-to-one, the G, H double coset, M_i in $G \times G$ goes over into the H', H'' double coset in G given by

$$(e \times G) \cap M_i.$$

We may choose our representative in the form (e, g_i) and then it is easy to see that the subgroup K_i in (11.3), which is now a subgroup of G, is given by

$$G_i = H' \cap g_i H'' g_i^{-1}. \tag{11.7}$$

The representations s^i on $\mathrm{Hom}(V', V'')$ given by (11.5) is

$$s^i(k)T = (g_i^{-1} k g_i) T k^{-1} \quad k \in G_i. \tag{11.8}$$

(Notice that $k \in H'$, and so acts on V' via s', and $g_i^{-1} k g_i \in H''$, and so acts on V'' via s'', so that (11.8) makes sense.)

Putting the various identifications together into (11.6) we obtain the following formula (due to Mackey):

> the representation of G on $\mathrm{Hom}(\Gamma(E'), \Gamma(E''))$ is equivalent to the direct sum of the induced representations: (11.9)
>
> $$(s^1 \uparrow G) \oplus \cdots \oplus (s^n \uparrow G)$$
>
> where the sum is taken over all H', H'' double cosets in G, and where the s^i are the representations given by (11.8) on the subgroups, G_i, defined by (11.7)

Let us be given two representations, r^1 and r^2, of the group G on the vector spaces W_1 and W_2. We denoted by $\mathrm{Hom}_G(W_1, W_2)$ the subspace of $\mathrm{Hom}(W_1, W_2)$ consisting of those T which satisfy

$$r_g^2 T = T r_g^1 \quad \text{for all} \quad g \in G.$$

Thus $\mathrm{Hom}_G(W_1, W_2)$ consists of all invariant elements of $\mathrm{Hom}(W_1, W_2)$ under the action of G given by the representation $\mathrm{Hom}(r^1, r^2)$. We wish to use (11.9) to compute $\mathrm{Hom}_G(\Gamma(E'), \Gamma(E''))$. Now (11.9) identifies the representation space $\mathrm{Hom}(\Gamma(E'), \Gamma(E''))$ as a direct sum. The space of invariants in a direct sum is the direct sum of the invariant elements in each summand. So to apply (11.9) we need a little lemma telling us how to compute the invariant elements in an induced representation.

Let s be a representation of a subgroup H of the group G. Then there is a vector space identification

$$\mathrm{Inv}_G(s \uparrow G) = \mathrm{Inv}_H s, \tag{11.10}$$

where the left-hand side is the space of invariant elements in the induced representation, $\Gamma(E)$ under the action of the group, G, and the right-hand side denotes the invariant elements under H of the representation s.

Proof Let f be an element of $\Gamma(E)$. To say that f is invariant means that

$$af(a^{-1}x) = f(x)$$

or

$$f(ax) = af(x)$$

for all $a \in G$. Since G acts transitively on $M = G/H$, this shows that an invariant f is determined by its values at one point, say the point $x = H$. Taking $a \in H$ shows that $f(x)$ must be invariant. We thus have a map, $f \rightsquigarrow f(x)$ of $\mathrm{Inv}_G(s \uparrow G) \rightsquigarrow \mathrm{Inv}_H s$, and it is easy to see this is a bijection.

If we now plug (11.10) into (11.9), we get the Mackey formula for invariants:

$$\mathrm{Hom}_G(\Gamma(E'), \Gamma(E'')) \sim \mathrm{Hom}_{G_1}(V', V'') \oplus \cdots \oplus \mathrm{Hom}_{G_n}(V', V''), \tag{11.11}$$

where the subgroup G_i is given by (11.7), and where $k \in G_i$ acts on V' via s'_k and on V'' via $s''(g_i^{-1} k g_i)$.

There are many important applications of the Mackey formula, (11.11). For example, suppose that we take s' and s'' to be the trivial one-dimensional representations of H' and H''. Then each of the summands on the right-hand side of (11.11) is one dimensional. The number of summands is exactly the number of G orbits on $M' \times M''$. Thus (11.11) reduces to

$$\dim \mathrm{Hom}(F(M'), F(M'')) = \#(\text{of } G \text{ orbits on } M' \times M'')$$

which is just equation (5.2), Chapter 2. At another extreme, let us consider the case where $H' = H$ is a subgroup of G and where we take $H'' = G$. In this case there is only one double coset so that in (11.9) and (11.11) we may take $g_1 = e$ and $G_1 = H$. We are now given a representation s of H and construct the vector bundle $\Gamma(E)$ which is the $\Gamma(E')$ in (11.11), and we are also given a representation r of G on some vector space W, and the $\Gamma(E'')$ in (11.11) is just W itself, and the induced representation is just r. The left-hand side is $\mathrm{Hom}_G(\Gamma(E), W)$ and the right-hand side reduces to the single summand $\mathrm{Hom}_H(V, W)$. Thus (11.11) reduces to the Frobenius reciprocity theorem

$$\mathrm{Hom}_G(\Gamma(E), W) \sim \mathrm{Hom}_H(V, W).$$

For the remainder of this section we wish to given some applications of the Mackey formula (11.11) to the theory of the representations of the symmetric group. We shall take $G = S_n$. Let λ be a Young diagram, and let t be a Young tableau of type λ, and let $\{t\}$ be the corresponding tabloid, so that $\{t\}$ is a partition of $\{1, \ldots, n\}$ into disjoint subsets. We shall take H' and H'' in (11.11) be of the type $G_{\{t\}}$ for various choices of λ and t. Thus the spaces M' and M'' will be the spaces M_λ of Section 2.8.

In order to apply (11.11) we must first describe the S_n orbits on $M_\lambda \times M_\mu$ for two Young diagrams λ and μ. Consider two tabloids, $\{t\} \in M_\lambda$ and $\{s\} \in M_\mu$. Then $\{t\}$ is a disjoint union of subsets $\{t\} = t_1 \cup \cdots \cup t_n$ of subsets of $\{1, \ldots, n\}$ with $\#t_i = \lambda_i$, and similarly for s and μ. Then $t_i \cap s_j$ is a subset of $1, \ldots, n$, and the number

$$m_{ij} = \#(t_i \cap s_j)$$

is clearly a function on $M_\lambda \times M_\mu$ which is invariant under the action of S_n, and hence is constant on S_n orbits. Conversely, we claim that the matrix of entries (m_{ij}) completely determines the orbit: since the values m_{ij} determine λ, as $\lambda_i = \sum_j m_{ij}$, and the group S_n acts transitively on M_λ, it is clearly enough to prove the following – suppose that

$$\#(t_i \cap s_j) = \#(t_i \cap s'_j).$$

Then there exists a permutation which preserves $\{t\}$ and carries $\{s\}$ into $\{s'\}$. But the condition implies that for each fixed i the decompositions

$$t_i = \bigcup (t_i \cap s_j) \quad \text{and} \quad t_i = \bigcup (t_i \cap s'_j)$$

are partitions of the set t_i into subsets of equal sizes. Hence we can find a permutation of the set t_i which carries $t_i \cap s_j$ into $t_i \cap s'_j$. Considering one such permutation for each i gives a $\tau \in S_n$ such that

$$\tau(t_i \cap s_j) = \tau(t_i \cap s'_j).$$

Thus

$$\tau(t) = t \quad \text{and} \quad \tau(s) = s'$$

as required. Thus we have proved that

> the set of G orbits on $M_\lambda \times M_\mu$ is in one-to-one correspondence with the set of all $n \times n$ matrices with non-negative integer entries (m_{ij}) such that the row sum $\sum_j m_{ij} = \lambda_i$ and the column sum $\sum_i m_{ij} = \mu_j$. The orbit through the point $(\{t\}, \{s\})$ is characterized by the matrix (m_{ij}), where
>
> $$m_{ij} = \#(t_i \cap s_j). \tag{11.12}$$
>
> The isotropy group of the point $(\{t\}, \{s\})$ is the group of those permutations which preserve all the intersections $t_i \cap s_j$.

We are going to apply (11.11) to two types of vector bundles, that is to representations induced from two types of representations of $S_{\{t\}}$ – the one-dimensional trivial and sign representation. For the trivial representation, the corresponding $\Gamma(E)$ is just the function space $F(M)$. We shall denote the $\Gamma(E)$ coming from the sign representation by $\tilde{F}(M)$. Then an immediate consequence of (11.12) is

$$\dim \mathrm{Hom}_{S_n}(F(M_\lambda), F(M_\mu)) = \#(\text{non-negative integer matrices } (m_{ij}) \text{ with row sums } \lambda_i \text{ and column sums } \mu_j). \tag{11.13}$$

Suppose we want to compute $\mathrm{Hom}_{S_n}(F(M_\lambda), \tilde{F}(M_\mu))$. On the right-hand side of (11.11) the various groups are the subgroups which preserve the $t_i \cap s_j$. If some $t_i \cap s_j$ contains more than one element, then the corresponding isotropy group contains at least one odd permutation, and the trivial and sign permutations are therefore inequivalent. Thus the only non-zero contributions to the right-hand side of (11.11) can come from pairs (t, s) for which all $t_i \cap s_j$ are either empty or contain one element, so the corresponding m_{ij} must be zero or one. Thus

$$\dim \mathrm{Hom}_{S_n}(F(M_\lambda), \tilde{F}(M_\mu)) = \#(\text{zero one matrices } (m_{ij}) \text{ with row sums } \lambda_i \text{ and column sums } \mu_j). \tag{11.14}$$

For a Young diagram λ, we shall denote by $\hat{\lambda}$ the Young diagram obtained from λ by interchanging the rows and columns. Thus if

$$\lambda = \square\square\square\square$$

then

$$\hat{\lambda} = \begin{array}{l}\square\square \\ \square \\ \square\end{array}$$

Now to say that the column sums of (m_{ij}) equal the rows of $\hat{\lambda}$ is the same as saying that the column sums of (m_{ij}) equal the columns of λ. If all the entries m_{ij} are either zero or one, the only way that this can happen is for the ones to occur at the positions of the

diagram, λ, and zeros elsewhere in the matrix. Thus there is only one such matrix and

$$\dim \mathrm{Hom}_{S_n}(F(M_\lambda), \tilde{F}(M_{\hat{\lambda}})) = 1. \tag{11.15}$$

Thus $F(M_\lambda)$ and $\tilde{F}(M_{\hat{\lambda}})$ have exactly one irreducible representation in common. Now in Section 2.8 we saw that there was a distinguished irreducible representation, F^λ, that occurs with multiplicity one in $F(M_\lambda)$. Let t be a tableau for the diagram λ, and C_t the corresponding group of column permutations. Then the space $\tilde{F}(M_{\hat{\lambda}})$ can be considered as being induced from the sign representation of C_t, that is C_t is the isotropy group of a point of $M_{\hat{\lambda}}$. But the element e_t of F^λ determines a one-dimensional subspace of F^λ which transforms under C_t as the sign representation. Hence $\dim \mathrm{Hom}_{C_t}(F^\lambda, \mathrm{sign}) \geqslant 1$. But, by the Frobenius reciprocity theorem

$$\mathrm{Hom}_{C_t}(F^\lambda, \mathrm{sign}) = \mathrm{Hom}_{S_n}(F^\lambda, \tilde{F}(M_{\hat{\lambda}})).$$

Thus F^λ occurs with multiplicity at least one in $\tilde{F}(M_{\hat{\lambda}})$. Hence it occurs with multiplicity exactly one there, and can be identified with the unique irreducible representation that $F(M_\lambda)$ and $\tilde{F}(M_{\hat{\lambda}})$ have in common. In short we have proved the following.

> The representation of S_n on $F(M_\lambda)$ and $\tilde{F}(M_{\hat{\lambda}})$ have exactly one irreducible representation in common, and that representation is (equivalent to) the irreducible representation, F^λ, that we have associated to the diagram λ in Section 2. (11.16)

Before proceeding let us state a general fact about induced representations which is interesting in its own right. Let (r, V) be a representation of a group G and let (s, W) be a representation of a subgroup, H, of G. We can restrict the representation r to H so as to obtain the representation $r{\downarrow}H$ of H. We can then form the tensor product $(r{\downarrow}H) \otimes s$ of H and then form the induced representation $[(r{\downarrow}H) \otimes s]{\uparrow}G$. We claim that

$$r \otimes (s{\uparrow}G) \sim [(r{\downarrow}H) \otimes s]{\uparrow}G \tag{11.17}$$

as representations of G. To prove this, it is convenient to use the description of the space of the induced representations as a space of functions on G. Thus the underlying space of the right-hand side of (11.17) consists of all functions F from G to $V \otimes W$ which satisfy

$$F(ah) = (r(h)^{-1} \otimes s(h)^{-1})F(a) \qquad h \in H$$

while the left-hand side has as its underlying space linear combinations of expressions of the form $v \otimes f$, where f is a function from G to W satisfying $f(ah) = s(h)^{-1}f(a)$. We can define a map φ from the left-hand side to the right-hand side by

$$\varphi(v \otimes f)(a) = r(a)^{-1}v \otimes f(a).$$

It is easy to check that this does define a linear map which is a G morphism and is in fact an isomorphism, proving (11.17). For the case of the trivial representation of H this is the isomorphism introduced at the end of Section 3.3. Let us apply (11.17) taking $r = A$ to be the sign representation of $S_n = G$ and H to be the subgroup S_t with s either the sign

or the trivial representation. It then follows from (11.17) that

$$A \otimes F(M_\lambda) = \tilde{F}(M_\lambda)$$

and

$$A \otimes \tilde{F}(M_\lambda) = F(M_\lambda).$$

It therefore follows from (11.16) that

$$A \otimes F^\lambda = F^{\hat{\lambda}}. \tag{11.18}$$

In particular, the characters of the representations F^λ and $F^{\hat{\lambda}}$ agree on even elements and differ by a sign on odd elements.

3.12 Exchange forces and induced representations

One of the main computational problems in approximate methods of quantum mechanics is finding the eigenvalues of a linear operator T on a finite-dimensional vector space W. In this section we describe a method for calculating the eigenvalues of T under the assumptions that $W = \Gamma(E)$, where $E \to M$ is a homogeneous vector bundle for a group G, induced from a representation of a subgroup H. Here G acts transitively on M, and H is the isotropy group of a point of M, and it is assumed that T commutes with the action of G.

The simplest non-trivial case is when $G = S_2$, $H = \{e\}$, so that M is the two-element set and the representation of H is trivial. In this case $\Gamma(E)$ is two dimensional and the representation of G sends (12) into

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The condition that

$$T = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

commutes with the action of G then requires $x = w$ and $z = y$ so

$$T = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

We can write

$$x = t(e) \quad \text{and} \quad y = t((12)).$$

In case T is a matrix of 'forces' the element $y = t((12))$ is called the 'exchange force'. The eigenvectors of T are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and these, of course, span the subspaces transforming according to the character

$$\chi_1 = \chi_{\square\square} \text{ and } \chi_2 = \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}.$$

Finally, the eigenvalues of T are

$$x + y = \chi_1(e)t(e) + \chi_1((12))t((12))$$