Math 126 Exercise set 2.

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1. What is the character of the regular representation?

2. Let $\square$ denote the set of vertices of the square, and label the vertices in cyclic order as $\{1,2,3,4\}$. The group D_4 acts on this 4 element set, and this gives rise to a representation of D_4 on $\mathcal{F}(\square)$ which is a four dimensional vector space. The one dimensional subspace of $\mathcal{F}(\square)$ consisting of all constant functions is an invariant subspace, and the restriction of the representation on $\mathcal{F}(\square)$ to this subspace is the trivial representation.

1. Show that the one dimensional subspace consisting of all h which satisfy $h(1) = h(3) = -h(2) = -h(4)$ is an invariant subspace, and compute the representation (and hence the character) of D_4 acting on this subspace.

2. Show that the two dimensional subspace of $\mathcal{F}(\square)$ spanned by the functions f and g where $1 = f(1) = f(2) = -f(3) = -f(4)$ and $1 = g(2) = g(3) = -g(1) = -g(4)$ is an invariant subspace, and compute the character of the representation of D_4 on this two dimensional subspace.

3. Conclude that this two dimensional representation is irreducible.

4. Conclude that D_4 has two further irreducible representations, both one dimensional.

The group S_n acts on the set $\{1,\ldots,n\}$ and hence has a representation on the n-dimensional space $\mathcal{F}(\{1,\ldots,n\})$ known as the "permutation representation". Let us (in this exercise set) denote this representation by p. We know that the constant functions form a one dimensional invariant subspace giving rise to the trivial representation, and that there is a complementary invariant subspace of dimension $n-1$ which is irreducible. Let us denote this representation by r_{n-1}. So

$$p = \text{trivial} \ \oplus \ r_{n-1}.$$

The map $S_n \to \mathbb{C}$ given by

$$a \mapsto \det p(a)$$

us a one dimensional representation known as the sign representation and denoted by sgn. So

$$\mathrm{sgn}(a) := \det p(a).$$

3. Show that $\mathrm{sgn}((12)) = -1$ so this representation sgn is not trivial.

4. Show that the $\mathrm{sgn}((12\ldots k)) = (-1)^{k-1}$. [Hint: use the factorization of the cycle $(12\cdots k)$ as

$$(12\cdots k) = (1k)\cdots(13)(12).]$$

5. What is the character of r_{n-1} evaluated on the element (12)?

6. Show that $r_{n-1} \otimes \mathrm{sgn}$ is irreducible, and is not equivalent to r_{n-1} if $n > 3$. What goes wrong with this argument when $n = 3$?

7. Conclude that S_4 has a two dimensional representation, and five irreducibles in all.

8. Construct the two dimensional irreducible representation of S_4 by exhibiting a homomorphism from S_4 onto S_3. [Hint. Exhibit a three element set on which S_4 acts as all permutations.]