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1 The action of $Gl(V)$ and S_r on $T^r(V)$.

1.1 $T^r(V)$ and the action of $\mathrm{End}(V)$ on it.

Let V be a finite dimensional vector space over the complex numbers, and let $T^r(V)$ denote the r-fold tensor product of V with itself:

$$T^k rV) := V \otimes \cdots \otimes V \qquad r \text{ factors.}$$

If $A \in \mathrm{End}(V)$ then $T_rA \in \mathrm{End}(T^r(V))$ is defined as that linear transformation determined by its action on decomposable tensors by

$$(T_rA)(v_1 \otimes \cdots \otimes v_r) := Av_1 \otimes \cdots \otimes Av_r.$$

From its definition it is clear that

$$T_r(AB) = T_r(A)T_r(B)$$

and so if we restrict to elements of $Gl(V)$ we see that $A \mapsto T_rA$ gives a representation of $Gl(V)$ of $T^r(V)$.

Our first task will be to show that this representation is completely reducible and to describe its decomposition into irreducibles.

1.2 The action of S_r on $T^r(V)$.

Let $s \in S_r$ act on the set of decomposable tensors by

$$s(v_1 \otimes \cdots \otimes v_r) := v_{s^{-1}1} \otimes \cdots \otimes v_{s^{-1}r}.$$

In other words, think of the entries of the decomposable as a function from the r element set $\{1, \ldots, r\}$ to V, and let S_r act as it usually does on functions. From this description it is clear that we get an action of S_r on the set of decomposable tensors.

For emphasis I repeat: the elements of S_r change the positions of the vectors in the tensor product but do not change the vectors themselves. For example, when $r = 3$

$$(123)(x \otimes y \otimes z) = z \otimes x \otimes y.$$

Therefor this action is linear in one vector when the others are fixed: for example

$$(123)(x \otimes y_1 \otimes z) + (123)(x \otimes y_2 \otimes z) = z \otimes x \otimes y_1 + z \otimes x \otimes y_2$$

$$= z \otimes x \otimes (y_1 + y_2) = (123)(x \otimes (y_1 + y_2) \otimes z).$$

Therefor this action of S_r on decomposable tensors extends to a representation of S_r on $T^r(V)$,

1.3 The actions of S_r and $Gl(V)$ on $T^r(V)$ commute.

This is clear, because the element of $Gl(V)$ act on the individual vectors, while the elements of S_r permute the vectors. For example

$$(123)T_r(A)(x\otimes y\otimes z) = (123)(Ax\otimes Ay\otimes Az) = Az\otimes Ax\otimes Ay = T_r(A)(z\otimes x\otimes y) = T_r(A)(123)(x\otimes y\otimes z).$$

1.4 Goal.

We thus get a representation of $Gl(V) \times S_r$ on $T^r(V)$. We shall show that under this representation of the product group, $T^(V)$ decomposes into a direct sum of irreducibles, with each irreducible occurring once.

1.5 Example: $r = 2$.

The group S_2 has two irreducible representations, the trivial representation and the sign representation. Decompose $T^2(V)$ into the direct sum

$$T^2(V) = S^2(V) \oplus \wedge^2(V).$$

Here $S^2(V)$ consists of the those $t \in T^2(V)$ which transform according to the trivial representation, i.e. which satisfy

$$(12)t = t.$$

These are called symmetric tensors (of degree two). And $\wedge^2(V)$ consists of those tensors which transform according to the sign representation, i.e. which satisfy

$$(12)t = -t.$$

These are called anti-symmetric tensors.

If V is one dimensional, then $\wedge^2(V) = \{0\}$. Otherwise both $S^2(V)$ and $\wedge^2(V)$ are non-zero, and we shall prove that they give irreducible representations of $Gl(V)$.

1.6 A look ahead.

For $r > 2$ we can still decompose $T^r(V)$ into irreducibles of S_r, and collect together all irreducibles which are equivalent to a given irreducible of S_r. If λ is a Young diagram with r boxes, then the summand we just described will be isomorphic to a direct sum of a number of copies of F^λ, the Specht module corresponding to the diagram λ. It will be isomorphic to

$$F^\lambda \oplus \cdots \oplus F^\lambda \sim \mathbb{C}^k \otimes F^\lambda$$

where k is the number of copies. We will write U^λ instead of $\mathbb{C}^k$, so the subspace we are looking at is equivalent to

$$U^\lambda \otimes F^\lambda$$

as a representation of S_r. Schur's lemma implies that any linear transformation on $U^\lambda \otimes F^\lambda$ which commutes with all the elements of S_r must be of the form $X \otimes I$ where X is a linear transformation of U^λ. Also, Schur's lemma implies that any transformation of $T^r(V)$ which commutes with all elements of S_r must preserve each of the subspaces $U^\lambda \otimes F^\lambda$. This means that for every λ, we get a representation of $Gl(V)$ on U^λ, provided that $U^\lambda \neq \{0\}$.

We shall show that these are irreducible representations of $Gl(V)$ and are indeed all the irreducible polynomial representations of $Gl(V)$.

2 $Gl(V)$ spans $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$.

The purpose of this section is to prove the assertion in the title: that the $T_r(A)$ as A ranges over $Gl(V)$ spans the space of those linear transformations of $T^r(V)$ which commute with all the $w \in S_r$. To do this we need some facts about the space of symmetric tensors.

2.1 The symmetric tensors.

We let $S^r(V)$ denote the subspace of $T^r(V)$ consisting of symmetric tensors, i.e. those which satisfy

$$wt = t \quad \forall w \in S_r.$$

Averaging over S_r gives a projection from $T^r(V)$ to $S^r(V)$. Namely, define the linear transformation $\mathcal{A}$ on $T^(V)$ by

$$\mathcal{A}t := \frac{1}{r!}\sum_{w\in S_r} wt.$$

Clearly

$$\mathcal{A}t \in S^r(V) \quad \forall t \in T^r(V)$$

and

$$\mathcal{A}s = s \quad \text{if} \;\; s \in S^r(V).$$

2.1.1 Bases of $S^r(V)$ from bases of V.

For a decomposable element $v_1 \otimes \cdots \otimes v_r$ we write

$$v_1 \cdot v_2 \cdots v_r := \mathcal{A}(v_1 \otimes \cdots \otimes v_r$$

and if some of the v's are repeated we use the power notation, so

$$e^3 f^2 := \mathcal{A}(e \otimes e \otimes e \otimes f \otimes f).$$

If $e_1, \ldots, e_n$ form a basis of V, then the $e_{i_1} \otimes \cdots \otimes e_{i_r}$ form a basis of $T^r(V)$ and so the

$$e_1^{k_1} \cdot e_2^{k_1} \cdots e_n^{k_n}, \qquad k_1 + \cdots + k_n = r$$

span $S^r(V)$, and since they are linearly independent, they form a basis of $S^r(V)$.

2.1.2 The dimension of $S^r(V)$.

We can count the number of elements in the above basis, and hence determine the dimension of $S^r(V)$ as follows: Write k_1 X's followed by a / followed by k_2 X's and so on. So, for example, if V is five dimensional and $r = 13$, the five-tuple $(7, 2, 0, 1, 3)$ would be written as

$$XXXXXXX/XX//X/XXX$$

and corresponds to

$$e_1^7 e_2^2 e_4 e_5^3.$$

In other words, we can recover the knowledge of the basis element from knowing where the slashes occur. There are r X's and $n-1$ /'s, a total of $n+r-1$ symbols in all, and the information is encoded in the location of the /' among these symbols. So the total number of basis elements is the total number of ways of selecting $n-1$ positions from among $n+r-1$ positions so

$$\dim S^r(V) = \begin{pmatrix} n+r-1 \\ n-1 \end{pmatrix} = \frac{n(n+1)\cdots(n+r-1)}{r!}.$$

2.1.3 The v^r span $S^r(V)$.

If $v = a_1e_2 + \cdots + a_rv_r \in V$, then

$$v^r = \mathcal{A}(v \otimes v \otimes \cdots \otimes v) = \sum k_1! \cdots k_n! a_1^{k_1} \cdots a_n^{k_n} e_1^{k_1} \cdots e_n^{k_n}.$$

The assertion in the title of this subsection is that these elements span $S^r(V)$. To prove this, it is enough to prove that the elements $e_1^{k_1} \cdots e_n^{k_n}$ lie in the subspace spanned by all the v^r. This subspace, like any subspace of a finite dimensional vector space is closed. So it is enough to show that the elements $e_1^{k_1} \cdots e_n^{k_n}$ can be written as limits of elements in this subspace. Taking the derivative is a limiting process, and

$$\left(\frac{\partial}{\partial a_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial a_n}\right)^{k_n} \left(\sum k_1! \cdots k_n! a_1^{k_1} \cdots a_n^{k_n} e_1^{k_1} \cdots e_n^{k_n}\right) = e_1^{k_1} \cdots e_n^{k_n},$$

proving the desired result.

2.2 The isomorphism of $T^r(\mathrm{Hom}(V,V))$ with $\mathrm{Hom}(T^r(V), T^r(V))$.

The space $\mathrm{Hom}(V,V)$ is a vector space, and so we can form its tensor powers such as

$$T^r(\mathrm{Hom}(V,V)) = \mathrm{Hom}(V,V) \otimes \cdots \otimes \mathrm{Hom}(V,V) \qquad r \text{ factors.}$$

We may define the linear map

$$\phi \quad T^r(\mathrm{Hom}(V,V)) \to \mathrm{Hom}(T^r(V), T^r(V))$$

by

$$\phi(A_1 \otimes \cdots \otimes A_r)(v_1 \otimes \cdots \otimes v_r) := A_1v_1 \otimes \cdots \otimes A_rv_v$$

on decomposable elements, and extending by linearity. It is clear that

$$\phi(A_1B_1 \otimes \cdots \otimes A_rB_r) = \phi(A_1 \otimes \cdots \otimes A_r)\phi(B_1 \otimes \cdots \otimes B_r)$$

so that ϕ is a homomorphism of algebras. If $\phi(A_1 \otimes \cdots \otimes A_r) = 0$, then $A_1v_1 \otimes \cdots \otimes A_nv_n = 0$ for all $v_1 \otimes \cdots \otimes v_r$, and letting the v_i range independently over all elements of a basis shows that $A_1 \otimes \cdots \otimes A_r = 0$. So ϕ is injective. Since the dimension of $T^r(\mathrm{Hom}(V,V))$ and $\mathrm{Hom}(T^r(V), T^r(V))$ are both equal to n^{2r} we see that ϕ is an isomorphism.

2.3 This isomorphism is an S_r morphism.

The group S_r acts on $T^r(V)$, and hence acts on $\mathrm{Hom}(T^r(V), T^r(V)$ by conjugation: If $Z \in \mathrm{Hom}(T^r(V), T^r(V)$ and $w \in S_r$ then

$$w: \qquad Z \mapsto wZw^{-1}.$$

The group S_r also acts on $T^r(\mathrm{Hom}(V,V))$ since it acts on the r-th tensor power of any vector space. We wish to show that

$$\phi \circ w = w \circ \phi$$

i.e. that

$$\phi(wZ) = w\phi(Z)w^{-1}$$

for all $Z \in T^r(\mathrm{Hom}(V,V)$ and all $w \in S_r$. It suffices to prove this for decomposable Z, i.e. for $Z = A_1 \otimes \cdots \otimes A_r$ since these span. We have

$$\begin{aligned}
wZ &= A_{w^{-1}(1)} \otimes \cdots \otimes A_{w^{-1}(r)} \quad \text{so} \\
\phi(wZ)(v_1 \otimes \cdots \otimes v_r) &= A_{w^{-1}(1)}v_1 \otimes \cdots \otimes A_{w^{-1}}(r)v_r \quad \text{while} \\
w^{-1}(v_1 \otimes \cdots \otimes v_r) &= v_{w(1)} \otimes \cdots \otimes v_{w(r)} \quad \text{so} \\
\phi(Z)w^{-1}(v_1 \otimes \cdots \otimes v_r) &= A_1 v_{w(1)} \otimes \cdots \otimes A_r v_{w(r)} \quad \text{and} \\
w\phi(Z)w^{-1}(v_1 \otimes \cdots \otimes v_r) &= A_{w^{-1}(1)}v_1 \otimes \cdots \otimes A_{w^{-1}(r)}v_r \\
&= \phi((wZ)(v_1 \otimes \cdots \otimes v_r))
\end{aligned}$$

as was to be proved.

2.4 The space $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$.

To say that $Y \in \mathrm{Hom}(T^r(V), T^r(V))$ commutes with all $w \in S_r$ is to say that $wYw^{-1} = Y$ for all $w \in S_r$. If we write $Y = \Phi(Z)$, this says that $wZ = Z$ for all $w \in S_r$, in other words that $Z \in S^r(\mathrm{Hom}(V,V)$. So we have shown that

$$\mathrm{Hom}_{S_r}(T^r(V), T^r(V)) = \phi(S^r(\mathrm{Hom}(V,V)).$$

Now $S^r(\mathrm{Hom}(V,V)))$ is spanned by elements of the form

$$A \bullet \cdots \bullet A = \mathcal{A}(A \otimes \cdots \otimes A)$$

and the image under ϕ of such an element is exactly $T_r(A)$. So we have shown that the subspace of $\mathrm{Hom}(T^r(V), T^r(V))$ spanned by all the $T_r(A)$ as A ranges over all of $\mathrm{Hom}(V,V)$ is exactly $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$. Now every element of $\mathrm{Hom}(V,V)$ can be written as the limit of a sequence of elements of $Gl(V)$. So once again, using the fact that linear subspaces of a finite dimensional vector space are closed, we see that the set $T_r(A), \quad A \in Gl(V)$ spans all of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$.

2.5 The representation of $Gl(V)$ on U^λ is irreducible if $U^\lambda \neq \{0\}$.

We have shown that the set $T_r(A), \quad A \in Gl(V)$ spans all of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$. This implies that the elements $r^\lambda(A), A \in G$ spans all of $\mathrm{Hom}(U^\lambda, U^\lambda)$ where r^λ denotes the representation of $Gl(V)$ on U^λ. So if $U^\lambda \neq \{0\}$, the representation r^λ is irreducible.

3 The decomposition of $T^r(V)$ into irreducibles.

3.1 Review of the decomposition of $\mathcal{F}(G)$.

Recall that if G is any finite group, then we had a decomposition

$$\mathcal{F}(G) = \bigoplus U \otimes U^*$$

where U ranges over representatives of each of the irreducible representations of G, and that this decomposition is a $G \times G$ morphism. The injection of $U \otimes U^* \to \mathcal{F}(G)$ was the map which sends

$$u \otimes \ell \mapsto f_u^\ell, \quad f_u^\ell(a) = \langle r(a^{-1}u, \ell \rangle.$$

Since this map is injective, its transpose $\mathcal{F}(G)^* \to (U \otimes U^*)^*$ is surjective. We can use the trace on $U \otimes U^* = \mathrm{Hom}(U,U)$ to identify $\mathrm{Hom}(U,U)$ with its dual space. That is, we can think of $X \in \mathrm{Hom}(U,U)$ as that linear function on $\mathrm{Hom}(U,U)$ which sends $Y \in \mathrm{Hom}(U,U)$ into $\mathrm{tr}\, XY$. If ρ denotes the representation of G on U, and if $f \in \mathcal{F}(G)$, let $\hat{\rho}(f) \in \mathrm{Hom}(U,U)$ be defined by

$$\hat{\rho}(f)(u) := \sum_{a \in G} f(a)\rho(a)u$$

or, more compactly,

$$\hat{\rho}(f) = \sum_{a \in G} f(a)\rho(a).$$

Then

$$\begin{aligned} \mathrm{tr}(\hat{\rho}(f))(u \otimes \ell) &= \sum_{a \in G} f(a)\, \mathrm{tr}\, \rho(a)(u \otimes \ell) \\ &= \sum_{a \in G} f(a) \langle au, \ell \rangle \\ &= \sum_{a \in G} f(a) f_u^\ell(a^{-1}). \end{aligned}$$

We can identify $\mathcal{F}(G)$ with $\mathcal{F}(G)^*$ if we consider $f \in \mathcal{F}(G)$ as that linear function on $\mathcal{F}(G)$ which sends $h \in \mathcal{F}(G)$ into

$$\sum_{a \in G} f(a)h(a^{-1}).$$

With this identification, we see that the above string of identities implies that the map $f \mapsto \hat{\rho}(f)$ is the transpose of the map $U \otimes U^* \to \mathcal{F}(G)$. To say that this transpose is surjective is the same as saying that

Proposition 3.1 *The* $\rho(a), \quad a \in G$ *span all of* $\mathrm{Hom}(U,U)$.

In fact we have proved more: Let $U_1, \ldots, U_n$ be representatives of the inequivalent irreducible representations of G. Then

Proposition 3.2 *Given* $T_1 \in \mathrm{Hom}(U_1, U_1), \ldots, T_n \in \mathrm{Hom}(U_n, U_n)$ *then there exists an* $f \in \mathcal{F}(G)$ *such that* $\hat{\rho}_1(f) = T_1, \ldots, \hat{\rho}_n(f) = T_n$.

3.2 The Centralizer of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$.

Let us now examine more closely the decompostion

$$T^r(V) = \bigoplus U^\lambda \otimes F^\lambda. \qquad (1)$$

We showed (from Schur's lemma) that

$$\mathrm{Hom}_{S_r}(T^r(V), T^r(V)) = \bigoplus \mathrm{Hom}(U^\lambda, U^\lambda),$$

i.e. that every element of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$ is a sum of terms of the form $P_\lambda \otimes I_\lambda$ where $P_\lambda \in \mathrm{Hom}(U^\lambda, U^\lambda)$.

Now suppose that $R \in \mathrm{Hom}(T^r(V), T^r(V))$ commutes with all elements of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$. In terms of the decomposition (1) we can write R in "block form" as

$$R = (R_{\lambda\mu}), \quad R_{\lambda\mu} \in \mathrm{Hom}(U^\lambda \otimes F^\lambda, U^\mu \otimes F^\mu).$$

The assumption that R commutes with all elements of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$ then implies that

$$R_{\lambda\mu}(P_\lambda \otimes I_{F^\lambda}) = (P_\mu \otimes I_{F^\mu})R_{\lambda\mu}$$

for any choice of $P_\lambda \in \mathrm{Hom}(U^\lambda, U^\lambda)$ and $P_\mu \in \mathrm{Hom}(U^\mu, U^\mu)$ where the P^μ and P^λ can be chosen independently if $\mu \neq \lambda$. If we choose $P_\lambda = I_{U^\lambda}$ and $P_\mu = 0$, we see that

$$R_{\lambda\mu} = 0 \quad \text{if} \quad \lambda \neq \mu.$$

So R maps each subspace $U^\lambda \otimes F^\lambda$ into itself. The restriction $R_{\lambda\lambda}$ of R to each such subspace, commutes with all $P \otimes I$ and hence must be of the form $I \otimes Q$. (I have suppressed the λ.) But each such Q is a linear combination of elements of the form $\rho^\lambda(w), w \in S_r$, and by Prop. 3.2 we conclude that

Proposition 3.3 *If $R \in \mathrm{Hom}(T^r(V), T^r(V))$ commutes with all elements of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$, then R is a linear combination of elements of S_r acting on $T^r(V)$.*

3.3 The representations r^λ are inequivalent for different λ.

Suppose $\lambda \neq \mu$ and that $L : U^\lambda \to U^\mu$ commutes with all elements of $Gl(V)$. Then for any linear transformation $B : F^\lambda \to F^\mu$, the map $L \otimes B$ commutes with all of $\mathrm{Hom}_{S_r}(T^r(V), T^r(V))$. By Proposition 3.3 this implies that $L \otimes B$ is a linear combination of elements of S_r. But every element of S_r leaves each of the spaces $U^\nu \otimes F^\nu$ invariant. so $L = 0$, proving the assertion in the title of the subsection.

3.4 Summary

We have decomposed

$$T^r(V) = \bigoplus U^\lambda \otimes F^\lambda.$$

Some of the U^λ might be $\{0\}$. (We shall soon see that this happens when the number of rows in λ is greater than the dimension of V.) The representation T_r of $Gl(V)$ on $T^r(V)$ then determines a representation r^λ of $Gl(V)$ on U^λ which is irreducible. In fact, the elements $r^\lambda(A),\ A \in Gl(V)$ span all of $\mathrm{Hom}(U^\lambda, U^\lambda)$. So the representation $r^\lambda \otimes \rho^\lambda$ of $Gl(V) \times S_r$ on $U^\lambda \otimes F^\lambda$ is irreducible. The representations r^λ when $U^\lambda \neq \{0\}$ are all inequivalent. In fact each irreducible of S_r determines a unique irreducible r^λ of $Gl(V)$ provided that $U^\lambda \neq \{0\}$.

4 The Littlewood-Richardson rule and the Clebsch Gordon decomposition.

We claim that

$$U^\mu \otimes U^\nu = \bigoplus c^\lambda_{\mu,\nu} U^\lambda, \tag{2}$$

where the $c^\lambda_{\mu\nu}$ are the coefficients which enter into the Littlewood-Richardson rule. Indeed, under the tensor product identification $T^k(V) \otimes T^\ell(V) \to T^{k+\ell}(V)$ we get a map

$$(U^\mu \otimes F^\mu) \otimes (U^\nu \otimes F^\nu) = (U^\mu \otimes U^\nu) \otimes F^\mu \otimes F^\nu \to T^{k+\ell}(V).$$

So under $S_k \times S_\ell$ we have the decomposition

$$T^{k+\ell}(V) = \sum_{\mu,\nu} (U^\mu \otimes U^\nu) \otimes (F^\mu \otimes F^\nu).$$

Decompose $T^{k+\ell}(V)$ into irreducibles under $Gl(V) \times S_{k+\ell}$. So we are applying (1) where $r = k + \ell$. Restrict each summand of (1) to be a representation of $Gl(V) \times (S_k \times S_\ell)$. By the Frobenius reciprocity theorem and the Littlewood-Richardson rule we know that

$$F^\lambda \downarrow (S_k \times S_\ell) = \bigoplus c^\lambda_{\mu\nu} F^\mu \otimes F^\nu.$$

These two decompositions of $T^{k+\ell}(V)$ are $Gl(V) \times (S_k \times S_\ell)$ morphisms. Comparing the "coefficients" of the irreducible representation $F^\mu \otimes F^\nu$ of $S_k \times S_\ell$ proves (2).

4.1 $U^\lambda = \{0\}$ if the number of rows of λ exceeds dim V.

If μ consists of a single column with k rows, we know that $U^\mu = \wedge^k(V)$, the space of anti-symmentric tensors of degree k. If $k > \dim V$ we know that this vanishes. If λ has k rows, then we know by the Pieri formula, that $\lambda = \mu \cdot \nu$ for some ν, and so by our Clebsch Gordon decompostion (2) that $U^\lambda = U^\mu \otimes U^\nu$. But $U^\mu = \{0\}$, proving our assertion.

4.2 Peeling off determinants.

Now let's look at the case where λ has n rows where $n = \dim V$. Let μ be the column with n-rows. The vector space $U^\lambda = \wedge^n(V)$ is one dimensional and the representation of $Gl(V)$ on this space is

$$A \mapsto \det(A).$$

If λ has p boxes in its n-th row, then by repeated application of Pieri, we can write $\lambda = \mu^p \otimes \nu$ where ν has only $n-1$ rows. Then

$$r^\lambda(A) = \det(A)^p r^\nu(A).$$

4.3 Representations of $Sl(V)$.

In particular, the restriction of r^λ and r^ν (continuing the notation of ths previous subsection) to $Sl(V)$ are the same. It is not hard to show that different Young diagrams with $n-1$ or fewer rows give rise to inequivalent representations.

For example, if V is two dimensional, we need only consider representations corresponding to a single row, and if this row has d boxes, this is just $S^d(V)$.

5 A hook formula for dim U^λ.

Put an n in the upper left hand corner of λ where $n = \dim V$. Then fill the remaining boxes on the top row with $n+1, n+2$, etc. Then put $n-1$ in the leftmost box in the second row, and keep adding one as you move to the right. Put $n-2$ in the leftmost box of the third row etc.

Multiply all these integers and then divide by the product of the hook lengths. This is dim U^λ.

Example. Take $N = 3$ and $\lambda = (2,1)$. Then the numerator is $3 \cdot 4 \cdot 2$ while the denominator is 3. So the dimesnion of U^λ is 8.