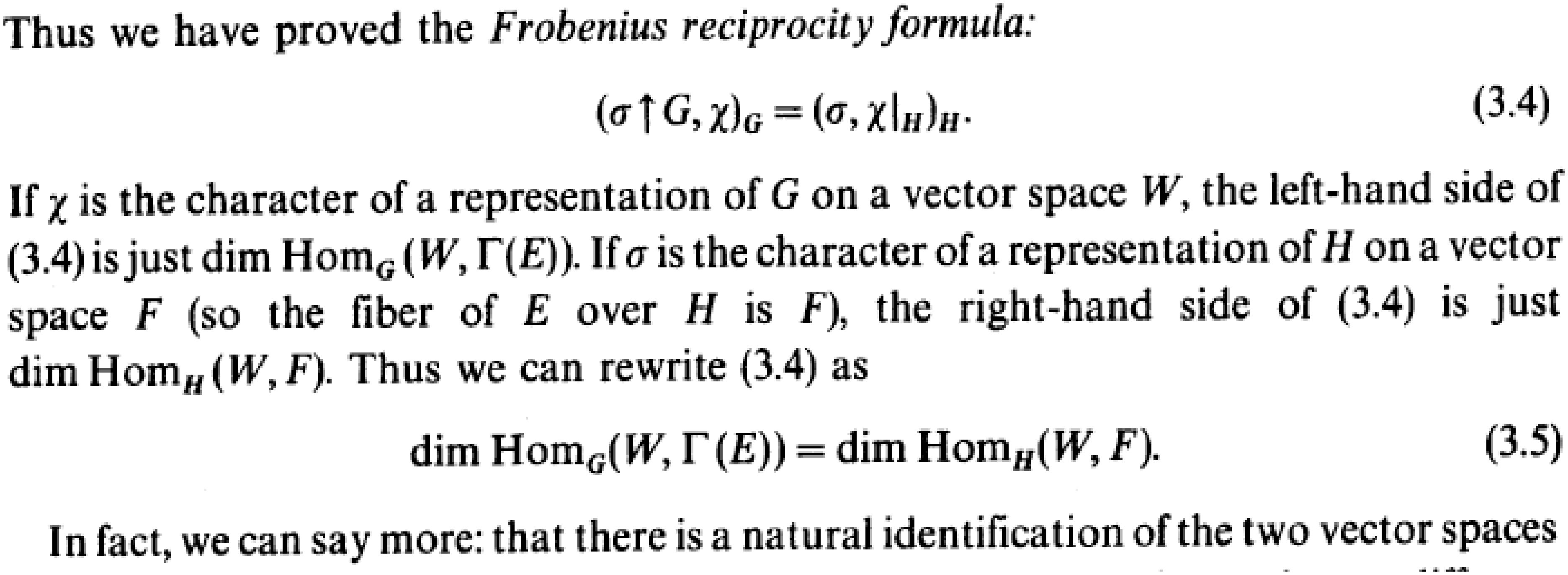
Induced representations

October 29, 2002

Frobenius reciprocity



Thus we have proved the *Frobenius reciprocity formula:*

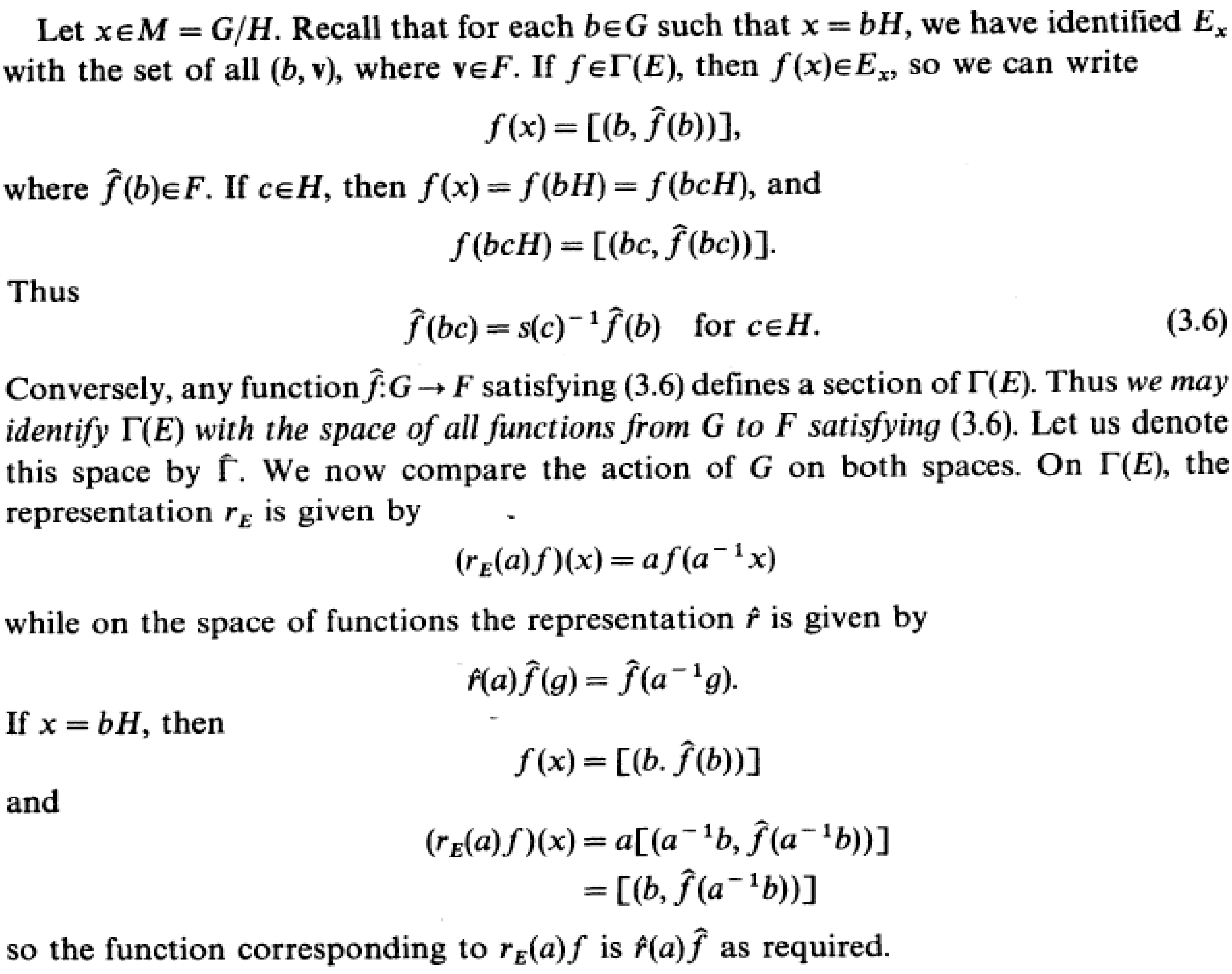
$$(\sigma \uparrow G, \chi)_G = (\sigma, \chi|_H)_H. \tag{3.4}$$

If χ is the character of a representation of G on a vector space W, the left-hand side of (3.4) is just $\dim \operatorname{Hom}_G(W, \Gamma(E))$. If σ is the character of a representation of H on a vector space F (so the fiber of E over H is F), the right-hand side of (3.4) is just $\dim \operatorname{Hom}_H(W, F)$. Thus we can rewrite (3.4) as

$$\dim \operatorname{Hom}_G(W, \Gamma(E)) = \dim \operatorname{Hom}_H(W, F). \tag{3.5}$$

In fact, we can say more: that there is a natural identification of the two vector spaces

A second construction of induced representations.



Let $x \in M = G/H$. Recall that for each $b \in G$ such that $x = bH$, we have identified E_x with the set of all $(b, \mathbf{v})$, where $\mathbf{v} \in F$. If $f \in \Gamma(E)$, then $f(x) \in E_x$, so we can write

$$f(x) = [(b, \hat{f}(b))],$$

where $\hat{f}(b) \in F$. If $c \in H$, then $f(x) = f(bH) = f(bcH)$, and

$$f(bcH) = [(bc, \hat{f}(bc))].$$

Thus

$$\hat{f}(bc) = s(c)^{-1}\hat{f}(b) \quad \text{for } c \in H. \tag{3.6}$$

Conversely, any function $\hat{f}: G \to F$ satisfying (3.6) defines a section of $\Gamma(E)$. Thus *we may identify* $\Gamma(E)$ *with the space of all functions from G to F satisfying* (3.6). Let us denote this space by $\hat{\Gamma}$. We now compare the action of G on both spaces. On $\Gamma(E)$, the representation r_E is given by

$$(r_E(a)f)(x) = af(a^{-1}x)$$

while on the space of functions the representation $\hat{r}$ is given by

$$\hat{r}(a)\hat{f}(g) = \hat{f}(a^{-1}g).$$

If $x = bH$, then

$$f(x) = [(b. \hat{f}(b))]$$

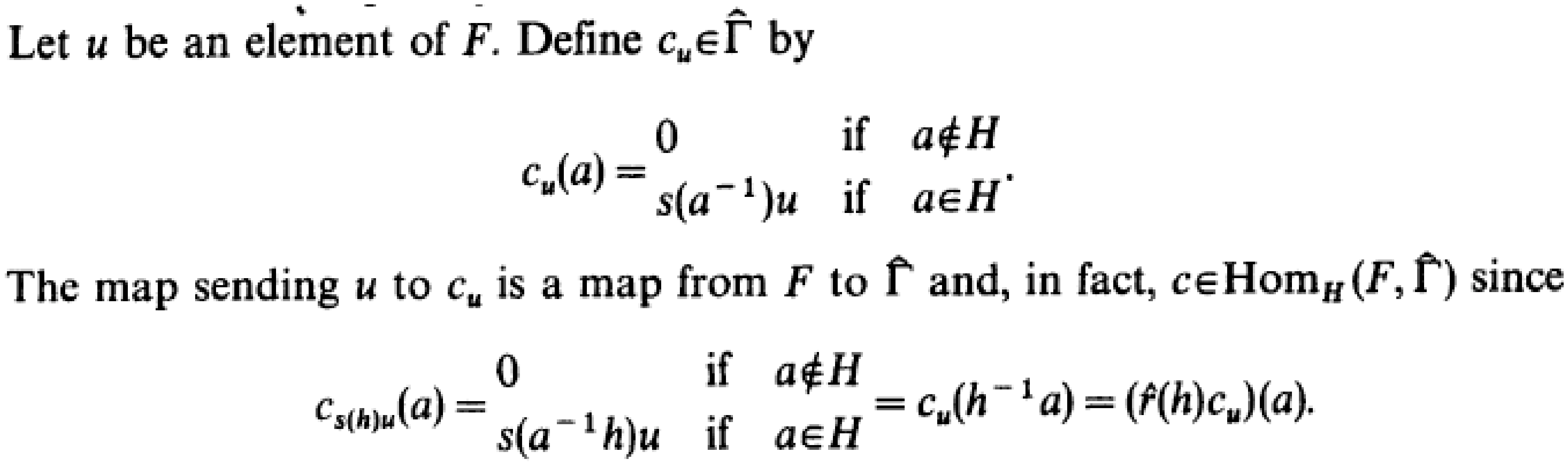
and

$$\begin{aligned}(r_E(a)f)(x) &= a[(a^{-1}b, \hat{f}(a^{-1}b))] \\ &= [(b, \hat{f}(a^{-1}b))]\end{aligned}$$

so the function corresponding to $r_E(a)f$ is $\hat{r}(a)\hat{f}$ as required.

Frobenius reciprocity

We shall use this alternative definition of the induced representation to give a proof of Frobenius reciprocity. We first locate the original representation of H on F inside the induced module:



Let u be an element of F. Define $c_u \in \hat{\Gamma}$ by

$$c_u(a) = \begin{array}{ll} 0 & \text{if} \quad a \notin H \\ s(a^{-1})u & \text{if} \quad a \in H \end{array}.$$

The map sending u to c_u is a map from F to $\hat{\Gamma}$ and, in fact, $c \in \mathrm{Hom}_H(F, \hat{\Gamma})$ since

$$c_{s(h)u}(a) = \begin{array}{ll} 0 & \text{if} \quad a \notin H \\ s(a^{-1}h)u & \text{if} \quad a \in H \end{array} = c_u(h^{-1}a) = (\hat{r}(h)c_u)(a).$$

Proof of Frobenius reciprocity

Now let (t, W) be any representation of G. Then, by restriction, we can think of W as a representation space of H. Let $S: F \to W$ be an element of $\mathrm{Hom}_H(F, W)$. Define

$T_S: \hat{\Gamma} \to W$ by $$T_S\hat{f} = \frac{1}{\#H}\sum_{a\in G} t(a)S\hat{f}(a)$$

We want to prove that $T_S \in \mathrm{Hom}_G(\hat{\Gamma}, W)$. To see this observe that

$$t(b)T_S\hat{f} = \frac{1}{\#H}\sum_{a\in G} t(ba)S\hat{f}(a)$$

$$= \frac{1}{\#H}\sum_{c\in G} t(c)S\hat{f}(b^{-1}c) \qquad a = b^{-1}c$$

$$= T_S\hat{t}(b)\hat{f}$$

which says that $T_S \in \mathrm{Hom}_G(\hat{\Gamma}, W)$.

The map s □ T_S is clearly linear. Also

$$T_S \cdot c_u = \frac{1}{\#H}\sum_{a\in H} t(a)Ss^{-1}(a)u$$

$$= \frac{1}{\#H}\sum_{a\in H} Ss(a)s^{-1}(a)u = Su.$$

Proof of Frobenius reciprocity.

Given $S \in \mathrm{Hom}_H(F, W)$, we thus get a $T_S \in \mathrm{Hom}_G(\hat{\Gamma}, W)$ with

$$T_S \circ c = S.$$

This gives an injection of $\mathrm{Hom}_H(F, W)$ into $\mathrm{Hom}_G(\hat{\Gamma}, W)$ which is an isomorphism, since we already know that the dimensions are the same.

Still more definitions of induced representation.

In our construction of $\Gamma(E)$ from a representation of H, the sections which vanish except at a point $x \in M=G/H$ are identified with elements of the fiber E_x . So if $m = H$, and $W:= E_m$, and $V:= \Gamma(E)$ then W is a subspace of the G-module V stable under the action of the elements of H and V is the direct sum of the images of W under the left cosets sH. This is the definition in Serre page 28. It suffices to use s which belong to a system of left coset representatives R.

The **group algebra** $\mathbf{C}[G]$ of a finite group G is the algebra which has a basis indexed by elements of G and whose multiplication extends that of G. Any G module becomes automatically a $\mathbf{C}[G]$ module.

If H is a subgroup of G then $\mathbf{C}[H]$ is a subalgebra of $\mathbf{C}[G]$ and the elements of R form a basis of $\mathbf{C}[G]$ considered as a $\mathbf{C}[H]$ module. If W is an H module then $\mathbf{C}[G] \otimes_{\mathbf{C}[H]} W$ is the induced module.

Induction followed by restriction.

- Let H and K be subgroups of G. We want to study the operation of inducing a module from H to G and then restricting to K.
- **Double cosets**: We let $K \times H$ act on G with K acting on the left and H on the right. The orbits for this action are called double cosets. We choose a set S of double coset representatives, which means that G is the disjoint union of the KsH as s ranges over S. For $s \in S$ we define $H_s := sHs^{-1} \cap K$ which is a subgroup of K.
- If r is a representation of H, we let r^s denote the representation of H given by $r^s(x) := r(s^{-1}xs)$.
- Claim $(r \uparrow G) \downarrow K = \bigoplus_s (r^s \uparrow K)$.

Proof that $(r\uparrow G)\downarrow K = \bigoplus(r^s\uparrow K)$.

V is the direct sum of xW for $x \in G/H$. Let V(s) be the sum over $x \in KsH$. So V is the direct sum of the V(s) and V(s) is invariant under K. We wish to show that each V(s) is induced from the representation r^s of H_s on W.

The subgroup of K fixing sW is H_s, and V(s) is the direct sum of the images xsW, $x \in K/H_s$.

The representation of H_s on sW is given by $k(sw) = (ks)w = s(s^{-1}ks)w$ and so is equivalent to r^s by the isomorphism $s: W \to sW$.

Mackey's irreducibility criterion.

This says that r↑G is irreducible if and only if
(a) r is irreducible and
(b) r^s and $r\square H_s$ are disjoint for all s □ H.

Let □ be the character of r and let □ be the character of r↑G . Irreducibility of r↑G says that $(\square, \square)_G = 1$. Frobenius recip. says that $(\square, \square)_G = (\square\square H, \square)_H$. If $\square^s$ denotes the character of r^s then by the previous slide

$$(r\uparrow G)\square H = \oplus\, r^s \uparrow H \qquad \text{or} \qquad \square\square H = \square\ \square^s \uparrow H.$$

By Frobenius reciprocity, $(\square\square H, \square)_H = (\square\ \square^s \uparrow H, \square)_H = (\square\ \square^s, \square\ \square H_s)_{H_s}$. Now when s=e, $H_s = H$ and $\square^s = \square$. Thus the summand corresponding to s =e must equal one, which is condition (a), and all other summands must vanish which is condition (b).

Representations of semi-direct products.

Let G be the semi-direct product of a group H and an abelian group A which means that every element of G can be written uniquely as ah with a □ A and h □ H. All irreductible representations of A are one dimensional and form a group X under multipication. The group H acts via conjuation on A and hence acts on X by

$$(s\square)(a)= \square(s^{-1}as).$$

Let $\{\square_i\}$be a system of representatives of the orbits of H on X and Let H_i be the isotropy group of $\square_i$. Let $G_i = A\, H_i$. Extend the function $\square_i$ to G_i by setting $\square_i\,(ah):= \square_i(a)$. We have

$$\square_i\,(ahbh')= \square_i\,(ahbh^{-1}\,hh')= \square_i\,(ahbh^{-1})= \square_i\,(a)\,\square_i\,(b)= \square_i\,(ah)\,\square_i\,(bh')$$

so this extension gives a one dimensional representation of G_i . Let r be an irreducible representation of H_i . It gives a representation of G_i via the homomorphism of G_i onto H_i . Take r $\square_i$ and induce up to G. Call this induced representation $R(r, \square_i)$. The claim is that these are all the irreducible representations of G.

Proof that R(r, $\square_i$) is irreducible.

Suppose that s $\square$ G_i and let $K_s := G_i \square s G_i s^{-1}$. Using Mackey's criterion, it is enough to show that the restriction of r $\square_i$ to K_s is disjoint from the representation (r $\square_i$ $)^s$. But already on A These are disjoint since $\square_i{}^s$ is not equivalent to $\square_i$.

Notice that R(r, $\square_i$) determines r and $\square_i$. Indeed, the restriction of R(r, $\square_i$) to A involves only the characters belonging to the H orbit of $\square_i$. So the orbit and hence i is determined. Let W be the space of R(r, $\square_i$) and let U be the subspace of W which Transfors under A according to the character $\square_i$. This space is stable under H_i and the restriction of H_i to this subspace acts via r.

Proof that the R(r, $\square_i$) are all the irreducibles.

Suppose that s is an irreducible representation of G on W. Decompose W according to the irreducibles of A: W =$\bigoplus$ W_i where W_i is the subspace of W consisting of those w which satisfy aw =$\square_i$(a) w, □ a□G. Since h W_i =W_j where h $\square_i$= $\square_j$ we see that the direct sum of those W_i where the $\square_I$ belong to a fixed orbit form an invariant subspace. So The direct sum is really over the $\square_i$ belonging to a single orbit, and hence is an induced representation from a subgroup of the form AH_i on W_i . By Mackey's theorem this representation must be irreducible and is of the form $\square_i$ r as desired.

The group AH_i is called the **little group**.

The irreducibles of D_4, or any even dihedral group.

Step 1. $D_4 = H \circledS N$, where $H = \{e, \alpha_1\}$ and $N = \{e, R_1, R_2, R_3\}$. Here $\square_1$ is a reflection and the R_i are rotations through 90, 180, and 270 degrees.

Step 2 Form N^*, the set of characters of N ($\cong C_4$) (see Table 21).

	e	R_1	R_2	R_3
χ_0	1	1	1	1
χ_1	1	i	-1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	-1	i

Step 3 Let H act on N^* by $h\chi_i(n) = \chi_i(h^{-1}nh)$ (see Table 22). The action breaks up N^* into three orbits as in Fig. 3.12.

	χ_0	χ_1	χ_2	χ_3
e	χ_0	χ_1	χ_2	χ_3
α_1	χ_0	χ_3	χ_2	χ_1

Fig. 3.12

The irreducibles of D_4, continued.

Step 4 Pick an orbit and a point in the orbit. Find L_i, the isotropy subgroup, which fixed the point. For the single-element orbits $L_i = H$. For the double-element orbit $L_i = \{e\}$.

Step 5 Consider the two-element orbit and choose χ_1 to be identified with the coset N. Then χ_3 is identified with the coset $\alpha_1 N$. Construct a vector bundle over the two points by taking as basis elements $e_1 = [(e, v_0)]$ and $e_2 = [(\alpha_1, v_0)]$.

Step 6 Calculate representation matrices by letting $G = D_4$ act on basis elements. Use

$[(bl,v)]=[(b, \square(l)v)]$ where $\sigma(l) = \chi_1(n)\rho(h)$ and $l = hn$. Since $H = \{e\}$, $\rho(h) \equiv 1$. So

$$R_1[(e, v_0)] = [(R_1, v_0)] = [(eR_1, v_0)] = [(e, \sigma(R_1)v_0)]$$
$$= [(e, \chi_1(R_1)v_0)] = [(e, iv_0)] = i[(e, v_0)] = ie_1$$

The irreducibles of D_4, continued.

Also

$$R_1[(\alpha_1, v_0)] = [(R_1\alpha_1, v_0)] = [(\alpha_1 R_3, v_0)] = [(\alpha_1, \sigma(R_3)v_0)]$$
$$= [(\alpha_1, \chi_1(R_3)v_0)] = [(\alpha_1, -iv_0)] = -ie_2.$$

Thus R_1 is represented by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

For α_1 we have

$$\alpha_1[(e, v_0)] = [(\alpha_1, v_0)] = e_2$$

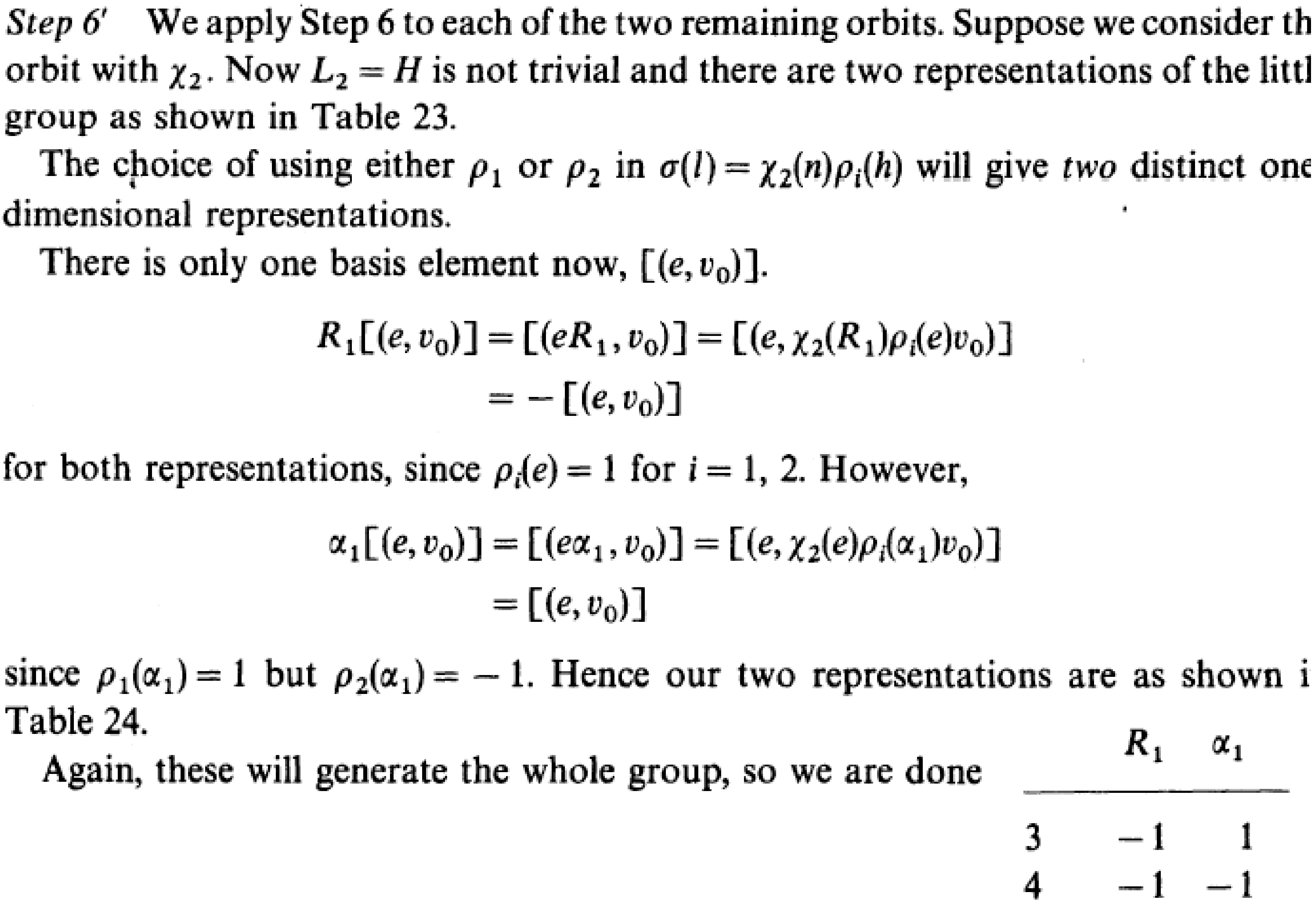
and

$$\alpha_1[(\alpha_1, v_0)] = [(e, v_0)] = e_1.$$

So α_1 is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Since $R_1\alpha_1 = \beta_1$ we can, with these two matrices, recover all of the matri representations of the group. So this orbit gave the two-dimensional representation c D_4.

The irreducibles of D_4, continued.



Step 6′ We apply Step 6 to each of the two remaining orbits. Suppose we consider th orbit with χ_2. Now $L_2 = H$ is not trivial and there are two representations of the littl group as shown in Table 23.

The choice of using either ρ_1 or ρ_2 in $\sigma(l) = \chi_2(n)\rho_i(h)$ will give *two* distinct one dimensional representations.

There is only one basis element now, $[(e, v_0)]$.

$$\begin{aligned} R_1[(e, v_0)] &= [(eR_1, v_0)] = [(e, \chi_2(R_1)\rho_i(e)v_0)] \\ &= -[(e, v_0)] \end{aligned}$$

for both representations, since $\rho_i(e) = 1$ for $i = 1, 2$. However,

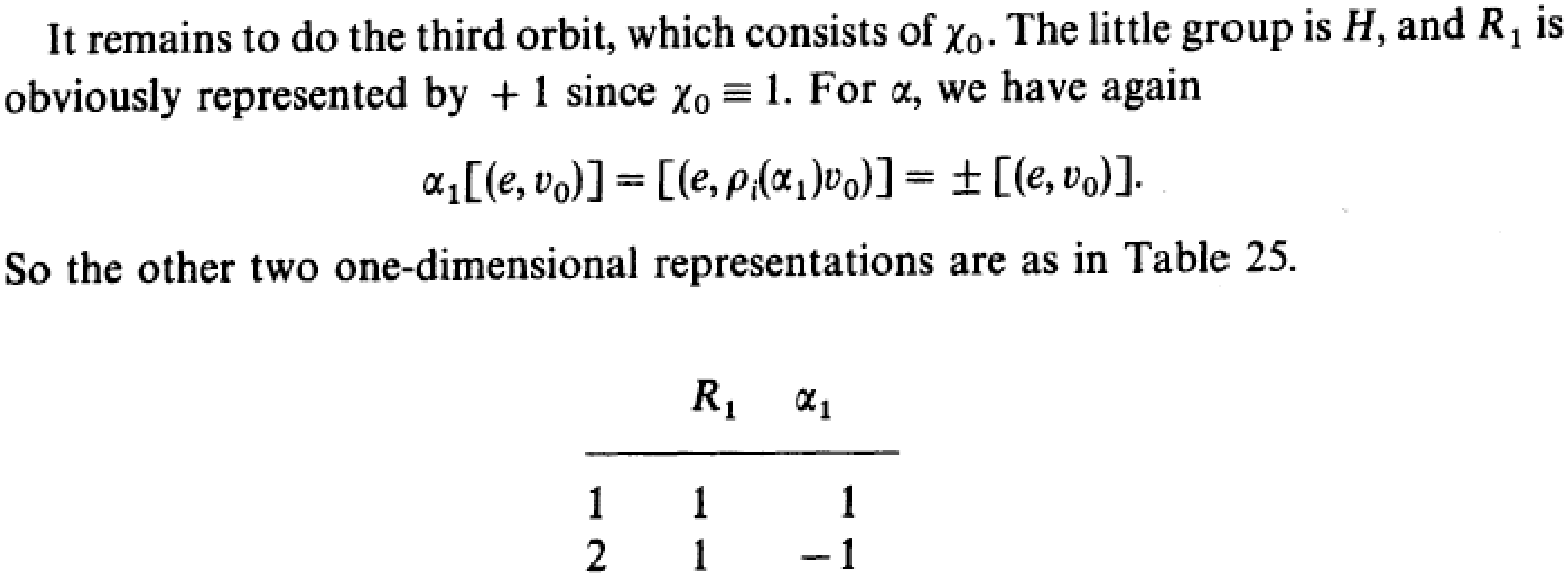
$$\begin{aligned} \alpha_1[(e, v_0)] &= [(e\alpha_1, v_0)] = [(e, \chi_2(e)\rho_i(\alpha_1)v_0)] \\ &= [(e, v_0)] \end{aligned}$$

since $\rho_1(\alpha_1) = 1$ but $\rho_2(\alpha_1) = -1$. Hence our two representations are as shown i Table 24.

Again, these will generate the whole group, so we are done

	R_1	α_1
3	-1	1
4	-1	-1

The irreducibles of D$_4$, continued.



It remains to do the third orbit, which consists of χ_0. The little group is H, and R_1 is obviously represented by $+1$ since $\chi_0 \equiv 1$. For α, we have again

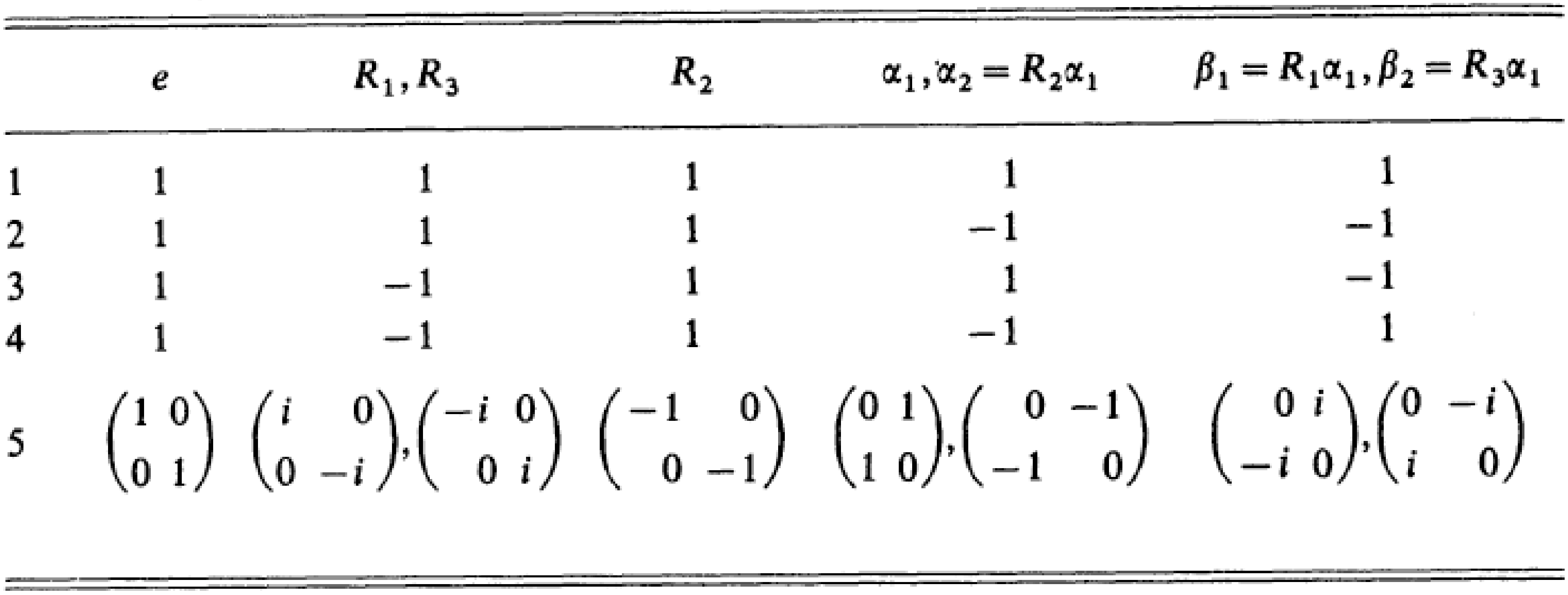
$$\alpha_1[(e, v_0)] = [(e, \rho_i(\alpha_1)v_0)] = \pm[(e, v_0)].$$

So the other two one-dimensional representations are as in Table 25.

	R_1	α_1
1	1	1
2	1	-1

The irreducibles of D_4, concluded.

Our results are summarized in the table



	e	R_1, R_3	R_2	$\alpha_1, \alpha_2 = R_2\alpha_1$	$\beta_1 = R_1\alpha_1, \beta_2 = R_3\alpha_1$
1	1	1	1	1	1
2	1	1	1	-1	-1
3	1	-1	1	1	-1
4	1	-1	1	-1	1
5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$